VISIBLE ACTIONS ON $SL(m+n,\mathbb{C})/(SL(m,\mathbb{C})\times SL(n,\mathbb{C}))$

ATSUMU SASAKI

Abstract. A holomorphic action of a Lie group G on a complex manifold D is called strongly visible if there exist an open set D' in D, a real submanifold S in D meeting every G-orbit in D' and an anti-holomorphic diffeomorphism σ of D' such that $\sigma|_S = \mathrm{id}_S$ and that σ preserves each G-orbit in D'. In this talk, we show that the action of $SU(m+n)$ on the homogeneous space $SL(m+n,\mathbb{C})/(SL(m,\mathbb{C})\times SL(n,\mathbb{C}))$ is strongly visible if $m \neq n$ by taking S with dimension equal to $\min(m,n)+1$.

1. Introduction and the statement of the main result

The notion of (strongly) visible actions on complex manifolds has been introduced by T. Kobayashi [3, Definition 2.3] as a basic assumption of the propagation theorem of multiplicity-free property from fibers to the space of holomorphic sections (see [4, 5]). Let us review the definition of strongly visible actions on complex manifolds.

Definition 1.1 ([4, Definition 3.3.1])**.** A holomorphic action of a Lie group G on a complex manifold D is called *strongly visible* if the following two conditions are satisfied:

(a) There exists a real submanifold S in D such that

$$D' := G\cdot S \text{ is open in } D. \tag{V.1}$$

(b) There exists an anti-holomorphic diffeomorphism σ of D' such that

$$\sigma|_S = \mathrm{id}_S, \tag{S.1}$$

$$\sigma \text{ preserves each } G\text{-orbit in } D'. \tag{S.2}$$

We say that this real submanifold S is a *slice* with respect to an anti-holomorphic diffeomorphism σ. The slice S is automatically totally real (see [4, Remark 3.3.3]).

One of the simplest example is that the standard action of a torus $\mathbb{T}$ on $\mathbb{C}$. In fact, one can take a positive half line $\mathbb{R}_+$ as a slice with respect to the standard complex conjugation of $\mathbb{C}$.

Symposium on Representation Theory 2008 at Wel Heartpia Atami, November 11–14, 2008 (organizers: K. Taniguchi, T. Yoshino).

Another example is that, for any compact symmetric pair (G, K), the G-action on the complex symmetric space $G_{\mathbb{C}}/K_{\mathbb{C}}$ is strongly visible ([4, Theorem 11]). Here, $G_{\mathbb{C}}$ and $K_{\mathbb{C}}$ denote the complexifications of compact Lie groups G and K, respectively.

In [7], we investigate visible actions on Hermitian symmetric spaces. Let G be a semisimple Lie group such that G/K is a Hermitian symmetric pair. Then any symmetric subgroup of G acts on D in a strongly visible fashion ([7, Theorem 1.5]).

Furthermore, we study the visible linear actions on complex vector spaces in [9, 10]. For a holomorphic representation of a connected complex reductive Lie group $G_{\mathbb{C}}$ on a complex vector space V, the representation of $G_{\mathbb{C}}$ on the polynomial ring $\mathbb{C}[V]$ is multiplicity-free if and only if the action of a compact real form G_U of $G_{\mathbb{C}}$ on V is strongly visible.

In this talk, we treat the case where a complex manifold D is the complex homogeneous space $SL(m+n,\mathbb{C})/(SL(m,\mathbb{C}) \times SL(n,\mathbb{C}))$. This is not a symmetric space.

Our main theorem is stated as follows:

Theorem 1.2. *If $m \neq n$, then the action of $SU(m+n)$ on the complex homogeneous space $SL(m+n,\mathbb{C})/(SL(m,\mathbb{C}) \times SL(n,\mathbb{C}))$ is strongly visible.*

Theorem 1.3. *For the strongly visible action of $SU(m+n)$ on $SL(m+n,\mathbb{C})/(SL(m,\mathbb{C}) \times SL(n,\mathbb{C}))$, we can find a slice S with dimension* $\min(m, n) + 1$.

Our goal of this talk is to give proofs of Theorems 1.2 and 1.3, and to constract an explicit description of a slice S and an anti-holomorphic diffeomorphism σ.

Throughout this paper, let us assume $m > n$, and let G, H, K be as follows:

$$\begin{aligned} G &= SL(m+n,\mathbb{C}), \\ H &= SL(m,\mathbb{C}) \times SL(n,\mathbb{C}), \\ K &= SU(m+n). \end{aligned}$$

2. Explicit construction of a slice

Let us consider the following subgroup of G

$$L := S(GL(m,\mathbb{C}) \times GL(n,\mathbb{C})).$$

H is the commutator subgroup of L. By using a diagonal matrix

$$I_{m,n} = \mathrm{diag}(\underbrace{1,\dots,1}_{m},\underbrace{-1,\dots,-1}_{n}),$$

we define an involutive automorphism τ of G by $\tau(g) = I_{m,n} g I_{m,n}$ $(g \in G)$. Then (G, L) is a symmetric pair with respect to τ. Similarly, (G, K) is also a symmetric pair with respect to a Cartan involution θ of G commuting with the involution τ.

First, we consider the triple of Lie groups $K \subset G \supset H$. Let $\mathfrak{g}, \mathfrak{k}, \mathfrak{l}$ be Lie algebras of G, K, L, respectively. We denote by $\mathfrak{g} = \mathfrak{l} + \mathfrak{q}$ the eigenspace decomposition of $\mathfrak{g}$ into 1 and -1 eigenspace for the differential of τ, and by $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ the corresponding Cartan decomposition of $\mathfrak{g}$. We take a maximal abelian subspace $\mathfrak{a}$ on $\mathfrak{p} \cap \mathfrak{q}$, and set $A = \exp \mathfrak{a}$. In our setting, A can be given of the form

$$A = \left\{ \begin{pmatrix} c(t_1, \dots, t_n) & & s(t_1, \dots, t_n) \\ & I_{m-n} & \\ s(t_1, \dots, t_n) & & c(t_1, \dots, t_n) \end{pmatrix} : t_1, \dots, t_n \in \mathbb{R} \right\},$$

where we set

$$\begin{aligned} c(t_1, \dots, t_n) &= \operatorname{diag}(\cosh t_1, \dots, \cosh t_n), \\ s(t_1, \dots, t_n) &= \operatorname{diag}(\sinh t_1, \dots, \sinh t_n). \end{aligned}$$

Lemma 2.1 ([2, Theorem 4.1 (i)]). *We have a generalized Cartan decomposition:*

$$G = KAL. \tag{2.1}$$

Second, we consider the pair of Lie groups $L \supset H$. We define

$$r(s) = \operatorname{diag}(\underbrace{e^s, \dots, e^s}_{n}, \underbrace{e^{-\frac{2ns}{m-n}}, \dots, e^{-\frac{2ns}{m-n}}}_{m-n}, \underbrace{e^s, \dots, e^s}_{n}),$$

$$t(\theta) = \operatorname{diag}(\underbrace{e^{\sqrt{-1}\theta}, \dots, e^{\sqrt{-1}\theta}}_{n}, \underbrace{e^{-\frac{2\sqrt{-1}n\theta}{m-n}}, \dots, e^{-\frac{2\sqrt{-1}n\theta}{m-n}}}_{m-n}, \underbrace{e^{\sqrt{-1}\theta}, \dots, e^{\sqrt{-1}\theta}}_{n}).$$

We set two subgroups of L by

$$\begin{aligned} Z_{\mathbb{R}} &= \{r(s) : s \in \mathbb{R}\}, \\ Z_{\mathbb{T}} &= \{t(\theta) : \theta \in \mathbb{R}/2\pi\mathbb{Z}\}. \end{aligned}$$

Clearly, $Z_{\mathbb{R}}$ and $Z_{\mathbb{T}}$ commute with each other.

Lemma 2.2. *We have a decompotision*

$$L = Z_{\mathbb{R}} Z_{\mathbb{T}} H. \tag{2.2}$$

Proof. It is obvious that $Z_{\mathbb{R}} Z_{\mathbb{T}} H$ is contained in L.

We will verify the inclusion $L \subset Z_{\mathbb{R}} Z_{\mathbb{T}} H$. Let us take an element

$$g = \begin{pmatrix} g_1 & O \\ O & g_2 \end{pmatrix} \in L$$

$(g_1 \in GL(m,\mathbb{C}),\ g_2 \in GL(n,\mathbb{C}))$. We write $\det g_2 = e^{ns}e^{\sqrt{-1}n\theta} \in \mathbb{C}^\times$ for some $s \in \mathbb{R}$ and $\theta \in \mathbb{R}/2\pi\mathbb{Z}$. Then $h := r(-s)t(-\theta)g$ is an element of H. This means that $g = r(s)t(\theta)h$ is an element of $Z_\mathbb{R}Z_\mathbb{T}H$. Hence L implies $Z_\mathbb{R}Z_\mathbb{T}H$.

Therefore the decomposition (2.2) has been proved. □

We are ready to give S which meets every K-orbit in G/H.

Proposition 2.3. *Let S be an $(n+1)$-dimensional submanifold of G/H defined by*

$$S := (Z_\mathbb{R}A)H/H. \tag{2.3}$$

Then S meets every K-orbit in G/H.

Proof. It follows from (2.1) and (2.2) that we have

$$G = KAL = KA(Z_\mathbb{R}Z_\mathbb{T}H) = KZ_\mathbb{T}Z_\mathbb{R}AH \tag{2.4}$$

because A, $Z_\mathbb{R}$ and $Z_\mathbb{T}$ commute with one another. Since $Z_\mathbb{T}$ is a subgroup of K, we have $G = KZ_\mathbb{T}Z_\mathbb{R}AH = K(Z_\mathbb{R}A)H$ from (2.4). This implies

$$G/H = K \cdot S,$$

which proves Proposition 2.3. □

3. Proof of the main result

Let us give an anti-holomorphic diffeomorphism of G/H. We define an anti-holomorphic involution σ_G of G by $\sigma_G(g) = \overline{g}$. Since σ_G stabilizes H, the involution σ_G induces an anti-holomorphic diffeomorphism σ of G/H by

$$\sigma(gH) = \sigma_G(g)H = \overline{g}H. \tag{3.1}$$

Now, we give a proof of Theorems 1.2 and 1.3 simultaneously.

Proof of Theorems 1.2 and 1.3. In order to prove Theorem 1.2, we need to verify the conditions (V.1), (S.1) and (S.2) in Definition 1.1 for the above S and σ (see (2.3) and (3.1) for definition). The condition (V.1) is an immediate consequence of Proposition 2.3. It is clear that $\sigma|_S = \mathrm{id}_S$, which is the condition (S.1).

We will check (S.2). We take an element $xH = k \cdot aH \in K \cdot S$ for some $k \in K$ and some $aH \in S$. Then we obtain

$$\sigma(xH) = \sigma_G(k) \cdot \sigma(aH) = \sigma_G(k) \cdot aH = (\sigma_G(k)k^{-1}) \cdot xH. \tag{3.2}$$

Since σ_G stabilizes K, we have $\sigma_G(k)k^{-1} \in K$. The equation (3.2) shows that the K-orbit through $\sigma(xH)$ coincides with that through xH. Hence the condition (S.2) has been verified. Therefore Theorem 1.2 has been proved.

Theorem 1.3 follows readily from the definition (2.3) of the slice S. $\square$

References

[1] C. Benson and G. Ratcliff, On multiplicity-free actions, *Representations of real and p-adic groups* (eds. E.-C. Tan and C.-B Zhu), Lect. Notes Ser. Inst. Math. Sci. Natl. Univ. Singap. **2**, Singapore Univ. Press, Singapore (2004) 221–304.

[2] M. Flensted–Jensen, Spherical functions of a real semisimple Lie group. A method of reduction to the complex case, *J. Funct. Anal.* **30** (1978), 106–146.

[3] T. Kobayashi, Geometry of multiplicity-free representations of $GL(n)$, visible actions on flag varieties, and triunity, *Acta. Appl. Math.* **81** (2004), 129–146.

[4] T. Kobayashi, Multiplicity-free representations and visible actions on complex manifolds, *Publ. Res. Inst. Math. Sci.* **41** (2005), 497–549, special issue commemorating the fortieth anniversary of the founding of RIMS.

[5] T. Kobayashi, Propagation of multiplicity-free property for holomorphic vector bundles, math.RT/0607004.

[6] T. Kobayashi, A generalized Cartan decomposition for the double coset space $(U(n_1)\times U(n_2)\times U(n_3))\backslash U(n)/(U(p)\times U(q))$, *J. Math. Soc. Japan* **59** (2007), 669–691.

[7] T. Kobayashi, Visible actions on symmetric spaces, *Transform. Groups* **12** (2007), 671–694.

[8] M. Krämer, Sphärische Untergruppen in kompakten zusammenhängenden Liegruppen, *Compositio Math.* **38** (1979), 129–153.

[9] A. Sasaki, Visible actions on irreducible multiplicity-free spaces, *preprint*.

[10] A. Sasaki, Visible actions on reducible multiplcity-free spaces, *preprint*.

Department of Applied Mathematics, Faculty of Science and Engineering, Waseda University, Okubo, Shinjuku-ku, Tokyo, 169-8555 Japan.

E-mail address: atsumu@aoni.waseda.jp