Multiplicity one theorem for Horn's hypergeometric functions*

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We consider asymptotic behaviors at the infinity of Horn's hypergeometric functions for the purpose of the application to the multiplicity theorem for generalized Whittaker models (cf. [3]).

Let $P_i(x)$ and $Q_i(x)$ are nonzero polynomials of variables $x = (x_1, \ldots, x_n)$ for $i = 1, \ldots, n$. Then the Horn's hypergeometric functions are defined as solutions of the system of linear partial differential equations

$$[x_i P_i(\vartheta) - Q_i(\vartheta)]f(x) = 0, \quad i = 1, \ldots, n. \tag{0.1}$$

Here $\vartheta_i = x_i \frac{\partial}{\partial x_i}$ and $\vartheta = (\vartheta_1, \ldots, \vartheta_n)$. In this note, we assume that P_i and Q_i can be decomposed by products of linear factors, i.e.,

$$P_i(s) = \prod_{k=1}^{p}(\langle A_k, s\rangle - c_i), \qquad Q_i(s) = \prod_{l=1}^{q}(\langle B_l, s\rangle - d_l)$$

for $s \in \mathbb{R}^n$, $A_k, B_l \in \mathbb{R}^n$, $c_k, d_l \in \mathbb{C}$ and $\langle\ ,\ \rangle$ denote the natural inner product in $\mathbb{R}^n$. We also assume $P_i(s)$, $Q_i(s + e_i)$ are relatively prime for $i = 1, \ldots, n$. Here $e_i = (0, \ldots, 0, 1, 0, \ldots, 0)$ (1 in the ith position).

We consider the following system of difference equations associated with the system of differential equations (0.1),

$$P_i(-(s + e_i))\phi(s + e_i) = Q_i(-s)\phi(s) \quad i = 1, \ldots, n. \tag{0.2}$$

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Remark 0.1. *Let ϕ be a solution of the system of difference equations (0.2). We consider the following integral,*

$$f(x) = \int_{\mathcal{C}} \phi(s) x^{-s} \, ds.$$

Then under the following assumptions, we can see that $f(x)$ is a solution of the system of differential equations (0.1).

1. *For any $i = 1, \ldots, n$, the translation of the contour $\mathcal{C}$ with respect to the basis e_i is homologically equivalent to $\mathcal{C}$ in the complement of the set of the singularities of the integrand $\phi(s)$ in $\mathbb{C}^n$.*

2. *The integral converges absolutely and it can be differentiated with respect to x sufficiently many times.*

We put

$$R_i(s) = \frac{Q_i(-s)}{P_i(-(s+e_i))} \quad i = 1, \ldots, n.$$

Theorem 0.2 (Ore [5], Sato [7], Sadykov [6])**.** 1. *The system of difference equations (0.2) is solvable if and only if*

$$R_i(s+e_j)R_j(s) = R_j(s+e_i)R_i(s), \quad i,j = 1, \ldots, n. \tag{0.3}$$

2. *If the system (0.2) is solvable, then its solution is unique up to an arbitrary periodic function $\psi(s)$ with respect to e_i, i.e.,*

$$\psi(s+e_i) = \psi(s),$$

for $i = 1, \ldots, n$. Furthermore, there exist $p', q' \in \mathbb{N}$, $A'_k, B'_l \in \mathbb{R}^n$ $(1 \le k \le p', 1 \le l \le q')$, $c'_k, d'_l \in \mathbb{C}$ $(1 \le k \le p', 1 \le l \le q')$ and $t_i \in \mathbb{R}$ $(i = 1, \ldots, n)$ such that the general solution of (0.2) *is written as follows,*

$$\phi(s) = t^{-s} \frac{\prod_{l=1}^{q'} \Gamma(\langle B'_l, s\rangle - d'_l)}{\prod_{k=1}^{p'} \Gamma(\langle A'_k, s\rangle - c'_i)} \psi(s),$$

where $t^{-s} = t_1^{-s_1} \cdots t_n^{-s_n}$ and $\psi(s)$ is an arbitrary periodic function satisfying $\psi(s+e_i) = \psi(s)$.

We put an assumption as follows for the multiplicity theorem.

(A). The system of difference equations (0.2) is solvable, i.e., the condition (0.3) is satisfied and we can choose a solution,

$$\phi(s) = t^{-s} \frac{\prod_{l=1}^{q'} \Gamma(\langle B'_l, s\rangle - d'_l)}{\prod_{k=1}^{p'} \Gamma(\langle A'_k, s\rangle - c'_i)}$$

which satisfies following conditions;

(i) We have the inequality,

$$\sum_{l=1}^{q'} |\langle B'_l, s\rangle| - \sum_{k=1}^{p'} |\langle A'_k, s\rangle| \geq \sum_{i=1}^{n} |s_i|$$

for $s \in \mathbb{R}^n$.

(ii) The function $\phi(s)$ has no zero if each $\mathrm{Re}(s_i)$ are sufficiently large for $i = 1, \ldots, n$.

Remark 0.3. *We consider the integral*

$$f(x) = \int_{\sigma_1-\sqrt{-1}\infty}^{\sigma_1+\sqrt{-1}\infty} \cdots \int_{\sigma_n-\sqrt{-1}\infty}^{\sigma_n+\sqrt{-1}\infty} \phi(s) x^{-s}\, ds,$$

for appropriate $\sigma_i \in \mathbb{R}$ $i = 1, \ldots, n$. Under the assumption (A)-(i), it follows that the integral is absolutely convergent in the set $\{x \in \mathbb{R}^n \mid (t_1x_1, \ldots, t_nx_n) \in (\mathbb{R}_{\geq 0})^n\}$.

The following theorem is a generalization of the theorem of Diaconu and Goldfeld (Theorem 6.1.6 in [2])

Theorem 0.4 (Multiplicity one)**.** *Suppose that the system of difference equations* (0.2) *associated with the one of differential equations* (0.1) *satisfies the assumption (A). Let $f(x)$ be a solution of the system* (0.1) *which satisfies the growth condition*

$$\sup_{x \in (\mathbb{R}_{\geq 0})^n} |x^\alpha f(tx)| < +\infty$$

for sufficiently large integers $\alpha_i \in \mathbb{N}$, $i = 1, \ldots, n$. Then it is unique up to constant multiple. Here $x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ and $tx = (t_1x_1, \ldots, t_nx_n)$.

Proof. We consider the Mellin transform of $f(tx)$ as the function of x,

$$\mathcal{M}[f,s] = \int_0^\infty \cdots \int_0^\infty f(tx)x^{s-1}\,dx.$$

This integral converges absolutely and $\mathcal{M}[f,s]$ is analytic function of s if each $\mathrm{Re}(s_i)$ is sufficiently large by the assumption of $f(x)$. Changing the variables x to $tx = (t_1x_1, \cdots, t_nx_n)$, then we have

$$\mathcal{M}[f,s] = t^{-s} \int_0^{t_1^{-1}\infty} \cdots \int_0^{t_n^{-1}\infty} f(x)x^{s-1}\,dx.$$

By the growth condition of $f(x)$, we have

$$\begin{aligned}\int_0^{t_1^{-1}\infty} \cdots \int_0^{t_n^{-1}\infty} & \frac{\partial^k}{\partial x_i^k} f(x)x^{s-1}\,dx \\ &= (-1)^k \int_0^{t_1^{-1}\infty} \cdots \int_0^{t_n^{-1}\infty} f(x)\frac{\partial^k}{\partial x_i^k} x^{s-1}\,dx,\end{aligned}$$

by integration by parts for $i = 1, \ldots, n$. Recall that $f(x)$ satisfies the system of the partial differential equations (0.1), then we have the system of the difference equations for $\mathcal{M}[f,s]$,

$$P_i(-(s+e_i))\mathcal{M}[f,s+e_i] = Q_i(-s)\mathcal{M}[f,s] \quad i = 1, \ldots, n.$$

Hence by Theorem 0.2, there is a periodic function $\psi(s)$ and we have

$$\frac{\prod_{l=1}^{q'} \Gamma(\langle B'_l, s\rangle - d'_l)}{\prod_{k=1}^{p'} \Gamma(\langle A'_k, s\rangle - c'_i)}\psi(s) = \int_0^{t_1^{-1}\infty} \cdots \int_0^{t_n^{-1}\infty} f(x)x^{s-1}\,dx. \tag{0.4}$$

By Stirling's formula and the assumption (A)-(i), we obtain the estimate for $\mathrm{Re}(s_i) > 0$ $(i = 1, \ldots, n)$,

$$\frac{\prod_{l=1}^{q} \Gamma(\langle B_l, s\rangle - d_l)}{\prod_{k=1}^{p} \Gamma(\langle A_k, s\rangle - c_i)} = O\left(\exp(-\frac{1}{2}\pi \sum_{i=1}^n |\mathrm{Im}(s_i)|)\right) \quad \text{as} \quad \sum_{i=1}^n |\mathrm{Im}(s_i)| \to +\infty.$$

Also by the Riemann-Lebesgue theorem, we have

$$\mathcal{M}[f,s] \to 0 \quad \text{as} \quad \sum_{i=1}^n |\mathrm{Im}(s_i)| \to +\infty.$$

Combining these estimates, we obtain the asymptotic behaviour of the periodic function

$$\psi(s) = O(\exp(\frac{1}{2}\pi|\mathrm{Im}(s_i)|)), \tag{0.5}$$

as $\mathrm{Im}(s_i) \to \infty$ and the other s_j $(i \neq j)$ are fixed. The right hand side of the equation (0.4) is the analytic function of s when $\mathrm{Re}(s_i)$ $(i = 1, \ldots, n)$ are sufficiently large. Thus if we recall that the assumption (A)-(ii) and the periodicy of $\psi(s)$, we can see that $\psi(s)$ is an entire function. We put $z_i = \exp 2\pi\sqrt{-1}s_i$ for $i = 1, \ldots, n$. And we consider the Laurant expansion of $\phi(s)$ with respect to z_1,

$$\psi(s) = \sum_{k=-\infty}^{\infty} c_k^{(1)}(s_2, \ldots, s_n) z_1^k.$$

Here $c_k^{(1)}(s_2, \ldots, s_n)$ are periodic and entire functions for $(s_2, \ldots, s_n) \in \mathbb{C}^{n-1}$. We write $s_i = \sigma_i + \sqrt{-1}\tau_i$ for $\sigma_i, \tau_i \in \mathbb{R}$, $i = 1, \ldots, n$. We consider an integration

$$\begin{aligned} \int_0^1 |\psi(s)|^2 \, d\sigma_i &= \sum_{k=-\infty}^{\infty} |c_k^{(1)}(s_2, \ldots, s_n)|^2 \exp(-4\pi k \tau_i) \\ &\geq |c_t^{(1)}(s_2, \ldots, s_n)|^2 \exp(-4\pi t \tau_i) \end{aligned}$$

for every $t = 0, \pm 1, \pm 2, \ldots$. However the estimate (0.5) tells us that there exist constants $M_i \in \mathbb{R}_{>0}$ and we have

$$\exp(\pi|\tau_i|) > M_i \int_0^1 |\psi(s)|^2 \, d\sigma_i$$

for sufficiently large τ_i. Thus we have $c_t^{(1)}(s_2, \ldots, s_n) = 0$ for $t = \pm 1, \pm 2, \ldots$. The remaining coefficient $c_0^{(1)}(s_2, \ldots, s_n)$ is also the periodic and entire functions for $(s_2, \ldots, s_n) \in \mathbb{C}^{n-1}$. Hence we apply the same argument for $c_0^{(1)}(s_2, \ldots, s_n)$ with respect to s_2. And also we can proceed inductively for $i = 3, \ldots, n$. Thus we can conclude $\psi(s)$ must be a constant. This completes the proof of the theorem. $\square$

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