鏡映群と一意化方程式[1]

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Abstract. このノートでは，対数的自由因子とそのような因子に沿って特異点をもつ一意化方程式を論じる．実鏡映群，複素鏡映群の判別式から定まる超曲面はそのような因子の例である．3 次元アフィン空間に限定して，このような因子に沿った一意化方程式を求めて，簡単な性質に言及する．また一意化方程式についての問題を定式化する．実鏡映群，複素鏡映群の判別式に関係しない場合にもいくつか得られた成果を述べる．([13] には関連した話題を論じられている．)

もとの予稿集原稿の主要部を英文に書き直したものが [14] に掲載してある．その原稿と重複する恐れもあるので，ホームページに公開するにあたり，一部を削除したりして内容を少し書き直したことをお断りしておく．

§1. Introduction. In this note, I report recent progress on systems of uniformization equations with respect to logarithmic free divisors. The hypersurface defined by the discriminant of a real reflection group is a typical example of logarithmic free divisors. I believe that the hypersurface defined by the discriminant of a complex reflection group is also a logarithmic free divisor, though I don't know its unified proof. My interests are to construct (1) logarithmic free divisors , (2) systems of uniformization equations, and (3) their solutions in a concrete manner. Restricting to the case of three dimensional affine space, I obtained some results to (1), (2). But it is difficult to attack (3) compared with (1), (2). The purpose of this note is to report my results on (1), (2) for the discriminants of irreducible complex reflection groups of rank three.

§2. Definition of logarithmic free divisors. Let $F(x) = F(x_1, x_2, \ldots, x_n)$ be a reduced polynomial. Then $D = \{x \in \mathbf{C}^n;\ F(x) = 0\}$ is a (weighted homogeneous) logarithmic free divisor if (C1)+(C2) hold.

(C1) There is a vector field

$$E = \sum_{i=1}^{n} m_i x_i \partial_{x_i}$$

such that $EF = dF$, where $m_1, m_2, \ldots, m_n, d$ are positive integers with $0 < m_1 \le m_2 \le \cdots \le m_n$.

(C2) There are vector fields

$$V^i = \sum_{j=1}^{n} a_{ij}(x)\partial_{x_j}\ (i = 1, 2, \ldots, n)$$

such that

(i) each $a_{ij}(x)$ is a polynomial of $x_1, x_2, \ldots, x_n$,
(ii) $\det(a_{ij}(x)) = cF(x)$ for a non-zero constant c,
(iii) $V^1 = E$, $V^iF(x) = c_i(x)F(x)$ for polynomials $c_i(x)$,
(iv) $[E, V^i] = k_iV^i$ for some constants k_i,
(v) V^i $(j = 1, 2, \ldots, n)$ form a Lie algebra over $R = \mathbf{C}[x_1, x_2, \ldots, x_n]$

Remark 1 *The terminology "logarithmic free divisor" is also called "Saito free divisor".*

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We now give examples of logarithmic free divisors.

Let
$$f(t) = t^n + x_2 t^{n-2} + x_3 t^{n-3} + \cdots + x_{n-1} t + x_n$$
be a polynomial of nth degree and let $\Delta(x_2, x_3, \ldots, x_n)$ be the discriminant of $f(t)$. Then $\Delta = 0$ is a logarithmic free divisor in $\mathbf{C}^{n-1}$.

More generally, the zero locus of the discriminant of an irreducible real reflection group is a logarithmic free divisor.

Basic reference of this section is [8].

§3. Irreducible complex reflection groups of rank three. In this section, we collect basic results on irreducible complex reflection groups of rank three. A basic reference on complex reflection groups is Shephard-Todd [17].

Reflection groups treated in this section are real reflection groups of types A_3, B_3, H_3 and complex reflection groups of No.24,No.25,No.26,No.27 in the sense of [17]. The real reflection group of type H_3 is same as the group No. 23 in [17].

Let G be one of the seven groups and we use the notation in the previous sections without any comment. Let P_1, P_2, P_3 be algebraically independent basic G-invariant polynomials and put $k_j = \deg_\xi(P_j)$. We may assume that $k_1 \le k_2 \le k_3$. Let r be the greatest common divisor of k_1, k_2, k_3 and put $k'_j = k_j/r$ $(j = 1, 2, 3)$. For the later convenience, we write x_1, x_2, x_3 for P_1, P_2, P_3. Let $\delta_G(x_1, x_2, x_3)$ be the discriminant of G expressed as a polynomial of x_1, x_2, x_3.

In the cases A_3, B_3, H_3, taking G-invariants x_1, x_2, x_3 suitably, $F_{W(A_3)}(x_1, x_2, x_3), F_{W(B_3)}(x_1, x_2, x_3)$, $F_{W(H_3)}(x_1, x_2, x_3)$ are discriminants for G up to a constant factor, respectively, where $F_{A,1}, F_{B,1}, F_{H,1}$ are the polynomials given in Theorem 1 which will be shown later.

	group	order	k_1, k_2, k_3	degree	(k'_1, k'_2, k'_3)
A_3	$W(A_3)$	24	$2, 3, 4$	12	$(2, 3, 4)$
B_3	$W(B_3)$	48	$2, 4, 6$	18	$(1, 2, 3)$
H_3	$W(H_3)$	120	$2, 6, 10$	30	$(1, 3, 5)$
No.24	G_{336}	336	$4, 6, 14$	42	$(2, 3, 7)$
No.25	G_{648}	648	$6, 9, 12$	36	$(2, 3, 4)$
No.26	G_{1296}	1296	$6, 12, 18$	54	$(1, 2, 3)$
No.27	G_{2160}	2160	$6, 12, 30$	90	$(1, 2, 5)$

The concrete forms of discriminants of $W(A_3)$, $W(B_3)$, $W(H_3)$ are as follows:

Type A_3: $16x_1^4x_3 - 4x_1^3x_2^2 - 128x_1^2x_3^2 + 144x_1x_2^2x_3 - 27x_2^4 + 256x_3^3$.

Type B_3: $x_3(x_1^2x_2^2 - 4x_2^3 - 4x_1^3x_3 + 18x_1x_2x_3 - 27x_3^2)$.

Type H_3: $-50x_3^3 + (4x_1^5 - 50x_1^2x_2)x_3^2 + (4x_1^7x_2 + 60x_1^4x_2^2 + 225x_1x_2^3)x_3 - \frac{135}{2}x_2^5$
$-115x_1^3x_2^4 - 10x_1^6x_2^3 - 4x_1^9x_2^2$.

The discriminants for the groups No.25, No.26 are same as those for $W(A_3)$, $W(B_3)$ respectively by taking the basic invariants suitably. On the other hand, those for the groups No.24, No.27 will be given in §9, §.10.

§4. Systems of uniformization equations. Let $F(x) = 0$ be a logarithmic free divisor and let V^j $(j = 1, 2, \ldots, n)$ be basic vector fields logarithmic along $F = 0$. Let $u = u(x_1, x_2, \ldots, x_n)$ be an unknown function. Assume that $V^1u = su$ for a constant $s(\ne 0)$ Put $\vec{u} = {}^t(u, V^2u, \ldots, V^nu)$. and consider the system of differential equations

(UE) $V^j\vec{u} = A_j(x)\vec{u}$ $\quad (j = 1, 2, \ldots, n)$

where $A_j(x)$ $(j = 1, 2, \ldots, n)$ are $n \times n$ matrices whose entries are polynomials of x.

If (UE) is integrable, it is called a system of uniformization equations with respect to the logarithmic free divisor $F = 0$.

The system (UE) is written by

$$
\text{(UEa)} \begin{cases} V^1 u = su \\ V^i V^j u = \sum_{k=1}^{n} h_{ij}^k(x) V^k u \quad (\forall i, j) \end{cases}
$$

where $h_{ij}^k(x)$ are polynomials of x. There are n number of fundamental solutions of (UEa). Let $u_j(x)\,(j = 1, 2, \ldots, n)$ be fundamental solutions outside the divisor $F = 0$. Then

$$
\varphi(x) = (u_1(x), u_2(x), \ldots, u_n(x))
$$

defines a map of $\mathbf{C}^n - \{F = 0\}$ to $\mathbf{C}^n$. The following two problems are fundamental in the study on systems of uniformization equations.

PROBLEM 1: Construct fundamental solutions $u_j(x)$ $(j = 1, 2, \ldots, n)$ of (UEa).

PROBLEM 2: Construct the inverse of $\varphi(x)$ in a concrete manner.

§5. Logarithmic free divisors of simple type. Let p, q, r be positive integers such that $p < q < r$. We assume that p, q, r have no common divisor > 1. Put $R = \mathbf{C}[x, y, z]$. Let $x,\ y,\ z$ be weighted variables with weights equal to p, q, r.

We introduce a triplet of vector fields on $\mathbf{C}^3$:

$$
\begin{array}{rcl}
V^0 & = & px\partial_x + qy\partial_y + rz\partial_z, \\
V^1 & = & qy\partial_x + h_{22}\partial_y + h_{23}\partial_z, \\
V^2 & = & rz\partial_x + h_{32}\partial_y + h_{33}\partial_z,
\end{array}
$$

where V^0 is the Euler field and $h_{ij} \in R$

The requirements on the triplet of vector fields V^0, V^1, V^2 are given in Condition 1 below:

Condition 1

(i) $[V^0, V^1] = (q - p)V^1$, $[V^0, V^2] = (r - p)V^2$*;*

(ii) $[V^1, V^2] = f_0V^0 + f_1V^1 + f_2V^2$, *(*$f_j \in R$ *for all* $j = 0, 1, 2$*);*

(iii) $h_{22} = az + g(x, y)$, $a \in \mathbf{C}^*$ *and constant*

$g(x, y)$ *is a polynomial depending on* x *and* y *only;*

(iv) $F = \det(M)$ *is not equivalent to the monomial* z^3 *under weighted changes of variables. (*M *is the matrix whose entries are coefficients of* V^0, V^1, V^2*.)*

Theorem 1. *(cf. [11])*

(i) If $(p, q, r) \neq (2, 3, 4),\ (1, 2, 3),\ (1, 3, 5)$, *then there are no triples* $\langle V^0, V^1, V^2 \rangle$ *of vector fields satisfying Condition 1.*

(ii) The remaining cases are described as follows.

$(\mathcal{D}_A)$ *if* $(p, q, r) = (2, 3, 4)$, *then up to weighted changes of variables there are two triples of vector fields satisfying Condition 1; the corresponding polynomials* $F = \det(M)$ *of degree 12 are the following:*

$F_{A,1} = 16x^4z - 4x^3y^2 - 128x^2z^2 + 144xy^2z - 27y^4 + 256z^3$;

$F_{A,2} = 2x^6 - 3x^4z + 18x^3y^2 - 18xy^2z + 27y^4 + z^3$.

$(\mathcal{D}_B)$ *if* $(p, q, r) = (1, 2, 3)$, *then up to weighted changes of variables there are seven triples of vector fields satisfying Condition 1: the corresponding polynomials* $F = \det(M)$ *of degree 9 are the following:*

$F_{B,1} = z(x^2y^2 - 4y^3 - 4x^3z + 18xyz - 27z^2)$;

$F_{B,2} = z(-2y^3 + 4x^3z + 18xyz + 27z^2)$;

$F_{B,3} = z(-2y^3 + 9xyz + 45z^2)$;

$F_{B,4} = z(9x^2y^2 - 4y^3 + 18xyz + 9z^2)$;

$F_{B,5} = xy^4 + y^3z + z^3$;

$F_{B,6} = 9xy^4 + 6x^2y^2z - 4y^3z + x^3z^2 - 12xyz^2 + 4z^3$;

$F_{B,7} = \frac{1}{2}xy^4 - 2x^2y^2z - y^3z + 2x^3z^2 + 2xyz^2 + z^3$.

$(\mathcal{D}_H)$ *if* $(p, q, r) = (1, 3, 5)$, *then up to weighted changes of variables there are eight triples of vector fields satisfying Condition 1 the corresponding polynomials* $F = \det(M)$ *of degree 15 are the following:*

$F_{H,1} = -50z^3 + (4x^5 - 50x^2y)z^2 + (4x^7y + 60x^4y^2 + 225xy^3)z - \frac{135}{2}y^5 - 115x^3y^4 - 10x^6y^3 - 4x^9y^2$;

$F_{H,2} = 100x^3y^4 + y^5 + 40x^4y^2z - 10xy^3z + 4x^5z^2 - 15x^2yz^2 + z^3$;

$F_{H,3} = 8x^3y^4 + 108y^5 - 36xy^3z - x^2yz^2 + 4z^3$;

$F_{H,4} = y^5 - 2xy^3z + x^2yz^2 + z^3$;

$F_{H,5} = x^3y^4 - y^5 + 3xy^3z + z^3$;

$F_{H,6} = x^3y^4 + y^5 - 2x^4y^2z - 4xy^3z + x^5z^2 + 3x^2yz^2 + z^3$;
$F_{H,7} = xy^3z + y^5 + z^3$;
$F_{H,8} = x^3y^4 + y^5 - 8x^4y^2z - 7xy^3z + 16x^5z^2 + 12x^2yz^2 + z^3$.

As we pointed out in §3, $F_{A,1}, F_{B,1}, F_{H,1}$ are discriminants of the reflection groups of types A_3, B_3, H_3, respectively.

§6. Logarithmic free divisors related with 14 families of exceptional singularities. It is possible to construct logarithmic free divisors from some of the 14 families exceptional singularities in the sense of Arnol'd.

We use the notation due to Arnol'd. Our interest is related with singular curves. Among the 14 exceptional families, there are 8 which are realized by curve singularities. First we give these 8 families in TABLE I. Putting $w = 0$, we obtain curves in uv-space. In TABLE I, the triplet of numbers (a, b, c) and h mean the type and the degrees of the polynomials in question.

TABLE I

Type		Equation	(a,b,c)	h	(p,q,r)
E_{12}	:	$u^7 + v^3 + w^2 = 0$	$(6,14,21)$	42	$(2,3,7)$
E_{13}	:	$u(u^2 + v^5) + w^2 = 0$	$(4,10,15)$	30	$(1,2,5)$
E_{14}	:	$u^8 + v^3 + w^2 = 0$	$(3,8,12)$	24	$(1,3,8)$
Z_{11}	:	$u(u^4 + v^3) + w^2 = 0$	$(6,8,15)$	30	$(2,3,4)$
Z_{12}	:	$uv(u^3 + v^2) + w^2 = 0$	$(4,6,11)$	22	$(1,2,3)$
Z_{13}	:	$u(u^5 + v^3) + w^2 = 0$	$(3,5,9)$	18	$(1,3,5)$
W_{12}	:	$u^5 + v^4 + w^2 = 0$	$(4,5,10)$	20	$(2,4,5)$
W_{13}	:	$v(u^4 + v^3) + w^2 = 0$	$(3,4,8)$	16	$(1,3,4)$

We now show an example of logrithmic free divisors related to the singularities above.

The case E_{12}

The following eight polynomials $f_1, \dots, f_8$ are logarithmic free polynomials each of which defines deformations of a curve on (y, z)-space with a singular point of type E_{12} at the origin regarding x as a parameter.

Let $F(x, y, z)$ be a weighted homogeneous logarithmic free polynomial with weight $(2, 3, 7)$ and degree 21. Assume that $F(0, y, z) = z^3 + y^7$. Then $F(x, y, z)$ coincides with one of the 8 polynomials $f_1, \dots, f_8$ up to weight preserving coordinate transformation.

```
f1=x^6*y^3/32+3*x^3*y^5/28+3*y^7/49-3/16*x^4*y^2*z-3/7*x*y^4*z+z^3;
f2=-1/864*x^6*y^3+5*x^3*y^5/84+3*y^7/49-1/48*x^4*y^2*z-3/7*x*y^4*z+z^3;
f3=x^3*y^5/21+3*y^7/49-3/7*x*y^4*z+z^3;
f4=3*y^7/49-3/7*x*y^4*z+z^3;
f5=78125*x^9*y/200120949+44375*x^6*y^3/4840416+107*x^3*y^5/1372+3*y^7/49
    -6250*x^7*z/3176523-1375*x^4*y^2*z/16464-3/7*x*y^4*z+z^3;
f6=64*x^9*y/823543+208*x^6*y^3/453789+68*x^3*y^5/1029+3*y^7/49+
     48*x^7*z/117649-40*x^4*y^2*z/1029-3/7*x*y^4*z+z^3;
f7=-448*x^9*y/243+16*x^6*y^3/9-4*x^3*y^5/7+3*y^7/49-112*x^7*z/27+
     8/3*x^4*y^2*z-3/7*x*y^4*z+z^3;
f8=-752*x^9*y/823543-2017*I*x^9*y/(823543*Sqrt[3])-397*x^6*y^3/33614+
   323*I*Sqrt[3]*x^6*y^3/33614+39*x^3*y^5/686+9/686*I*Sqrt[3]*x^3*y^5+3*y^7/49+
   1763*x^7*z/235298-249*I*Sqrt[3]*x^7*z/235298+3/686*x^4*y^2*z-
   37/686*I*Sqrt[3]*x^4*y^2*z-3/7*x*y^4*z+z^3;
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Remark 2 *The weights of variables x, y, z are 2,3,7, respectively. It can be shown that by taking the basic invariants of the group G_{336} which is No.24 group, the discriminant of G_{336} coincides with f_6.*

For the details of the results of this section, see [12].

§7. The discriminant of the Coxeter group $W(H_3)$ of type H_3. A part of the argument in the case of $W(A_3)$ in §1 is also applicable to the case of the logarithmic free divisor defined by the zero locus of the discriminant of the Coxeter group of type H_3. For the details of results in this section, see [15].

The discriminant of the polynomial $P(t)$ defined by

$$(1) \quad P(t) = t^6 + y_1 t^5 + y_2 t^3 + y_3 t + \frac{1}{20} y_2^2 - \frac{1}{4} y_1 y_3$$

is Δ^2 up to a constant factor, where

$$(2) \quad \Delta = 125 y_1^3 y_2^4 + 864 y_2^5 - 1250 y_1^4 y_2^2 y_3 - 9000 y_1 y_2^3 y_3 + 3125 y_1^5 y_3^2 + 25000 y_1^2 y_2 y_3^2 + 50000 y_3^3.$$

The polynomial Δ is regarded as the discriminant of the group $W(H_3)$. In fact, the substitution of the variables (y_1, y_2, y_3) with (x_1, x_2, x_3) defined by the relations

$$(3) \quad \left\{ \begin{array}{lcl} y_1 & = & -4x_1 \\ y_2 & = & 10x_1^3 - 25x_2 \\ y_3 & = & -4x_1^5 + 50x_1^2 x_2 - 50x_3 \end{array} \right.$$

implies that Δ coincides with the determinant of the matrix M up to a constant factor, where M is defined by

$$(4) \quad M = \begin{pmatrix} x_1 & 3x_2 & 5x_3 \\ 3x_2 & 2x_3 + 2x_1^2 x_2 & 7x_1 x_2^2 + 2x_1^4 x_2 \\ 5x_3 & 7x_1 x_2^2 + 2x_1^4 x_2 & \frac{1}{2}(15x_2^3 + 4x_1^4 x_3 + 18x_1^3 x_2^2) \end{pmatrix}$$

and $\det M$ is the discriminant of $W(H_3)$. In the sequel, we always regard $P(t)$ as a polynomial of t and x. The hypersurface defined as the zero set of the polynomial $f_0 = \det M$ is an example of logarithmic free divisors. To show this, we define vector fields V_0, V_1, V_2 by

$$\begin{pmatrix} V_0 \\ V_1 \\ V_2 \end{pmatrix} = M \begin{pmatrix} \partial_{x_1} \\ \partial_{x_2} \\ \partial_{x_3} \end{pmatrix}$$

Then we have

$$[V_0, V_1] = 2V_1, \ [V_0, V_2] = 4V_2,$$
$$[V_1, V_2] = (4x_1^3 x_2 + 2x_2^2) V_0 + 4x_1 x_2 V_1$$

and

$$\begin{array}{l} V_0 f_0 = 15 f_0, \\ V_1 f_0 = 2x_1^2 f_0, \\ V_2 f_0 = 2x_1(2x_1^3 + 5x_2) f_0 \end{array}$$

We note that

$$(5) \quad P(-x_1) = \frac{125}{4} x_2^2$$

Consider the system of differential equations

$$(6) \quad V_i \begin{pmatrix} u \\ V_1 u \\ V_2 u \end{pmatrix} = B_{i+1} \begin{pmatrix} u \\ V_1 u \\ V_2 u \end{pmatrix} \quad (i = 0, 1, 2)$$

The system (6) is a system of uniformization equations with respect to the logarithmic free divisor $f_0(x) = 0$. Here B_j are defined as follows:

$$B_1 = \begin{pmatrix} s_0 & 0 & 0 \\ 0 & 2+s_0 & 0 \\ 0 & 0 & 4+s_0 \end{pmatrix}$$

$$B_2 = \begin{pmatrix} 0 & 1 & 0 \\ -\frac{2}{225}x_1 \left\{\begin{matrix} (8+70s_1-100s_1^2+8s_0 \\ +35s_1s_0+2s_0^2)x_1^3 \\ +(-180+825s_1-750s_1^2 \\ -90s_0+75s_1s_0)x_2 \end{matrix}\right\} & \frac{1}{15}(8+5s_1+4s_0)x_1^2 & s_1 \\ \frac{1}{900}\left\{\begin{matrix} (-128+80s_1+100s_1^2-128s_0 \\ +40s_1s_0-32s_0^2)x_1^6 \\ +10(-32-400s_1+550s_1^2+328s_0 \\ -20s_1s_0-8s_0^2)x_1^3x_2 \\ +(-4500s_1+5625s_1^2+1800s_0)x_2^2 \\ +(2400-3000s_1+1200s_0)x_1x_3 \end{matrix}\right\} & \frac{1}{15}x_1\left\{\begin{matrix} (8+5s_1+4s_0)x_1^3 \\ +10(8+5s_1+s_0)x_2 \end{matrix}\right\} & \frac{1}{15}(4-5s_1+2s_0)x_1^2 \end{pmatrix}$$

$$B_3 = \begin{pmatrix} 0 & 0 & 1 \\ \frac{1}{900}\left\{\begin{matrix} (-128+80s_1+100s_1^2-128s_0 \\ +40s_1s_0-32s_0^2)x_1^6 \\ +10(-32-400s_1+550s_1^2-32s_0 \\ -20s_1s_0-8s_0^2)x_1^3x_2 \\ +s_1(-4500+5625s_1)x_2^2 \\ +100(24-30s_1+12s_0)x_1x_3 \end{matrix}\right\} & \frac{1}{15}x_1\left\{\begin{matrix} (8+5s_1+4s_0)x_1^3 \\ +10(2+5s_1+s_0)x_2 \end{matrix}\right\} & \frac{1}{15}(4-5s_1+2s_0)x_1^2 \\ \frac{1}{450}\left\{\begin{matrix} (-128+80s_1+100s_1^2-128s_0 \\ +40s_1s_0-32s_0^2)x_1^8 \\ +(80-500s_1+500s_1^2-280s_0 \\ +200s_1s_0-160s_0^2)x_1^5x_2 \\ +25(-104-130s_1+325s_1^2+40s_0 \\ +25s_1s_0-8s_0^2)x_1^2x_2^2 \\ +100(12-15s_1+24s_0)x_1^3x_3 \\ +50(60-75s_1+30s_0)x_2x_3 \end{matrix}\right\} & \frac{1}{4}(4+5s_1)x_2(4x_1^3+5x_2) & \frac{1}{15}x_1\left\{\begin{matrix} (16-5s_1+8s_0)x_1^3 \\ +(40-50s_1+20s_0)x_2 \end{matrix}\right\} \end{pmatrix}$$

Remark 3 *In the case $s_0 = \frac{1}{2},\ s_1 = 1$, the monodromy group of the system $V_j\vec{u} = B_{j+1}\vec{u}\,(j = 0,1,2)$ coincides with $W(H_3)$. This case is treated in Haraoka-Kato [5].*

We consider the system $V_j\vec{u} = B_{j+1}\vec{u}\,(j = 0,1,2)$ with $s_0 = -2,\ s_1 = 0$. Then we obtain

$$(7) \quad \begin{cases} V_0v &= -2v \\ V_1V_1v &= 0 \\ V_2V_1v &= 0 \\ V_2V_2v &= -4x_1^2(3x_2^2+2x_1x_3)v + x_2(4x_1^3+5x_2)V_1v \end{cases}$$

Theorem 2 *(cf. [15]) The function $v(x)$ defined by*

$$v(x) = \int_{\infty}^{-x_1} P(t)^{-1/2}dt$$

is a solution of (7).

The proof of this theorem is given by an argument similar to the case of type A_3.

If $u(x)$ is a solution of (7) such that $V_1u = 0$, then u is a solution of

$$(8) \quad \begin{cases} V_0u = -2u \\ V_1u = 0 \\ \{V_2^2 + 4x_1^2(3x_2^2+2x_1x_3)\}u = 0 \end{cases}$$

Taking two paths C_1, C_2 appropriately and define

$$w_j(x) = \int_{C_j} \varphi_1(t)dt \quad (j = 1,2)$$

Then each $w_j(x)$ is also a solution of (7) and in this manner we can construct solutions of (8). In this case it is not clear whether solutions of (8) are expressed by special functions or not. Moreover **PROBLEM 2** (the construction of the inverse mapping) is still open.

Remark 4 講演では G_{336}, G_{2160} の場合もとりあげたが，それらについては，*[13],[14]* を参照してください.

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