MULTITEMPORAL WAVE EQUATIONS ON SYMMETRIC SPACES

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1. The Wave Equation in $\mathbb{R}^n$

Consider the Cauchy problem for the wave equation on $\mathbb{R}^n$

$$L_x\, u(x,t) = \frac{\partial^2 u(x,t)}{\partial t^2} \tag{1.1}$$
$$u(x,0) = f_1(x), \qquad u_t(x,0) = g(x)$$

where f_1 and f_2 are given functions in, say $\mathcal{S}(\mathbb{R}^n)$.

The solution to (1.1) may be arrived at by using three methods: Fourier transforms, Radon transforms, and Asgeirsson's mean value theorem.

By taking the Fourier transform of (1.1) with respect to the space variable x and solving the resulting second order ordinary differential equation in t we arrive at the well-known solution

$$u(x,t) = (f_1 * T_t')(x) + (f_2 * T_t)(x), \tag{1.2}$$

where T_t and T_t' are tempered distributions whose Fourier transforms are

$$\widetilde{T}_t(\xi) = \sin(t|\xi|)/|\xi|, \quad \widetilde{T_t'}(\xi) = \cos(t|\xi|)$$

The distributions T_t and T_t' are called the *fundamental solutions* to the wave equation.

One may also use the Radon transform to solve (1.1). If Rf denotes the Radon transform of a function $f \in \mathcal{S}(\mathbb{R}^n)$, then one may recover f from Rf by means of the inversion formula

$$c\, f(x) = \int_{\omega \in S^{n-1}} \Lambda(Rf)(\omega, \langle \omega, x\rangle)\, d\omega \tag{1.3}$$

Here c is a constant depending only on n and Λ is the operator on the space of hyperplanes in $\mathbb{R}^n$ given by

$$\Lambda\varphi(\omega,p) = \begin{cases} \dfrac{\partial^{n-1}\varphi}{\partial p^{n-1}}(\omega,p) & n \text{ odd} \\ \mathcal{H}_p \dfrac{\partial^{n-1}\varphi}{\partial p^{n-1}}(\omega,p) & n \text{ even,} \end{cases}$$

where $\mathcal{H}$ denotes the Hilbert transform.

An elementary application of (1.3) shows that the Cauchy problem (1.1) has solution

$$c\, u(x,t) = \int_{\omega \in S^{n-1}} \left(\frac{\partial^{n-1} Rf_1}{\partial t^{n-1}} + \frac{\partial^{n-2} Rf_2}{\partial t^{n-2}} \right)(\omega, \langle \omega, x\rangle + t)\, d\omega, \tag{1.4}$$

when n is odd, and

$$c\,u(x,t) = \int_{\omega\in S^{n-1}} \left(\mathcal{H}\frac{\partial^{n-1}Rf_1}{\partial t^{n-1}} + \mathcal{H}\frac{\partial^{n-2}Rf_2}{\partial t^{n-2}}\right)(\omega, \langle\omega, x\rangle + t)\,d\omega, \tag{1.5}$$

in the case when n is even. We remark that it is easy to deduce Huygen's principle from the solution (1.4).

Finally, Asgeirsson's mean value theorem allows us to obtain mean value solutions to (1.1). If we introduce "dummy" space variables $t_2, \dots, t_n$ to $t = t_1$ and apply the aforementioned theorem, we can obtain the solution

$$c_n\,u(x,t) = \frac{d}{dt}\circ\left(\frac{d}{d(t^2)}\right)^{\frac{n-3}{2}}(t^{n-2}\,M^t f_1(x)) + \left(\frac{d}{d(t^2)}\right)^{\frac{n-3}{2}}(t^{n-2}\,M^t f_2(x)) \tag{1.6}$$

when n is odd, and

$$\begin{aligned} c_n\,u(x,t) &= \frac{d}{dt}\int_0^t (t^2-s^2)^{-1/2}\left[\left(\frac{d}{d(t^2)}\right)^{\frac{n-3}{2}}(t^{n-2}\,M^t f_1(x))\right] s\,ds \\ &\quad + \int_0^t (t^2-s^2)^{-1/2}\left[\left(\frac{d}{d(t^2)}\right)^{\frac{n-3}{2}}(t^{n-2}\,M^t f_2(x))\right] s\,ds \end{aligned} \tag{1.7}$$

for n even. Here c_n is another constant depending only on n. Huygen's principle is then immediate from the first equation above.

The advantage of the mean value solutions (1.6) and (3.1) is that no decay restrictions are imposed on the initial data f_1 and f_2; they are, in a sense, the most general and explicit among the solutions to the Cauchy problem.

2. Multitemporal Wave Equations

While the wave equation may be extended to general Riemannian manifolds X, there is a more natural analogue when X is a Riemannian symmetric space. This analogue was introduced by Semenov-Tjan-Shansky [14] and further developed by Shahshahani [13], Phillips and Shahshahani [11], Helgason [7], and Helgason and Schlichtkrull [8].

While the problem may be formulated equally well for X of compact or noncompact type, let us assume for the moment that X is noncompact, with $X = G/K$, where G is a noncompact real semisimple Lie group with Iwasawa decomposition $G = NAK$. We adopt the notation in Helgason's book *Groups and Geometric Analysis* [5]. We may, when convenient, identify the Lie algebra $\mathfrak{a}$ of A with its dual $\mathfrak{a}^*$ via the Killing form on $\mathfrak{g}$.

Let $\mathbb{D}(X)$ denote the algebra of left G-invariant differential operators on X, and let

$$\Gamma : \mathbb{D}(X) \to \mathbb{D}_W(\mathfrak{a})$$

denote the Harish-Chandra isomorphism, where $\mathbb{D}_W(\mathfrak{a})$ is the algebra of W-invariant elements of the symmetric algebra $\mathbb{D}(\mathfrak{a})$ on $\mathfrak{a}$.

We now fix a basis $p_1, \dots, p_w$ ($w = |W|$) of the vector space of W-harmonic polynomials on $\mathfrak{a}$. If $\lambda \in \mathfrak{a}^*$ is regular, then the $w \times w$ matrix $(p_j(is\lambda))$ (with $s \in W, 1 \le j \le w$) is nonsingular, with inverse $(q^j(is\lambda))$, where the q^j are rational functions on $\mathfrak{a}^*$ given by

$$q^j(i\lambda) = \frac{h_j(i\lambda)}{\pi(i\lambda)}, \tag{2.1}$$

where $\pi = \Pi_{\alpha\in\Sigma_0^+} H_\alpha$, the product of indivisible root vectors. (See [1].) Here $h_j \in \mathbb{D}(\mathfrak{a})$.

For fixed functions $f_j \in \mathcal{E}(X)$ $(1 \leq j \leq w)$, we consider the Cauchy problem for the multitemporal system

$$D_x\, u(x,H) = \partial(\Gamma(D))_H\, u(x,H) \qquad D \in \mathbb{D}(X) \tag{2.2}$$

satisfying the initial data

$$\partial(p_j)_H\, u(x,0) = f_j(x) \qquad (1 \leq i \leq w) \tag{2.3}$$

As an example, when X is the real hyperbolic space $\mathbb{H}^n = SO(n,1)/SO(n)$, (2.2)-(2.3) reduce to the initial value problem for the modified wave equation

$$L_x(u(x,t) = \left(\frac{\partial^2}{\partial t^2} - \left(\frac{n-1}{2}\right)^2\right) u(x,t) \tag{2.4}$$

$$u(x,0) = f_1(x), \qquad u_t(x,0) = f_2(x). \tag{2.5}$$

Going back to the general multitemporal system (2.2)-(2.3), let us suppose that the inital data have compact support; i.e., $f_j \in \mathcal{D}(X)$. Then we can apply the Helgason Fourier transform to both sides of (2.2) with respect to the space variable x and, in a manner analogous to the method for $\mathbb{R}^n$, obtain the solution

$$u(x,H) = \sum_{j=1}^{w} \left(f_j \times S_j^H\right)(x). \tag{2.6}$$

Here S_j^H is a K-invariant distribution on X whose spherical Fourier transform is

$$\begin{aligned}\widetilde{\left(S_j^H\right)}(\lambda) &= \sum_{s\in W} q^j(is\lambda) \\ &= \frac{1}{\pi(i\lambda)} \sum_{s\in W} \det(s) h_j(is\lambda)\end{aligned}$$

The last expression above is easily seen to extend to a holomorphic W-invariant function on $(a^*)^c$ of polynomial growth in λ and exponential type $|H|$. Thus, by the Paley-Wiener theorem for the spherical Fourier transform on distributions (see [5]) it follows that the solution (2.6) moves at unit speed. Specifically, if o represents the identity coset $\{K\}$ in G/K and $B_\epsilon(o)$ is the ball of radius ϵ centered at o, then

$$\text{supp } f_j \subset B_\epsilon(o) \quad \forall j \implies u(x,H) = 0 \text{ for } d(o,x) > \epsilon + |H|.$$

When X is a compact symmetric space U/K, we obtain a convolution solution written in the same way as (2.6). In this case, each S_j^H is a K-invariant distribution on X with Fourier series

$$\sum_{\pi\in\widehat{U}_K} d(\pi) \sum_{\sigma\in W} q_j(\sigma(\mu_\pi + \rho))\, e^{\sigma(\mu_\pi+\rho)(H)} \phi_{\pi^*} \tag{2.7}$$

The outer sum above is taken over the set $\widehat{U}_K$ of all (equivalence classes of) irreducible K-spherical representations of the compact Lie group U. For $\pi \in \widehat{U}_K$, we have denoted the corresponding zonal spherical function and restricted highest weight by ϕ_{π^*} and μ_π, respectively.

Now the expression (2.7) and the Paley-Wiener theorem for K-invariant functions on U/K, suitably modified for distributions (see [10]; [2] for the case when

X is a compact semisimple Lie group) then shows that, again, $u(x,H)$ moves with unit speed: whenever $|H|$ is smaller than the injectivity radius of X, then

$$\text{supp}\; f_j \subset B_\epsilon(o) \quad \forall j \implies u(x,H)=0 \text{ for } d(o,x) > \epsilon + |H|.$$

When the symmetric space X is noncompact, there is a Radon transform based solution of the Cauchy problem (2.2)-(2.3) which is analogous to the Euclidean one (1.4)-(1.5):

$$u(x,H) = e^{\rho(H)} \sum_{j=1}^{w} \int_{K/M} \Lambda_j (Rf_j)(gk\, \exp(H)\cdot \xi_o)\, dk_M \tag{2.8}$$

In the expression on the right Rf_j denotes the horocycle Radon transform of f_j and Λ_j is the operator on the horocycle space $\Xi = G/MN = K/M \times A$ defined in the following way. Let $\psi \to \mathcal{F}\psi$ be the Euclidean Fourier transform on A:

$$\psi(\lambda) = \int_A f(a)\, e^{-i\lambda(\log a)}\, da \qquad (\psi \in \mathcal{S}(A),\; \lambda \in \mathfrak{a}^*)$$

For each j let σ_j be the tempered distribution on A whose Fourier transform is

$$\mathcal{F}(\sigma_j) = q^j(i\lambda)\,|c(\lambda)|^{-2}$$

translation-invariant operator on A defined by

$$t_j\psi = \psi * \sigma_j \qquad (\psi \in \mathcal{S}(A))$$

If all restricted root multiplicites are even, then t_j is a differential operator on A. Next let T_j be the operator on Ξ given by $T_j\varphi(kM,a) = (t_j)_a\,\varphi(kM,a)$, and finally put

$$J_j = e^{-\rho} \circ T_j \circ e^{\rho}.$$

J_j is then a G-invariant pseudodifferential operator on Ξ, and is a differential operator whenever all multiplicities are even. In such a case, an elementary argument using the Radon transform solution (2.8) results in Huygens' principle:

$$\text{supp}\, f_j \subset B_\epsilon(o)\; \forall j \implies u(x,H) = 0 \text{ whenever } d(o,x) < |H| - \epsilon.$$

We note that the Radon transform solution (2.8), as well as (1.4)-(1.5) in the Euclidean case, make use of the so-called *shifted dual transform* (see [12]), which is often used to obtain inversion formulas for Radon transforms arising from homgeneous spaces in duality.

3. Mean Value Solutions

We are interested in obtaining mean value type solutions to the multitemporal system (2.2)-(2.3) in both the noncompact and compact cases. The reason is twofold. For one thing, there is at present no useful Radon transform based solution when X is a general compact symmetric space, and thus no means for obtaining Huygen's principle using such a solution. For another, mean value solutions offer us more explicit solution formulas which allow us in particular to remove decay restrictions on the initial data.

There is, unfortunately, at present no general mean value type solution to the aforementioned multitemporal system. However, we have obtained such solutions in the case when the multiplicities are all even. Our formula reads as follows.

First, for any $y \in X$, let us recall the *mean value operator* M^y:

$$M^y f(g\cdot o) = \int_K f(gk\cdot y)\,dk$$

Then in both the compact and noncompact cases, when all the restricted root multiplicities are even, the solution to (2.2)-(2.3) is given by

$$u(x,H) = \sum_{j=1}^{w} \left(\partial(h_j)\circ E\right)_H \; M^{\exp H}(f_j(x)) \tag{3.1}$$

In the above, the polynomial h_j is the numerator of the rational function q^j (see (2.1)) and E is a W-skew differential operator on $\mathfrak{a}$ with C^∞ coefficients. Essentially, E is the adjoint of a *shift operator* on $\mathfrak{a}$. (For a nice introduction to the shift principle, see [4].) Note that when X is compact, the solution (3.1) above is "periodic" in H. Huygens' principle is also apparent from the formula, as is the observation that $u(g\cdot o, H)$ depends only on the values of the initial data f_j on the translated orbit $gK\exp(H)\cdot o$. (See [4], [9]).

A special case occurs when X is a compact semisimple Lie group U (all multiplicities $= 2$). Here the solution is periodic in H and can be written as follows:

$$u(x,H) = \frac{1}{\pi(\rho)} \sum_{i=1}^{w} \partial(h_i)_H \left(\delta^{1/2}(t)\, M^{\exp H} f_i(x)\right),$$

where $\delta^{1/2}$ is the Weyl denominator $\sum_{s\in W} \det s\, e^{s\rho}$.

Another special case occurs when n is an odd-dimensional hyperbolic space $\mathbb{H}^n$. Then the solution to the initial value problem (2.4)-(2.5) is

$$\begin{aligned} u(x,t) \;=\; c_n \;\Bigg\{ \frac{d}{dt}\circ\left(\frac{d}{d(\cosh t)}\right)^{\frac{n-3}{2}} &\left((\sinh t)^{n-2}\, M^t f_1(x)\right) \\ + \left(\frac{d}{d(\cosh t)}\right)^{\frac{n-3}{2}} &\left((\sinh t)^{n-2}\, M^t f_2(x)\right)\Bigg\}. \end{aligned}$$

This solution can also be obtained using Asgeirsson's mean value theorem (see [5]). See also [3] for its spherical counterpart.

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