# DEFORMATION OF DISCONTINUOUS SUBGROUPS ACTING ON SOME NILPOTENT HOMOGENEOUS SPACES

ALI BAKLOUTI

## 1. Introduction

Let $H$ be an arbitrary closed connected subgroup of a connected, simply connected exponential Lie group $G$. The main subject of this note is the description of the deformation space $\mathcal{T}(\Gamma, G, H)$ of a discontinuous subgroup $\Gamma$ of $G$ for the homogeneous space $G/H$. The problem of describing explicitly deformations for Clifford-Klein forms in general settings was initiated by T. Kobayashi in [8] and was formalized as *Problem C* [5]. The deformation space $\mathcal{T}(\Gamma, G, H)$ was also introduced by the same author in ([5], (5.3.1)) for general homogeneous space $G/H$ and discontinuous groups $\Gamma$. The problem of determining explicitly such deformations is in general pretty tricky. Beyond some known setups, the hurdles involved in this problem are considerable. It is rather hopeless to obtain an accurate description of such objects by means of the accessible ingredients we are submitted to cope with, even when we stick for instance to the case where $G$ is two-step nilpotent. However, such a description is sometimes one fundamental genesis for being relevant in understanding the local geometric structures of these deformations. Such structures might be pretty complex when the Clifford-Klein forms in question are not necessarily compact. We specifically focus attention in this note on the case where the group $G$ is the $2n+1$ Heisenberg group.

## 2. Backgrounds

We begin this section with fixing some notation and terminology and recording some basic facts about deformations. Concerning the entire subject, we strongly recommend the papers [6, 7].

2.1. **Proper and fixed point actions.** Let $\mathcal{M}$ be a locally compact space and $K$ a locally compact topological group. The action of the group $K$ on $\mathcal{M}$ is said to be:

(1) Proper (in the sense of Palais [12]) if, for each compact subset $S \subset \mathcal{M}$ the set $K_S = \{k \in K : \ k \cdot S \cap S \neq \emptyset\}$ is compact.

(2) Fixed point free (or merely free) if, for each $m \in \mathcal{M}$, the isotropy group $K_m = \{k \in K : \ k \cdot m = m\}$ is trivial.

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(3) Properly discontinuous if, $K$ is discrete and for each compact subset $S \subset \mathcal{M}$ the set $K_S$ is finite.

In the case where $\mathcal{M} = G/H$ is a homogeneous space and $K$ a subgroup of $G$, then it is well known that the action of $K$ on $\mathcal{M}$ is proper if $SHS^{-1} \cap K$ is compact for any compact set $S$ in $G$. Here, for two sets $A$ and $B$ of the locally compact topological group $G$, the product $AB$ is the subset $\{ab : \ a \in A, \ b \in B\}$. Likewise the action of $K$ on $\mathcal{M}$ is free if for every $g \in G$, $K \cap gHg^{-1} = \{e\}$. In this context, the subgroup $K$ is said to be a discontinuous group for the homogeneous space $\mathcal{M}$, if $K$ is a discrete subgroup of $G$ and $K$ acts properly and fixed point freely on $\mathcal{M}$.

2.2. **Clifford-Klein forms.** For any given discontinuous subgroup $\Gamma$ for the homogeneous space $G/H$, the quotient space $\Gamma\backslash G/H$ is said to be a *Clifford-Klein form* for the homogeneous space $G/H$. It is then well-known that any Clifford-Klein form is endowed through the action of $\Gamma$ with a manifold structure for which the quotient canonical surjection

$$\pi : G/H \to \Gamma\backslash G/H \tag{1}$$

turns out to be an open covering and particularly a local diffeomorphism. On the other hand, any Clifford-Klein form $\Gamma\backslash G/H$ inherits any $G$-invariant geometric structure (e.g. complex structure, pseudo-Riemanian structure, conformal structure, symplectic structure,...) on the homogeneous space $G/H$ through the covering map $\pi$ defined as in equation (1) below.

2.3. **Parameters and deformation spaces.** The material dealt with in this subsection comes from [6]. The reader could consult that reference and some references therein for more detailed information. We designate by $\mathrm{Hom}(\Gamma, G)$ the set of groups homomorphisms from $\Gamma$ to $G$ endowed with the point wise convergence topology. The same topology is obtained by taking generators $\gamma_1, ..., \gamma_k$ of $\Gamma$, then using the injective map:

$$\mathrm{Hom}(\Gamma, G) \to G \times \cdots \times G, \ \varphi \mapsto (\varphi(\gamma_1), ..., \varphi(\gamma_k))$$

to equip $\mathrm{Hom}(\Gamma, G)$ with the relative topology induced from the direct product $G \times \cdots \times G$. We consider then the parameter space $R(\Gamma, G, H)$ of $\mathrm{Hom}(\Gamma, G)$ defined by:

$$\begin{aligned} R(\Gamma, G, H) = \{\varphi \in \mathrm{Hom}(\Gamma, G) : \varphi \text{ is injective and } \varphi(\Gamma) \text{ acts properly} \\ \text{discontinuousely and freely on } G/H\}. \end{aligned}$$

According to this definition, for each $\varphi \in R(\Gamma, G, H)$, the space $\varphi(\Gamma)\backslash G/H$ is a Clifford-Klein form which is a Hausdorff topological space and even equipped with a structure of a manifold for which, the quotient canonical map is an open covering. Let now $\varphi \in R(\Gamma, G, H)$ and $g \in G$, we consider the element $\varphi^g := g^{-1} \cdot \varphi \cdot g$ of $\mathrm{Hom}(\Gamma, G)$ defined by:

$$\varphi^g(\gamma) = g^{-1}\varphi(\gamma)g, \gamma \in \Gamma.$$

It is then clear that the element $\varphi^g \in R(\Gamma, G, H)$ and that the map:

$$\varphi(\Gamma)\backslash G/H \longrightarrow \varphi^g(\Gamma)\backslash G/H, \ \ \varphi(\Gamma)xH \mapsto \varphi^g(\Gamma)g^{-1}xH$$

is a natural diffeomorphism. We consider then the orbits space:

$$\mathcal{T}(\Gamma, G, H) = R(\Gamma, G, H)/G$$

instead of $R(\Gamma, G, H)$ in order to avoid the unessential part of deformations arising inner automorphisms and to be quite precise on parameters. We call the set $\mathcal{T}(\Gamma, G, H)$ as the space of the deformation of the action of $\Gamma$ on the homogeneous space $G/H$.

2.4. **Topological features of deformations.** We keep the same notation and hypotheses. A. Weil [13] introduced the notion of local rigidity of homomorphisms in the case where the subgroup $H$ is compact. T. Kobayashi [8] generalized it in the case where $H$ is not compact. The distinguishing feature for non-compact setting is that the local rigidity does not hold in general in the non-Riemannian case and has been studied in [1, 3, 6, 7, 10]. We briefly recall here some details. For a comprehensible information, we refer the readers to the references [1, 3, 4, 5, 6, 7, 8, 9, 10, 11] and some references therein. For $\varphi \in R(\Gamma, G, H)$, the discontinuous subgroup $\varphi(\Gamma)$ for the homogeneous space $G/H$ is said to be *locally rigid* in the sense of Kobayashi ([8]) as a discontinuous group of $G/H$ if the orbit of $\varphi$ through the inner conjugation is open in $\mathrm{Hom}(\Gamma, G)$(respectively in the set $R(\Gamma, G, H)$). This means equivalently that any point sufficiently close to $\varphi$ should be conjugate to $\varphi$ under an inner automorphism of $G$. When every point in $R(\Gamma, G, H)$ is locally rigid, the deformation space turns out to be discrete and then we say that the Clifford-Klein form $\Gamma\backslash G/H$ can not deform continuously through the deformation of $\Gamma$ in $G$. If a given $\varphi \in R(\Gamma, G, H)$ is not locally rigid, we say that it admits a *continuous deformation* and that the related Clifford-Klein form is continuously deformable.

The homomorphism $\varphi$ is said to be *topologically stable* or merely *stable* in the sense of Kobayashi-Nasrin ([10]), if there is an open set in $\mathrm{Hom}(\Gamma, G)$ which contains $\varphi$ and is contained in $R(\Gamma, G, H)$. When the set $R(\Gamma, G, H)$ is an open subset of $\mathrm{Hom}(\Gamma, G)$, then obviously each of its elements is stable, which is the case for irreducible Riemannian symmetric spaces. Furthermore, we point out in this setting that the concept of stability may be one fundamental genesis to understand the local structure of the deformation space.

## 3. The main results

3.1. **Algebraic interpretation of deformations.** We keep our hypotheses and notation. We first prove the following preliminary algebraic interpretation of the parameter and deformation spaces. Let $G$ be a connected simply connected completely solvable Lie group, $H = \exp(\mathfrak{h})$ a connected subgroup of $G$ and $\Gamma$ a discrete subgroup of $G$ acting properly discontinuously on $G/H$. We designate by $L = \exp(\mathfrak{l})$ the syndetic hull of $\Gamma$ and we pose

$$R(\mathfrak{l}, \mathfrak{g}, \mathfrak{h}) = \left\{ \varphi \in \mathrm{Hom}(\mathfrak{l}, \mathfrak{g}) \;\middle|\; \begin{array}{l} \dim \varphi(\mathfrak{l}) = \dim \mathfrak{l} \text{ and} \\ \exp \varphi(\mathfrak{l}) \text{ acts properly on } G/H \end{array} \right\}.$$

The group $G$ acts on the spaces $\mathrm{Hom}(\Gamma, G)$, $\mathrm{Hom}(L, G)$ and $\mathrm{Hom}(\mathfrak{l}, \mathfrak{g})$ respectively through the following laws:

$$(g \cdot \varphi)(\gamma) = g\varphi(\gamma)g^{-1},\ \gamma \in \Gamma(\text{or } L),\ \ \varphi \in \mathrm{Hom}(\Gamma, G)(\text{or } \mathrm{Hom}(L, G)), g \in G$$
$$g \cdot \psi = \mathrm{Ad}_g \circ \psi,\ \psi \in \mathrm{Hom}(\mathfrak{l}, \mathfrak{g}), g \in G.$$

**Theorem 3.1.** *Let $G$ be a connected simply connected completely solvable Lie group, $H$ a connected subgroup of $G$ and $\Gamma$ a discrete subgroup of $G$ acting properly discontinuously on $G/H$. Then up to a homeomorphism we have,*

$$R(\Gamma, G, H) = R(\mathfrak{l}, \mathfrak{g}, \mathfrak{h}).$$

*In particular if $\Gamma$ and $\Gamma'$ have the same syndetic hull, then $R(\Gamma, G, H)$ and $R(\Gamma', G, H)$ are homeomorphic. Furthermore, up to homeomorphism, the deformation space $\mathcal{T}(\Gamma, G, H)$ coincides with $R(\mathfrak{l}, \mathfrak{g}, \mathfrak{h})/G$.*

3.2. **On Grassmannians.** Let $k = \dim \mathfrak{l}$, $s = \dim \mathfrak{h}$ and $n = \dim \mathfrak{g}$. We fix a basis $\{X_1, \ldots, X_n\}$ of $\mathfrak{g}$ passing through $\mathfrak{h}$. We identify the vector spaces $\mathfrak{g}$ to $\mathbb{R}^n$, $\mathfrak{l}$ to $\mathbb{R}^k$, $\mathfrak{h}$ to the $s$ dimensional subspace $\mathbb{R}^s \times 0_{\mathbb{R}^{n-s}}$ of $\mathbb{R}^n$, $\mathcal{L}(\mathfrak{l}, \mathfrak{g})$ to $M_{n,k}(\mathbb{R})$ the real vector space of $n \times k$ matrices with real entries and $\mathrm{Hom}(\mathfrak{l}, \mathfrak{g})$ to a closed subset of $M_{n,k}(\mathbb{R})$. Let $M^\circ_{n,k}(\mathbb{R})$ be the open set of $M_{n,k}(\mathbb{R})$ consisting of rank $k$ matrices in $M_{n,k}(\mathbb{R})$, which is also identified to the set $\{\varphi \in \mathcal{L}(\mathfrak{l}, \mathfrak{g}),\ \varphi \text{ injective}\}$. We define the set

$$I(n, k) = \{(i_1, \ldots, i_k) \in \mathbb{N}^k,\ 1 \le i_1 < \cdots < i_k \le n\}.$$

For

$$M = \begin{pmatrix} L_1 \\ \vdots \\ L_n \end{pmatrix} \in M_{n,k}(\mathbb{R}) \text{ and } \alpha = (i_1, \ldots, i_k) \in I(n, k),$$

we denote by $M_\alpha$ the $k \times k$ relative minor $\begin{pmatrix} L_{i_1} \\ \vdots \\ L_{i_k} \end{pmatrix}$. Let

$$U_\alpha = \{M \in M^\circ_{n,k}(\mathbb{R}) :\ M_\alpha = I_k\} \cong M_{n-k,k}(\mathbb{R}),$$

where $I_k$ designates the identity element of $M_k(\mathbb{R})$. The group $GL_k(\mathbb{R})$, acting on $M_{n,k}(\mathbb{R})$ through a right multiplication and the Grassmannian $G_{n,k}(\mathbb{R})$ of the $k-$dimensional subspaces of $\mathbb{R}^n$ is identified to the quotient topological space $M^\circ_{n,k}(\mathbb{R})/GL_k(\mathbb{R})$. Let

$$\begin{array}{rccc} \eta : & M^\circ_{n,k}(\mathbb{R}) & \to & G_{n,k}(\mathbb{R}) \\ & M & \mapsto & M(\mathbb{R}^k) \end{array}$$

be the canonical surjection. It is easy to see that the restriction $\eta_\alpha$ of $\eta$ to the set $U_\alpha$ is an homeomorphism between $U_\alpha$ and its image. We get then quite easily that

$$G_{n,k}(\mathbb{R}) = \bigcup_{\alpha \in I(n,k)} \eta_\alpha(U_\alpha).$$

It is well known that these bijections define a compatible affine charts on $G_{n,k}(\mathbb{R})$ which endow this space with a structure of a manifold of dimension $k(n-k)$.

3.3. **On abelian discontinuous subgroups.** Throughout the present section, $\mathfrak{g} := \mathfrak{h}_{2n+1}$ designates the Heisenberg Lie algebra of dimension $2n+1$ and $G := H_{2n+1}$ the corresponding Lie group. $\mathfrak{g}$ can be defined as a real vector space endowed with a skew-symmetric bilinear form $b$ of rank $2n$ and a fixed generator $Z$ belonging to the kernel of $b$. The center $\mathfrak{z}$ of $\mathfrak{g}$ is then the kernel of $b$ and it is the one dimensional subspace $[\mathfrak{g}, \mathfrak{g}]$. For any $X, Y \in \mathfrak{g}$, the Lie bracket is given by

$$[X, Y] = b(X, Y)Z.$$

The following result is proved in ([3], Proposition 3.1) and provides a method to construct a symplectic basis of $\mathfrak{g}$ starting from a given subalgebra $\mathfrak{l}$ of $\mathfrak{g}$ and referred to be adapted to $\mathfrak{l}$.

**Proposition 3.2.** *Let $\mathfrak{l}$ be a Lie subalgebra of $\mathfrak{g}$. Then there exists a basis $\mathscr{B}_{\mathfrak{l}} = \{Z, X_1 \ldots, X_n, Y_1, \ldots, Y_n\}$ of $\mathfrak{g}$ with the Lie commutation relations*

$$[X_i, Y_j] = \delta_{i,j} Z,\ \ i, j = 1, ..., n$$

*and satisfying:*
*1) If $\mathfrak{z} \subset \mathfrak{l}$, then there exist two integers $p$, $q \geq 0$ such that the family*

$$\{Z, X_1, \ldots, X_{p+q}, Y_1, \ldots, Y_p\}$$

*constitutes a basis of $\mathfrak{l}$.*
*2) If $\mathfrak{z} \not\subset \mathfrak{l}$, then $\dim \mathfrak{l} \leq n$ and $\mathfrak{l}$ is generated by $X_1, \ldots, X_s$, where $s = \dim \mathfrak{l}$. The symbol $\delta_{i,j}$ here designates the Kronecker symbol. The basis $\mathscr{B}_{\mathfrak{l}}$ is said to be a symplectic basis of $\mathfrak{g}$ adapted to $\mathfrak{l}$.*

For any $\alpha \in I_s(n, k) = \{(i_1, \ldots i_k) \in I(n, k),\ i_1 > s\}$, we consider the set

$$\mathscr{V}_\alpha := \left\{ M = \begin{pmatrix} 0 \\ A \end{pmatrix},\ A \in M_{2n,k}(\mathbb{R}), \quad M_\alpha = \mathrm{I}_k \text{ and } {}^tMJ_bM = 0 \right\},$$

where $J_b$ is the matrix of $b$ in $\mathscr{B}_{\mathfrak{h}}$. Let also

$$I_s^1(2n+1, k) = \{(i_1, \ldots, i_k),\ i_1 = 1 \text{ and } i_2 > s+1\}.$$

The following theorem is proved in ([3]) and provides a description of the deformation space in this context.

**Theorem 3.3.** *Let $G$ be the Heisenberg Lie group of dimension $2n+1$, $H$ a connected Lie subgroup of dimension $s$ and $\Gamma$ a rank $k$ discontinuous subgroup for $G/H$. Then*
*1) If $H$ contains the center of $G$, then*

$$\mathcal{T}(\Gamma, G, H) = \bigcup_{\alpha \in I_s(2n+1,k)} \mathcal{T}_\alpha \quad \textit{and}$$

*where for every $\alpha \in I_s(2n+1, k)$, the set $\mathcal{T}_\alpha$ is an open subset of $\mathcal{T}(\Gamma, G, H)$ homeomorphic to the product $GL_k(\mathbb{R}) \times \mathscr{V}_\alpha$.*

*2) If $H$ does not meet the center of $G$, then*

$$\mathcal{T}(\Gamma, G, H) = \bigcup_{\alpha \in I_{s+1}(2n+1,k)} \mathcal{T}_\alpha \bigcup_{\alpha \in I_s^1(2n+1,k)} \mathcal{T}_\alpha,$$

*where for every $\alpha \in I_{s+1}(2n+1,k)$, the set $\mathcal{T}_\alpha$ is open in $\mathcal{T}(\Gamma, G, H)$ and homeomorphic to the product $GL_k(\mathbb{R}) \times \mathscr{V}_\alpha$. On the other hand, for any $\alpha \in I_s^1(2n+1,k)$, the set $\mathcal{T}_\alpha$ is open in $\mathcal{T}(\Gamma, G, H)$ and is homeomorphic to the product $O_k \times \mathbb{R}^k \times N_k \times \mathscr{V}_\alpha$, $N_k$ designates here the set of upper triangular unipotent matrices.*

We now describe the deformation space for compact Clifford-Klein forms. We have the following:

**Theorem 3.4.** *Let $G$ be the Heisenberg Lie group of dimension $2n+1$, $H$ a connected Lie subgroup of dimension $s$ and $\Gamma$ a rank $k$ discontinuous subgroup for $G/H$. Assume in addition that the Clifford-Klein form $\Gamma\backslash G/H$ is compact. Then*

*1) If $H$ contains the center of $G$, then $k < s$ and*

$$\mathcal{T}(\Gamma, G, H) = GL_k(\mathbb{R}) \times M_{p,q}(\mathbb{R})^2 \times Sym(\mathbb{R}^q) \times Sp(p, \mathbb{R})/Sp(p-r, \mathbb{R}),$$

*where $q+1 = \dim(\ker b_{|\mathfrak{h}})$, $2p+q+1 = \dim \mathfrak{h}$ and $p+q+r = n$.*

*2) If $H$ does not contain the center of $G$ then,*

$$\mathcal{T}(\Gamma, G, H) = O_{n+1} \times \mathbb{R}^{n+1} \times N_n \times Sym(\mathbb{R}^n).$$

3.4. **On non-abelian discontinuous subgroups.** We keep the same notation and hypotheses and we now assume that $\Gamma$ is not abelian. The group $\mathrm{Aut}(\mathfrak{l})$ of automorphisms of $\mathfrak{l}$ is a closed subgroup of $GL_k(\mathbb{R})$, then the homogeneous space $GL_k(\mathbb{R})/\mathrm{Aut}(\mathfrak{l})$ is endowed with a manifold structure for which the quotient map $p : GL_k(\mathbb{R}) \to GL_k(\mathbb{R})/\mathrm{Aut}(\mathfrak{l})$ admits local sections. Consider an open covering $\{V_\beta\}_{\beta \in I}$ of $GL_k(\mathbb{R})/\mathrm{Aut}(\mathfrak{l})$ such that for any $\beta \in I$, there is a section $s_\beta : V_\beta \to GL_k(\mathbb{R})$ satisfying $p \circ s_\beta = Id_{|V_\beta}$.

Let us denote by $I_s^1(2n+1,k)$ the set of all element $\alpha \in I_s(2n+1,k)$ of the form $\alpha = (s+1, i_2, \dots, i_k)$. The following theorem ([2]) describes the structure of the deformation space in this context.

**Theorem 3.5.** *Let $G$ be the $2n+1-$dimensional Heisenberg group, $H$ a connected Lie subgroup of $G$ and $\Gamma$ a non-abelian discontinuous subgroup of $G$ for $G/H$. Let $L = \exp(\mathfrak{l})$ be the syndetic hull of $\Gamma$. There exists a set of a local sections $(s_\beta)_{\beta \in I}$ for the canonical surjection $GL_k(\mathbb{R}) \to GL_k(\mathbb{R})/Aut(\mathfrak{l})$ such that, the deformation space of $\Gamma$ acting on $G/H$ reads*

$$\mathcal{T}(\Gamma, G, H) = \bigcup_{\substack{\beta \in I \\ \alpha \in I_s^1(2n+1,k)}} \mathscr{T}_{\alpha\beta},$$

*where for $\beta \in I$ and $\alpha \in I_s^1(2n+1,k)$, the set $\mathscr{T}_{\alpha\beta}$ is open in $\mathcal{T}(\Gamma, G, H)$ and homeomorphic to the set*

$$\mathbb{R}^\times \times \mathbb{R}^{2pq} \times Sp(2p) \times GL_q(\mathbb{R}) \times \mathscr{A}_{\alpha,\beta},$$

*where $\mathscr{A}_{\alpha,\beta}$ is a closed subset of $s_\beta(V_\beta) \times U_\alpha$ for any $\alpha \in I_s^1(2n+1,k)$ and $\beta \in I$. Here $q+1 = \dim \mathfrak{z}(\mathfrak{l})$ and $1+2p+q = k$.*

As a direct consequence, we get the following:

**Corollary 3.6.** *Let $H$ be a connected subgroup of the Heisenberg group $G = H_{2n+1}$ and $\Gamma$ a discontinuous subgroup for the homogeneous space $G/H$. Then $R(\Gamma, G, H)$ is an open set in $\mathrm{Hom}(\Gamma, G)$. That is, every element of $R(\Gamma, G, H)$ is stable. Moreover, the local rigidity propriety fails to hold globally on $R(\Gamma, G, H)$.*

## References


1. Baklouti. A and Kédim. I. On the deformation space of Clifford-Klein forms of some exponential homogeneous spaces. Int. J. Math. To appear.
2. Baklouti. A and Kédim. I. Deforming non-abelian discontinuous subgroups for some exponential solvable homogeneous spaces. Preprint.
3. Baklouti. A, Kédim. I. and Yoshino. T. On the deformation space of Clifford-Klein forms of Heisenberg groups. Int. Math. Res. Not. To appear
4. Kobayashi. T. Criterion of proper action on homogeneous space of reductive type, J. Lie theory 6, 147-163(1996)
5. Kobayashi. T. Deformation of compact Clifford-Klein forms of indefinite Riemannian homogeneous manifolds, Math. Ann. 310, 394-408(1998)
6. Kobayashi. T. Discontinuous groups for non-Riemannian homogeneous space, Mathematics Unlimited-2001 and Beyond, edited by B. Engquist and W. Schmid. Springer, 723-747(2001)
7. Kobayashi. T. Discontinuous groups acting on homogeneous spaces of reductive type. Proceeding of the conference on representation theorie of Lie Groups and Lie algebras held in 1990 August-september at Fuji- Kawaguchiko (ICM-90 Satellite Conference). Word Scientific, 59-75(1992)
8. Kobayashi. T. On discontinuous groups on homogeneous space with noncompact isotropy subgroups, J. Geom. Phys. 12, 133-144(1993)
9. Kobayashi. T. Proper action on homogeneous space of reductive type. Math. Ann. 285, 249-263(1989)
10. Kobayashi. T. And Nasrin S. Deformation of properly discontinuous action of $\mathbb{Z}^k$ on $\mathbb{R}^{k+1}$, Int. J. Math. 17, 1175-1193(2006)
11. Kobayashi. T and Yoshino. T. Compact Clifford-Klein forms of symmetric spaces-Revisited, Pure and applied Mathematics Quaterly, Volume 1, Number 3 (Special Issue: In memory of Armand Borel, Part 2 of 3), 591-663(2005)
12. Palais. R. S. On the existence of slices for actions of non-compact Lie groups, Ann. of Math. 73, 295-323(1961)
13. Weil A. Remarks on the cohomology of groups, Ann. Math. 80, 149-157(1964)



Ali Baklouti: Department of Mathematics, Faculty of Sciences of Sfax, Route de Soukra, 3038, Sfax, Tunisia.

*E-mail address*: `Ali.Baklouti@fss.rnu.tn`

*E-mail address*: `bakloutia@yahoo.fr`