RESONANCES AND RESIDUE OPERATORS FOR SYMMETRIC SPACES OF RANK ONE

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Extended Abstract

Let G be a connected noncompact real semisimple Lie group with finite center and let K be a maximal compact subgroup of G. Then the homogeneous space $X = G/K$ is a symmetric space of noncompact type. As X is complete with respect to its canonical G-invariant Riemannian structure, the Laplace-Beltrami operator of X is a self-adjoint operator on the Hilbert space $L^2(X)$ of square integrable function on X. We shall consider in the sequel the operator Δ which is the negative of the Laplace-Beltrami operator. In this way, the spectrum of Δ is the half-line $[||\rho||^2, +\infty[$, where 2ρ is the sum of the positive restricted roots associated with X. The resolvent $R(z) = (\Delta - z)^{-1}$ of Δ is thus a holomorphic function on $\mathbb{C} \setminus [||\rho||^2, +\infty[$ with values in the space of bounded operators on $L^2(X)$. If a meromorphic continuation of $R(z)$ across the spectrum of Δ is possible, then the poles of the meromorphically extended resolvent are called the resonances.

The problem of the meromorphic continuation of the resolvent $R(z)$ have been studied in many frameworks, such as the asymptotically hyperbolic manifolds (see for instance [4], [3], [2] and references therein) and the locally symmetric spaces of rank-one (see for instance [6] and [1]). For an arbitrary symmetric space of the noncompact type G/K, Mazzeo and Vasy [5] and Strohmaier [7] have determined a domain where the analytic continuation of the resolvent is possible. This domain is related to the singularities of the Plancherel measure occurring in the spectral decomposition of $L^2(G/K)$.

More detailed information on the nature of the resonances is so far available only in very specific cases. In [3], Guillopé and Zworski have studied the resonances for the real hyperbolic space $\mathbb{H}^n = \mathrm{SO}(n,1)/\mathrm{SO}(n)$. They proved that there are no resonances for n even; for n odd, there are resonances (which are explicitely determined), and the corresponding residue operators are shown to have finite rank. A representation theoretical approach to the study of the resolvent residue operator was presented for upper half place $\mathrm{SL}(2,\mathbb{R})/\mathrm{SO}(2)$ in [8] by Maciej Zworski. He proved that the resonant states continue analytically to the spherical harmonics, that is to the eigenfunctions of the Laplace operator of the dual compact symmetric space. This shows, in particular, that the rank of the residues operators is finite and related to the dimension of the finite dimensional spherical representations. In [8], the author asked for the general pattern for symmetric spaces. We provide the answer for the symmetric spaces G/K of rank-one: as in the case of $\mathrm{SL}(2,\mathbb{R})/\mathrm{SO}(2)$, we prove an explicit formula for the resolvent residue operators. This allows us to show that each resolvent residue operator maps onto the space of a finite dimensional K-spherical representation of G and it is therefore of finite rank. Our main tool is the Fourier-Helgason transform on the symmetric space G/K together with a detailed analysis of the singularities of Plancherel measure for $L^2(G/K)$ and of the joint eigenspaces of the G-invariant differential operators on G/K.

Even our main result is in rank-one, we keep the higher rank notation to underline some complementary results which we expect to be interesting on their own. First of all, because of additional cancellations, the list of singularities of the Plancherel measure is shorter than those given in [5] or in [7]. Consequently, the resolvent of the Laplace-Beltrami operator of general symmetric spaces

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extends analytically to a larger domain. Secondly, we prove a formula allowing us to compute the dimension of the finite dimensional spherical representations using the restricted root systems. One can therefore use directly the parametrization of these representations provided by the Cartan-Helgason theorem, without having to reconstruct the root systems and highest weights as required by Weyl's dimension formula. The resulting dimension formula does not seem to appear previously in the literature. Also our explicit description of the joint eigenspaces seems to be new. Finally, we expect that the singularities of the Plancherel measure and the joint eigenspaces will play a role in the description of the singularities of the analytic continuation of the resolvent in the higher rank situation as well. We hope to come back to this point in some future work.

References

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