On a relation between canonical bases and Enyang's bases of B-M-W algebras

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Abstract

In this note, we give an explicit expression of the canonical basis of the Birman-Murakami-Wenzl algebras $\mathcal{B}_3$ and $\mathcal{B}_4$, in the sense of S.Fishel and I.Grojnowski. We also compare this basis and Enyang's basis, which is analogous to the Murphy basis of the Iwahori-Hecke algebra of type A.

1 Introduction

The Birman-Murakami-Wenzl algebra (B-M-W algebra for short) was introduced independently by J.Birman and H.Wenzl in [1] and J.Murakami in [6] in order to investigete an algebraic formulation of some new polynomial invariants of links in knot theory. This is a finite dimensional algebra defined over a rational function field in two variables, and can be considered as a deformation of the Brauer algebra. The specialized B-M-W algebra appears as the centralizer algebra for the quantum group associated to the symplectic Lie algebra. And the Iwahori-Hecke algebra is the quotient algebra of the B-M-W algebra. Therefore the B-M-W algebra has been studied by using the theory of Iwahori-Hecke algebra.

S.Fishel and I.Grojnowski [4] introduced the canonical basis of the B-M-W algebra as an analogue of the Kazhdan-Lusztig basis of the Iwahori-Hecke algebra of type A. C.Xi [7] proved that this basis is cellular. But the recipe of the explicit construction of this basis was not given in [4]. In this note we will provide an explicit expression of the canonical basis of $\mathcal{B}_3$ and $\mathcal{B}_4$.

On the other hand, J.Enyang constructed another cellular basis of the same algebra by using some combinatorial methods in [2] and [3], which is analogous to the Murphy basis of the Iwahori-Hecke algebra of type A. For the Iwahori-Hecke algebra of type A, M.Geck [5] gives an explicit relation between the Kazhdan-Lusztig basis and the Murphy basis. Motivated by this fact, we will investigate the relation between Fishel-Grojnowski's canonical basis and Enyang's basis in the cases of $\mathcal{B}_3$ and $\mathcal{B}_4$, and it will be observed that the transition matrices from Enyang's basis to Fishel-Grojnowski's canonical basis are triangular.

2 Some known results

Let q, r be indeterminates over $\mathbb{Z}$, and let $R = \mathbb{Z}[q^{\pm 1}, r^{\pm 1}, (q-q^{-1})^{-1}]$ be the localization of $\mathbb{Z}[q^{\pm 1}, r^{\pm 1}]$ at $(q-q^{-1})$ and define the element $x \in R$ to be

$$x = \frac{r - r^{-1}}{q - q^{-1}} + 1.$$

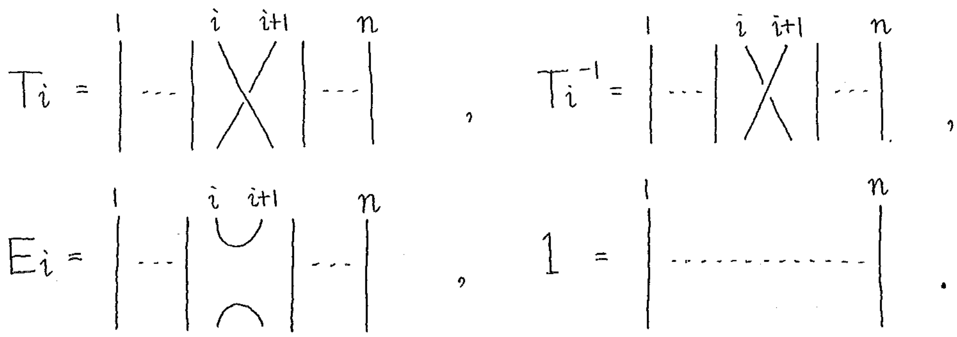
We define the generic B-M-W algebra $\mathcal{B}_n(q, r)$ or just $\mathcal{B}_n$ over R as the unital associative R-algebra generated by the elements $\{T_i \mid 1 \leq i < n\}$, subject to the following relations:

$$\begin{gathered}
(T_i - q)(T_i - r^{-1})(T_i + q^{-1}) = 0 \qquad for \qquad 1 \leq i < n \\
T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1} \qquad for \qquad 1 \leq i < n-1 \\
T_i T_j = T_j T_i \qquad for \qquad 2 \leq |i-j|\,,\, 1 \leq i, j < n \\
E_{i+1} T_i^{\pm 1} E_{i+1} = r^{\pm 1} E_{i+1} \qquad for \qquad 1 \leq i < n-1 \\
E_{i-1} T_i^{\pm 1} E_{i-1} = r^{\pm 1} E_{i-1} \qquad for \qquad 2 \leq i < n \\
E_i T_i = T_i E_i = r^{-1} E_i \qquad for \qquad 1 \leq i < n
\end{gathered}$$

where E_i is defined by Kauffman's skein relation

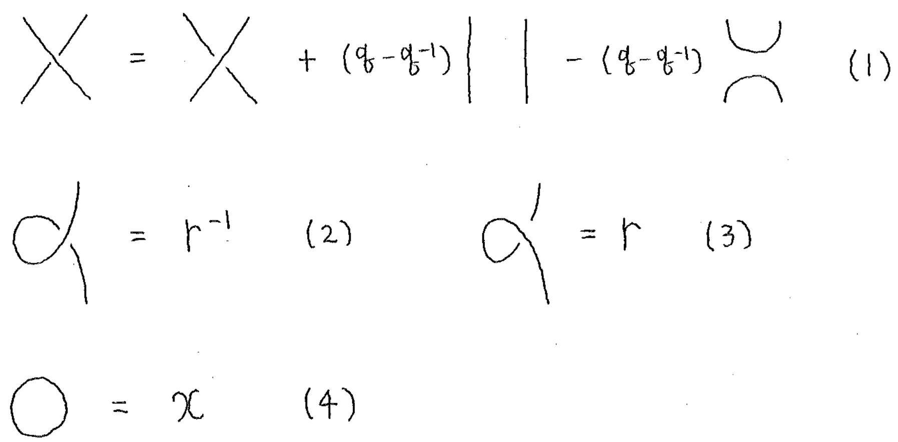
$$(q - q^{-1})(1 - E_i) = T_i - T_i^{-1}.$$

The algebra $\mathcal{B}_n(q,r)$ has also a geometric interpretation using Kauffman's tangle monoid. We set



These are examples of "n-tangle" defined in [4] and the set of all n-tangles forms a monoid. In this case, the multiplication is defined by concatenation of two diagrams, placing the left element above the right element, and we allow the Reidemeister moves of types II and III in an n-tangle.

Proposition 1. *Let M_n be the set of n-tangles which are generated by the above $T_1^{\pm 1}, \cdots, T_{n-1}^{\pm 1}$, $E_1, \cdots, E_{n-1}$ and 1. Then, $\mathcal{B}_n$ is the quotient algebra of the monoid algebra $R[M_n]$, subject to the following relations:*



Here, we apply these relations locally in an n-tangle.

Next, we define a "section" in order to produce the natural basis of the B-M-W algebra obtained from the set of n-Brauer diagrams. Here, an n-Brauer diagram d is a set-partition of $\{1, 2, \cdots, 2n\}$ into two-element subsets, and can be represented by a picture drawn in the plane as follows: we consider a rectangle in the plane and put n vertices labelled by $1, 2, \cdots, n$ on the top and n vertices labelled by $2n, 2n-1, \cdots, n+1$ on the bottom. If $\{i, j\} \in d$, then we join two vertices i and j by an edge so that two egdes intersect at most once, there are no self-intersections and at most two edges intersect at any point.

Then, the section is a map T obtained by replacing each crossing

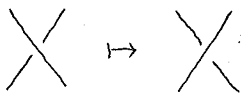
,

where the edge α (resp. β) connects vertices i and k (resp. j and l) and $i < j < k < l$. Then the following fact is known.

Proposition 2. *Let B_n be the set of all n-Brauer diagrams. Then, the set $\{T_d \mid d \in B_n\}$ forms a basis of $\mathcal{B}_n$.*

We will call this basis the "natural basis" of the B-M-W algebra afterwards.

It is known that there exists an algebra automorphism $\overline{} : \mathcal{B}_n \to \mathcal{B}_n$, called the "bar-involution", satisfying



and $q \mapsto q^{-1}$, $r \mapsto r^{-1}$.

Then, because of the relation (1), we have the following equation:

$$\overline{T_d} = T_d + \sum_{d' : l(d') < l(d)} r_{d'd} T_{d'},$$

where, $l(d)$ is the number of crossings contained in a diagram $d \in B_n$.

With respect to this involution, S.Fishel and I.Grojnowski proved in [4] the next statement.

Proposition 3. *For the above bar-involution, there exists a unique basis* $\{C_{T_d} \mid d \in B_n\}$ *which satisfies the following properties:*

- $\overline{C_{T_d}} = C_{T_d}$
- $C_{T_d} = T_d + \sum_{d':l(d')<l(d)} P_{d'd} T_{d'}$, *where* $P_{d'd} \in q^{-1}\mathbb{Z}[q^{-1}]$

We will call this basis the "canonical basis" of the B-M-W algebra afterwards.

3 Calculation of canonical basis

In this section, we give an explicit expression of the canonical basis in $\mathcal{B}_3$ and $\mathcal{B}_4$.

First, by using Kauffman's skein relation, we get

$$T_i + q^{-1} - q^{-1}E_i = T_i^{-1} + q - qE_i,$$

which leads to the expression

$$C_{T_i} := T_i + q^{-1} - q^{-1}E_i.$$

We also find out that

$$C_{E_i} := E_i.$$

Now we go on to construct the canonical basis from the natural basis. We obtain the following expressions in $\mathcal{B}_3$ and $\mathcal{B}_4$:

canonical basis of $\mathcal{B}_3$

1, E_1, E_2, E_1E_2, E_2E_1, $T_1 + q^{-1} - q^{-1}E_1$, $T_2 + q^{-1} - q^{-1}E_2$,
$T_1E_2 + q^{-1}E_2 - q^{-1}E_1E_2$, $E_2T_1 + q^{-1}E_2 - q^{-1}E_2E_1$,
$T_1E_2E_1 + q^{-1}E_2E_1 - q^{-1}E_1$, $E_1E_2T_1 + q^{-1}E_1E_2 - q^{-1}E_1$,
$T_1T_2 + q^{-1}T_1 + q^{-1}T_2 - q^{-1}T_1E_2 - q^{-1}E_1E_2T_1 - q^{-2}E_2 + q^{-2}$,
$T_2T_1 + q^{-1}T_2 + q^{-1}T_1 - q^{-1}E_2T_1 - q^{-1}T_1E_2E_1 - q^{-2}E_2 + q^{-2}$,
$T_1E_2T_1 + q^{-1}T_1E_2 + q^{-1}E_2T_1 - q^{-1}T_1E_2E_1 - q^{-1}E_1E_2T_1 + q^{-2}E_1 + q^{-2}E_2 - q^{-2}E_1E_2 - q^{-2}E_2E_1$,
$T_1T_2T_1 + q^{-1}T_1T_2 + q^{-1}T_2T_1 - q^{-1}T_1E_2T_1 + q^{-2}T_1 + q^{-2}T_2 - q^{-2}T_1E_2 - q^{-2}E_2T_1 - q^{-1}E_1 - q^{-3}E_2 + q^{-3}$

canonical basis of $\mathcal{B}_4$

$1,\quad E_1,\quad E_2,\quad E_3,\quad E_1E_2,\quad E_2E_3,\quad E_2E_1,\quad E_3E_2,\quad E_1E_3,$
$E_1E_2E_3,\quad E_3E_2E_1,\quad E_1E_3E_2,\quad E_2E_3E_1,\quad E_2E_1E_3E_2,$
$T_1+q^{-1}-q^{-1}E_1,\quad T_2+q^{-1}-q^{-1}E_2,\quad T_3+q^{-1}-q^{-1}E_3,$
$T_1E_2+q^{-1}E_2-q^{-1}E_1E_2,\quad T_2E_3+q^{-1}E_3-q^{-1}E_2E_3,$
$E_2T_1+q^{-1}E_2-q^{-1}E_2E_1,\quad E_3T_2+q^{-1}E_3-q^{-1}E_3E_2,$
$T_1E_2E_1+q^{-1}E_2E_1-q^{-1}E_1,\quad T_2E_3E_2+q^{-1}E_3E_2-q^{-1}E_2,$
$E_1E_2T_1+q^{-1}E_1E_2-q^{-1}E_1,\quad E_2E_3T_2+q^{-1}E_2E_3-q^{-1}E_2,$
$T_1E_3+q^{-1}E_3-q^{-1}E_1E_3,\quad E_1T_3+q^{-1}E_1-q^{-1}E_1E_3,$
$T_1E_2E_3+q^{-1}E_2E_3-q^{-1}E_1E_2E_3,\quad E_3E_2T_1+q^{-1}E_3E_2-q^{-1}E_3E_2E_1,$
$T_1E_3E_2+q^{-1}E_3E_2-q^{-1}E_1E_3E_2,\quad T_3E_1E_2+q^{-1}E_1E_2-q^{-1}E_1E_3E_2,$
$E_2E_3T_1+q^{-1}E_2E_3-q^{-1}E_2E_3E_1,\quad E_2E_1T_3+q^{-1}E_2E_1-q^{-1}E_2E_3E_1,$
$E_1E_2E_3T_1+q^{-1}E_1E_2E_3-q^{-1}E_1E_3,\quad E_1E_2E_3T_2+q^{-1}E_1E_2E_3-q^{-1}E_1E_2,$
$T_1E_3E_2E_1+q^{-1}E_3E_2E_1-q^{-1}E_1E_3,\quad T_2E_3E_2E_1+q^{-1}E_3E_2E_1-q^{-1}E_2E_1,$
$T_1E_2E_3E_1+q^{-1}E_2E_3E_1-q^{-1}E_1E_3,\quad T_1E_2E_1E_3E_2+q^{-1}E_2E_1E_3E_2-q^{-1}E_1E_3E_2,$
$E_1E_3E_2T_1+q^{-1}E_1E_3E_2-q^{-1}E_1E_3,\quad E_2E_1E_3E_2T_1+q^{-1}E_2E_1E_3E_2-q^{-1}E_2E_3E_1,$
$E_2T_1E_3E_2+q^{-1}E_2-q^{-1}E_2E_1E_3E_2,$
$T_1T_2+q^{-1}T_1+q^{-1}T_2-q^{-1}T_1E_2-q^{-1}E_1E_2T_1-q^{-2}E_2+q^{-2},$
$T_2T_3+q^{-1}T_2+q^{-1}T_3-q^{-1}T_2E_3-q^{-1}E_2E_3T_2-q^{-2}E_3+q^{-2},$
$T_2T_1+q^{-1}T_2+q^{-1}T_1-q^{-1}E_2T_1-q^{-1}T_1E_2E_1-q^{-2}E_2+q^{-2},$
$T_3T_2+q^{-1}T_3+q^{-1}T_2-q^{-1}E_3T_2-q^{-1}T_2E_3E_2-q^{-2}E_3+q^{-2},$
$T_1T_3+q^{-1}T_1+q^{-1}T_3-q^{-1}T_1E_3-q^{-1}E_1T_3-q^{-2}E_1-q^{-2}E_3+q^{-2}E_1E_3+q^{-2},$

$$\vdots$$

$T_1T_2T_3T_1T_2T_1+q^{-1}T_1T_2T_3T_1T_2+q^{-1}T_1T_2T_3T_2T_1+q^{-1}T_2T_1T_3T_2T_1-q^{-1}T_1T_2T_1E_3T_2T_1+$
$q^{-2}T_1T_2T_1T_3+q^{-2}T_1T_2T_3T_2+q^{-2}T_2T_1T_3T_2+q^{-2}T_2T_3T_2T_1+q^{-2}T_3T_1T_2T_1-$
$q^{-2}T_1T_2E_3T_1T_2-q^{-2}T_1T_2E_3T_2T_1-q^{-2}T_2T_1E_3T_2T_1+q^{-3}T_1T_2T_3+q^{-3}T_1T_3T_2+$
$q^{-3}T_1T_2T_1+q^{-3}T_2T_3T_2+q^{-3}T_2T_3T_1+q^{-3}T_3T_2T_1-q^{-3}T_1T_2T_1E_3-q^{-3}T_1T_2E_3T_2-$
$q^{-3}T_2T_1E_3T_2-q^{-3}T_2E_3T_2T_1-q^{-3}E_3T_1T_2T_1-q^{-1}T_1E_2T_1E_3E_2T_1+q^{-4}T_1T_2+$
$q^{-4}T_2T_3+q^{-4}T_1T_3+q^{-4}T_3T_2+q^{-4}T_2T_1+q^{-2}T_1E_2E_1E_3E_2T_1-q^{-4}T_1T_2E_3-$
$q^{-4}T_2T_1E_3-q^{-4}T_2E_3T_2-q^{-4}E_3T_1T_2-q^{-4}E_3T_2T_1-q^{-2}T_1E_2T_1-q^{-2}T_1E_2E_1T_3E_2-$
$q^{-2}E_2T_3E_1E_2T_1+q^{-5}T_1+q^{-5}T_2+q^{-5}T_3-q^{-5}T_1E_3-q^{-5}T_2E_3-q^{-5}E_3T_2+$
$q^{-3}T_1E_2E_1E_3E_2+q^{-3}E_2E_1E_3E_2T_1-q^{-3}T_1E_2-q^{-3}E_2T_1-q^{-3}E_2T_1E_3E_2-$
$q^{-1}E_1T_3-q^{-2}E_1-q^{-4}E_2-q^{-6}E_3+q^{-2}E_1E_3+q^{-4}E_2E_1E_3E_2+q^{-6}$

4 Comparison with Enyang's basis

Finally, we compare Fishel-Grojnowski's canonical basis and Enyang's basis in $\mathcal{B}_3$. The transition matrix becomes triangular. The same phenomenon occurs in the case of $\mathcal{B}_4$, but we omit the explicit form of the transition matrix here.

We use Enyang's basis introduced in [3], and review it succinctly. Define $\mathcal{B}_n^f$ to be the two sided ideal of $\mathcal{B}_n$ generated by the element $E_1E_3\cdots E_{2f-1}$. Then,

$$\{0\} = \mathcal{B}^{[\frac{n}{2}]+1} \subset \cdots \subset \mathcal{B}^1 \subset \mathcal{B}^0 = \mathcal{B}_n$$

gives a filtration of $\mathcal{B}_n$, which enables us to use the cellular structure of the Iwahori-Hecke algebra of type A.

For an integer f with $0 \leq f \leq [\frac{n}{2}]$ and a partition $\lambda = (\lambda_1, \cdots, \lambda_k)$ of $n-2f$, define the element

$$m_\lambda = E_1E_3\cdots E_{2f-1}\sum_{w\in\mathfrak{S}_\lambda} q^{l(w)}T_w,$$

where $\mathfrak{S}_\lambda = \mathfrak{S}_{\lambda_1}\times\cdots\times\mathfrak{S}_{\lambda_k}$ is a subgroup of the permutation group on $\{2f+1, 2f+2, \cdots, n\}$, and l denotes the usual length function.

We also define $\mathfrak{T}_n(\lambda)$ to be the set of paths of shape λ in the Bratteli diagram of $\mathcal{B}_n$. Then, J.Enyang proved in [3] the next statement.

Proposition 4. *$\mathcal{B}_n$ is freely generated as an R-module by the following set which becomes a cellular basis with respect to the algebra anti-involution $*$: $T_i \mapsto T_i$, $E_i \mapsto E_i$:*

$$\{m_{\mathfrak{s}\mathfrak{t}} = b_\mathfrak{s}^* m_\lambda b_\mathfrak{t} \,|\, \mathfrak{s},\mathfrak{t}\in\mathfrak{T}_n(\lambda),\ \lambda\vdash n-2f \textit{ and } 0\leq f\leq [\frac{n}{2}]\},$$

where $b_\mathfrak{t}$ is determined uniquely by lifting bases for cell modules of $\mathcal{B}_{n-1}$.

Now, we label Enyang's basis elements and Fishel-Grojnowski's basis elements of $\mathcal{B}_3$ as follows:

Enyang's basis

$\boxed{1}$ 1, $\boxed{2}$ E_1, $\boxed{3}$ E_2, $\boxed{4}$ E_1E_2, $\boxed{5}$ E_2E_1, $\boxed{6}$ $1+qT_1$,

$\boxed{7}$ $(1+qT_1)E_2$, $\boxed{8}$ $E_2(1+qT_1)$, $\boxed{9}$ $(1+qT_1)E_2E_1$, $\boxed{10}$ $E_1E_2(1+qT_1)$,

$\boxed{11}$ $(1+qT_1)T_2$, $\boxed{12}$ $T_2(1+qT_1)$, $\boxed{13}$ $(1+qT_1)E_2(1+qT_1)$,

$\boxed{14}$ $(1+qT_1)(1+qT_2+q^2T_2T_1)$, $\boxed{15}$ $T_2(1+qT_1)T_2$

Fishel-Grojnowski's basis

① 1, ② E_1, ③ E_2, ④ E_1E_2, ⑤ E_2E_1, ⑥ $T_1+q^{-1}-q^{-1}E_1$,

⑦ $T_1E_2+q^{-1}E_2-q^{-1}E_1E_2$, ⑧ $E_2T_1+q^{-1}E_2-q^{-1}E_2E_1$,

⑨ $T_1E_2E_1+q^{-1}E_2E_1-q^{-1}E_1$, ⑩ $E_1E_2T_1+q^{-1}E_1E_2-q^{-1}E_1$,

⑪ $T_1T_2+q^{-1}T_1+q^{-1}T_2-q^{-1}T_1E_2-q^{-1}E_1E_2T_1-q^{-2}E_2+q^{-2}$,

⑫ $T_2T_1+q^{-1}T_2+q^{-1}T_1-q^{-1}E_2T_1-q^{-1}T_1E_2E_1-q^{-2}E_2+q^{-2}$,

⑬ $T_1E_2T_1+q^{-1}T_1E_2+q^{-1}E_2T_1-q^{-1}T_1E_2E_1-q^{-1}E_1E_2T_1+q^{-2}E_1+q^{-2}E_2$
$-q^{-2}E_1E_2-q^{-2}E_2E_1$,

⑭ $T_1T_2T_1+q^{-1}T_1T_2+q^{-1}T_2T_1-q^{-1}T_1E_2T_1+q^{-2}T_1+q^{-2}T_2-q^{-2}T_1E_2$
$-q^{-2}E_2T_1-q^{-1}E_1-q^{-3}E_2+q^{-3}$,

⑮ $T_2+q^{-1}-q^{-1}E_2$

Then, the transition matrix is discribed in the following way:

Transition matrix in $\mathcal{B}_3$

$$
\begin{pmatrix} ① \\ ② \\ ③ \\ ④ \\ ⑤ \\ ⑥ \\ ⑦ \\ ⑧ \\ ⑨ \\ ⑩ \\ ⑪ \\ ⑫ \\ ⑬ \\ ⑭ \\ ⑮ \end{pmatrix}
=
\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -q^{-1} & 0 & 0 & 0 & q^{-1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -q^{-1} & 0 & 0 & q^{-1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -q^{-1} & 0 & 0 & q^{-1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -q^{-1} & 0 & 0 & 0 & 0 & 0 & 0 & q^{-1} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -q^{-1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & q^{-1} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & q^{-2} & 0 & q^{-2} & -q^{-2} & 0 & 0 & -q^{-2} & q^{-1} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & q^{-2} & q^{-2} & 0 & -q^{-2} & -q^{-2} & 0 & 0 & q^{-1} & 0 & 0 & 0 \\
0 & q^{-2} & 0 & 0 & 0 & 0 & 0 & 0 & -q^{-2} & -q^{-2} & 0 & 0 & q^{-2} & 0 & 0 \\
0 & -q^{-1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -q^{-3} & q^{-3} & 0 \\
0 & 0 & -q^{-1}-q^{-2}r^{-1}+r^{-1} & 0 & 0 & q^{-3} & 0 & 0 & 0 & 0 & q^{-2} & q^{-2} & 0 & -q^{-3} & q^{-1}
\end{pmatrix}
\begin{pmatrix} \boxed{1} \\ \boxed{2} \\ \boxed{3} \\ \boxed{4} \\ \boxed{5} \\ \boxed{6} \\ \boxed{7} \\ \boxed{8} \\ \boxed{9} \\ \boxed{10} \\ \boxed{11} \\ \boxed{12} \\ \boxed{13} \\ \boxed{14} \\ \boxed{15} \end{pmatrix}
$$

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