BRANCHING LAWS FOR SQUARE INTEGRABLE REPRESENTATIONS

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Abstract. This note considers irreducible square integrable representation of semisimple Lie group with admissible restriction to some reductive subgroup. It is mostly a report on joint work with Michel Duflo, to propose algorithms to compute multiplicity for each irreducible factor of the restriction, analogue to the multiplicity formula due to Kostant-Heckman or Blattner.

1. Introduction

From now on G is a connected simple Lie group, H denotes a connected reductive subgroup of G and we assume there exists an irreducible square integrable representation π of G which has an admissible restriction to H. Thus, we may express as a Hilbert sum,

$$\pi_{|_H} = \sum_{i \in \mathbb{N}} n_i \sigma_i.$$

Here, $0 \leq n_i < \infty$ and σ_i are irreducible square integrable representations for H [11], [3].

A Theorem of Harish-Chandra states: a semisimple Lie group admits irreducible square integrable representations if and only if it has a compact Cartan subgroup. Henceforth, since we assume G admits a square integrable representation with admissible restriction to H we may assume G has a compact Cartan subgroup T and we fix a maximal compact subgroup K of G satisfying:

$T \subseteq K$, $L := H \cap K$ is maximal compact subgroup of H,
$U := H \cap T$ is a compact Cartan subgroup of H.

Let W_S denote the Weyl group of a compact connected Lie group S. Harish-Chandra parameterized the set of equivalence classes of irreducible square integrable representations by means of the set of regular characters of T divided by the action of the Weyl group W_K. For each

1991 *Mathematics Subject Classification.* Primary 22E46; Secondary 32E45.
Key words and phrases. Discrete Series, branching laws, multiplicity formula.
Partially supported by FONCYT, CONICET, SECYTUNC, AgenciaCbaCiencia (Argentina), University of Paris VII(France), ICTP (Italy).

regular character[1] $e^\lambda \in \hat{T}$ (resp. $e^\mu \in \hat{U}$) let π_λ (resp. σ_μ) denote the corresponding irreducible square integrable representation of G (resp. H).

For π_λ we consider the system of positive roots determined by λ

$$\Psi = \{\alpha \in \Phi(\mathfrak{g}, \mathfrak{t}) : (\lambda, \alpha) > 0\}$$

and $\Psi_K := \Psi \cap \Phi(\mathfrak{k}, \mathfrak{t})$. In order to avoid cumbersome statements, π_λ will always denote the square integrable representation of Harish-Chandra parameter λ dominant with respect to Ψ. From now on, τ_μ denotes an irreducible representation of K of infinitesimal character μ dominant with respect to Ψ_K. By definition, the set of noncompact roots in Ψ is $\Psi_n := \Psi - \Psi_K$. Since Ψ_n spans a strict cone, the partition function p_{Ψ_n} associated to $\Psi_n := \{\beta_1, \ldots, \beta_q\}$ is well defined. As usual,

$$\rho := \frac{1}{2}\sum_{\alpha \in \Psi} \alpha, \;\; \rho_c := \frac{1}{2}\sum_{\alpha \in \Psi_K} \alpha, \;\; \rho_n := \rho - \rho_c.$$

The formula of Blattner for the multiplicity of the irreducible representation τ_μ of K in the square integrable function, π_λ, as shown by Hecht-Schmid, [5] is:

$$b(\lambda, \mu) := \sum_{w \in W_K} \epsilon(w) p_{\Psi_n}(w\mu - \lambda - \rho_n),$$

$$\pi_{\lambda_{|K}} = \sum_{\mu \text{ dominant w.r.t. } \Psi_K} b(\lambda, \mu)\tau_\mu.$$

Next, we recall the formula of Heckman-Kostant for compact G. Hence, $G = K, H = L$. We fix the systems of positive roots Ψ_K and $\Psi_L \subset \Phi(\mathfrak{l}, \mathfrak{u})$ to be compatible. That is,

$$\alpha \in \Phi(\mathfrak{k}, \mathfrak{t}) \text{ and } \alpha_{|\mathfrak{u}} \in \Psi_L, \text{ implies } \alpha \in \Psi_K.$$

and if $\Phi_\mathfrak{z} := \{\alpha \in \Phi(\mathfrak{k}, \mathfrak{t}) : \alpha(\mathfrak{u}) = 0\}$, the simple roots for $\Psi_\mathfrak{z} := \Psi \cap \Phi_\mathfrak{z}$ are simple for Ψ. We set for $\gamma \in \mathfrak{t}^\star$

$$\varpi(\gamma) = \frac{\prod_{\alpha \in \Psi_\mathfrak{z}} \gamma(h_\alpha)}{\prod_{\alpha \in \Psi_\mathfrak{z}} \rho_{\Psi_\mathfrak{z}}(h_\alpha)}$$

Let $q_\mathfrak{u} : \mathfrak{t}^\star \longrightarrow \mathfrak{u}^\star$ denote restriction map. Owing to our choices, the multiset, $q_\mathfrak{u}(\Psi_K - \Phi_\mathfrak{z}) - \Psi_L$ is contained in a strict cone in $\mathfrak{u}^\star$. Thus, partition function p_L with respect the multiset $q_\mathfrak{u}(\Psi_K - \Phi_\mathfrak{z}) - \Psi_L$ is defined. After we choose representatives of minimal length on each coset of W_L/W_K, the formula due to Heckman-Kostant [4] is

$$c(\mu, \nu) = \sum_{w \in W_L/W_K} \epsilon(w)\varpi(w\mu)p_L(q_\mathfrak{u}(w\mu) - \nu - (q_\mathfrak{u}(\rho_G) - \rho_H)),$$

[1] Unexplained notation is as in [13].

$$\tau_{\mu_{|_L}} = \sum_{\nu\,\text{dominant w.r.t } \Psi_L} c(\mu,\nu)\phi_\nu.$$

Here, ϕ_ν denotes the irreducible representation of L of infinitesimal character ν.

Whenever G, π_λ, H are chosen so that $\pi_{\lambda_{|_H}}$ is admissible, T. Kobayashi [8] has shown

$$\pi_{\lambda_{|_L}} \text{ as an admissible representation.}$$

In our case, this follows from a result proved by Harish-Chandra: an irreducible representation of L is $L-$type of finitely many square integrable representations for H. Therefore, we may express

$$\pi_{\lambda_{|_L}} = \sum n(\lambda,\nu)\phi_\nu$$

$$\pi_{\lambda_{|_H}} = \sum m(\lambda,\mu)\sigma_\mu$$

Here, $n(\lambda,\nu), m(\lambda,\mu)$ are nonnegative integers. In section 4, Theorem 6, we show a way to obtain $m(\lambda,\mu)$ from $n(\lambda,\nu)$. Another way is: we apply the formula of Blattner to compute multiplicities for $\pi_{\lambda_{|_K}}$ and the formula of Heckman-Kostant to the irreducible $K-$submodules of $\pi_{\lambda_{|_K}}$. Thus, we obtain a formula for $n(\lambda,\nu)$. In principle, this final formula involves two Weyl groups and two partition functions. In section 2 we introduce condition (C) for Ψ, L. Under this condition we show a formula for $m(\lambda,\mu)$ (resp. $n(\lambda,\mu)$) which involves one partition function and the group W_K.

For (G, H) a symmetric pair and Ψ satisfies certain restrictions, Gross-Wallach [3] obtained a formula for $m(\lambda,\mu)$ which involves two partition functions and two Weyl groups. For a scalar type holomorphic discrete series and (G, H) symmetric pair, Kobayashi [7] obtained a completely explicit formula for $m(\lambda,\mu)$ which resembles the formula of Schmid [12] for the highest weight of a $K-$type of a holomorphic scalar-type discrete series.
For (G, H) a symmetric pair we obtain the equivalence:

$$\pi_{\lambda_{|_H}} \text{ is admissible if and only if condition (C) holds for } \Psi, L.$$

This statement yields a result of a Ph.D. student in Córdoba, S. Simondi. Namely

For a symmetric pair (G, H), H different from K and K a simple Lie group, a square integrable irreducible representation of G restricted to H is not admissible.

However, for $G = SO(2p,1)$ and L a subgroup of $SO(2p)$ which acts transitively on the unit sphere of $\mathbb{R}^{2p}$, the subquotient theorem of Harish Chandra together with Mackey's restriction theorem yields that any irreducible admissible representation of G is an admissible representation of L. In Proposition 2 we obtain: for a proper subgroup L of $SO(2p)$, condition (C) does not hold for Ψ. Another example of admissible restriction to L and in which condition (C) is not satisfied is $G = SU(n,1)$, π_λ a holomorphic discrete series, L is a proper subgroup of $U(n)$ so that the representation of L in the polynomial algebra of $\mathbb{C}^n$ is admissible. This follows from [6], Example 4.6. Certainly, there are many more examples of the previous type.

The problem of understanding branching laws has received much attention, for an update on the subject we refer to T. Kobayashi [11] and references therein.

This note is not in a final version, the proofs will appear elsewhere. A preliminary version can be found in http://www.famaf.unc.edu.ar/ vargas/proper maps and multiplicities.pdf.

The author would like to thank, T. Kobayashi and the organizers of the Congress at Atami for their generous invitation to participate. We are indebted to D. Vogan for crucial help on Proposition 6.

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2. Condition C

In this section we state a condition which yields a formula for multiplicity function $m(\lambda,\mu)$ by means of one partition function and one Weyl group. Notation $G, H, K, L, T, U, q_{\mathfrak{u}}$ is as before, Ψ is a system of positive roots in $\Phi(\mathfrak{g},\mathfrak{t})$,

Definition 1. *Condition (C) holds for Ψ and L if there exists a system of positive roots Δ_1 in $\Phi(\mathfrak{k},\mathfrak{t})$ satisfying the following two conditions*

$$q_{\mathfrak{u}}(\Delta_1 - \Phi_{\mathfrak{z}}) \text{ is strict,}$$

for each w in the compact Weyl group of G, the multiset

$$q_{\mathfrak{u}}(w\Psi_n) \cup q_{\mathfrak{u}}(\Delta_1 - \Phi_{\mathfrak{z}}) - \Phi(\mathfrak{l},\mathfrak{u})$$

is strict.

Theorem 1. *Fix λ regular dominant with respect to Ψ. Assume condition (C) holds for Ψ and L, then π_λ restricted to H (resp. to L) is admissible.*

Example: We assume the center of $\mathfrak{k}$, $\mathfrak{z}_K$, is non trivial. Then, condition (C) for Ψ and Z_K holds if and only if Ψ is a holomorphic system. Thus, if Z_K is contained in L and Ψ is holomorphic, then condition (C) holds for Ψ, L.

Condition (C) holds for Ψ, L and $\mathfrak{z}_K$ non trivial do not imply the center of $\mathfrak{k}$ is contained in L as the following examples show:
• We fix Ψ holomorphic and let ϖ_β denote the fundamental weight associated to the noncompact simple root in Ψ. Thus ϖ_β spans $\imath\mathfrak{z}_K$. Choose $Y \in \mathfrak{t}$ orthogonal to the center of $\mathfrak{k}$ in such a way $\imath\varpi_\beta + Y$ is the infinitesimal generator of a one dimensional torus T_Y and every root in Ψ takes a positive value on $(J + \imath Y)$. Then, condition (C) holds for T_Y.

• Let G be locally isomorphic to $SO(2p, 2)$. Let $\epsilon_1, \dots, \epsilon_p, \delta$ be a basis of $\imath\mathfrak{t}^\star$ so that compact positive roots are $\Delta := \{\epsilon_j \pm \epsilon_k, j < k\}$ and noncompact roots are $\pm\epsilon_j \pm \delta$. Consider the systems of positive roots $\Psi_a := \Delta \cup \{\epsilon_j \pm \delta, j \leq a, \delta \pm \epsilon_r, a < r\}$. It readily follows condition (C) holds for $\Psi_p, SO(2p)$. Thus, π_λ is an admissible restriction of $SO(2p)$.

• Let K_s denote the semisimple factor of K. Then, condition (C) holds for Ψ, K_s if and only if $\mathbb{R}^+\Psi_n \cap \imath\mathfrak{z}_K^\star = \{0\}$.

• Condition (C) holds for Ψ, L implies condition (C) is true for Ψ and any subgroup in between L and K.

The following technical result is useful in order to write multiplicity formulae in terms of partition functions.

As before, Ψ system of positive roots in $\Phi(\mathfrak{g}, \mathfrak{t})$ and we consider Δ_1, Δ_2 systems of positive roots for $\Phi(\mathfrak{k}, \mathfrak{t})$. Assume, each $q_\mathfrak{u}(\Delta_j - \Phi_\mathfrak{z})$ is strict multiset and the center of $\mathfrak{k}$ is a subset of $\mathfrak{l}$

Proposition 1. *If condition (C) holds for Ψ, L by means of Δ_1, then it holds for Ψ, L by means of Δ_2.*

Example: G is a noncompact real form for G_2.
$L = SU_2$(short compact root), Ψ is the system of positive roots whose short simple is a noncompact root and the long simple is compact, then condition (C) holds for Ψ, L. Hence, π_λ restricted to L is admissible.
$L = SU_2(long\, root)$, Ψ is the system where short simple is compact and long simple noncompact, then condition (C) holds. Thus, π_λ restricted to L is admissible

In the next paragraph we analyze the constraints on K when we assume Condition (C) holds for certain subgroup.
To Ψ we associate and ideal $\mathfrak{k}_1(\Psi)$ of $\mathfrak{k}$ as follows:

$$\mathfrak{k}_1(\Psi)_\mathbb{C} := \text{ideal spanned by } \sum_{\alpha,\beta\in\Psi_n} \mathbb{C}X_{\alpha+\beta}$$

Proposition 2. *If condition (C) holds for Ψ and L. Then, $K_1(\Psi) \subseteq L$. When $K_1(\Psi)\, Z_K$ is included in L, then condition (C) holds for Ψ, L.*

The converse to the first statement is not true as shows $G = SO(2p, 2)$, $L = SO(2p), \Psi_a, 1 \leq a < p$ and $3 \leq p$.
However, whenever Ψ is small [3], then the semisimple factor of the subgroup K_1 defined by Gross and Wallach agrees with $K_1(\Psi)$, and condition (C) holds for $\Psi, K_1(\Psi)$.
We now assume Ψ is a non holomorphic system. Then, $\mathbb{R}^+\Psi_n \cap \imath\mathfrak{z}_K^\star = 0$ if and only if condition (C) holds for Ψ and $K_1(\Psi)$.

Let $K_2(\Psi)$ denote the semisimple factor of the complementary ideal to $K_1(\Psi)$. After a case by case computation we show:
π_λ restricted to $K_2(\Psi)$ is admissible if and only if G is locally isomorphic to $Sp(p, 1)$ and Ψ is so that $K_1(\Psi) = Sp(1), K_2 = Sp(q)$. Actually, $\Psi = \{\delta > \epsilon_1 > \cdots > \epsilon_p\}$.

Next we relate condition (C) to some of the Theorems of T. Kobayashi on admissible restriction.

Let G, K, H be as usual. For the following paragraph, we assume (G, H) is a symmetric pair. Let σ denote the involution determined by the symmetric pair (G, H). Thus, σ stabilizes K and (K, L, σ) is a symmetric pair. In order to state one of the equivalent statements of T. Kobayashi, we choose a compact Cartan subgroup $B \subset K$ as follows:

$\sigma(\mathfrak{b}) = \mathfrak{b}$,

$\mathfrak{b}_- := \{X \in \mathfrak{b} : \sigma X = -X\}$ is a maximal Cartan subspace for the symmetric pair (K, L).
Henceforth, we fix a system of positive roots $\Delta \subset \Phi(\mathfrak{k}, \mathfrak{b})$ and in such a way $\{\alpha_{|\mathfrak{b}_-} \neq 0 : \alpha \in \Delta\}$ is a system of positive roots for $\Phi(\mathfrak{k}, \mathfrak{b}_-)$. For a system of positive roots $\tilde{\Psi}$ in $\Phi(\mathfrak{g}, \mathfrak{b})$ which contains Δ,
we define,

$$\mathbb{R}^+\tilde{\Psi}_n := \sum_{\beta \in \tilde{\Psi}_n} \mathbb{R}^+\beta = \{\sum_{\beta \in \tilde{\Psi}_n} n_\beta \beta : n_\beta \geq 0\} \subset i\mathfrak{b}^*.$$

Theorem 2 (T. Kobayashi). *Assume λ is dominant with respect to $\tilde{\Psi}$. Then, π_λ restricted to H is admissible if and only if $\mathbb{R}^+\tilde{\Psi}_n \cap i\mathfrak{b}_-^* = \{0\}$.*

Proposition 3. *Assume $\mathbb{R}^+\tilde{\Psi}_n \cap i\mathfrak{b}_-^* = \{0\}$. Then $K_1(\tilde{\Psi})$ is a subgroup of L.*

The example $(SO(2p, 2), SO(2p, 1)), \Psi_p$ shows the hypothesis of the lemma does not implies that the center of K is contained in L.

Corollary 1. *(G,H) is a symmetric pair with H a noncompact group, such that there exists a non holomorphic discrete series of G with admissible restriction to H, then K is not a simple Lie group.*

Condition (C) is stated by means of compact Cartan subgroup which is maximally compact for the symmetric pair (K, L) whereas the condition of admissibility due to Kobayashi is expressed through a maximally split Cartan subgroup for (K, L). After we relate both compact Cartan subgroups via a Cayley transform, we show

Proposition 4. *Assume (G, H) is a symmetric pair. Then, condition (C) for Ψ and L holds if and only if Kobayashi's condition in Theorem 2 holds for the image under the Cayley transform of Ψ.*

Corollary 2. *For (G,H) symmetric pair and Ψ non holomorphic, condition (C) for Ψ and L implies condition (C) for Ψ and $K_1(\Psi)$. Thus, π_λ has an admissible restriction to $K_1(\Psi)$.*

Example: $\mathfrak{g} = \mathfrak{so}(4, 2q)$, $q \geq 3$, Let $\epsilon_1, \epsilon_2, \delta_1, \cdots, \delta_q$ be an ordered basis of $i\mathfrak{t}^\star$ so that

$$\Delta = \{\epsilon_1 \pm \epsilon_2, \delta_r \pm \delta_s,\ 1 \leq r < s \leq q\}, \Phi_n = \{\pm\epsilon_i \pm \delta_j\}.$$

We define

$$\Psi_{\pm a} = \{\delta_1 > \cdots > \delta_a > \epsilon_1 > \pm\epsilon_2 > \delta_{a+1} > \cdots > \delta_q\}$$

Then $\Psi_{\pm 0}, \Psi_{\pm q}$ are small. No other system of positive roots is small.
$\mathfrak{k}_1(\Psi_{\pm 0}) = \mathfrak{su}_2(\epsilon_1 \pm \epsilon_2)$, $\mathfrak{k}_1(\Psi_{\pm q}) = \mathfrak{so}(2q)$, for $1 \leq a < q$

$$\mathfrak{k}_1(\Psi_{\pm a}) = \mathfrak{su}_2(\epsilon_1 \pm \epsilon_2) \oplus \mathfrak{so}(2q).$$

For any other system of positive roots $\mathfrak{k}_1(\Psi) = \mathfrak{k}$. Every ideal of $\mathfrak{k}$ is equal to a $\mathfrak{k}_1(\Psi)$ but $\{0\}, \mathfrak{so}(4)$.
It follows from Proposition 4 and the classification of symmetric pairs $(SO(4, 2q), H)$ that π_λ is an admissible representation of H if and only if (H, Ψ) is one of: $\Psi = \Psi_{\pm 0}$, $\mathfrak{h} = \mathfrak{u}(\epsilon_1 \pm \epsilon_2) + \mathfrak{u}(q)$; $\Psi = \Psi_{\pm q}$ and $\mathfrak{h} = \mathfrak{so}(r) + \mathfrak{so}(s, 2q)$ where $r + s = 4$; $\Psi = \Psi_{\pm 0}$, $\mathfrak{h} = \mathfrak{so}(4, s) + \mathfrak{so}(r)$, here $r + s = 2q$;

3. Multiplicity formulae

Let $G, H, \Psi, K, L, U, q_\mathfrak{u}$ as before. We have expressed the formula of Heckman-Kostant or the formula of Blattner in terms of the infinitesimal character of the representations rather than their highest weights. We can also hide the presence of the $\rho' s$. This can be achieved by means of twisted partition functions.
For a finite multiset $\Sigma = \{\sigma_1, \cdots, \sigma_r\}$ of $i\mathfrak{u}^\star$, let Q_Σ denote the twisted partition function defined by

$$Q_\Sigma(\nu) = card\{(n_j)_{j=1}^r : \nu = \sum_j (\frac{1}{2} + n_j)\sigma_j\}.$$

Next, we assume π_λ has an admissible restriction to H. Let $m(\lambda,\mu)$ denote multiplicity of σ_μ in $\pi_{\lambda_{|_H}}$. Let ϖ denote the volume form in the formula of Heckman-Kostant for the data K, L, Ψ_K. For short, we write $\Phi_{\mathfrak{s}}$ for the root system of a semisimple Lie algebra $\mathfrak{s}$. Theorem 1 yields $\pi_{\lambda_{|_H}}$ is admissible under the hypothesis of condition (C) for Ψ, L. We have

Theorem 3. *Assume condition (C) holds for Ψ, L. Then,*

$$m(\lambda,\mu) = \sum_{w\in W} \epsilon(w)\,\varpi(w\lambda)\,Q_{q_{\mathfrak{u}}(-w\Psi_n)\cup q_{\mathfrak{u}}(\Psi_K-\Phi_{\mathfrak{z}})-\Phi_{\mathfrak{h}}}(q_{\mathfrak{u}}(w\lambda)-\mu).$$

We have the decomposition in ideals

$$\mathfrak{k} = \mathfrak{k}_1(\Psi) + \mathfrak{k}_2(\Psi) + \mathfrak{z}_K$$

For short, we write $W_j = W_{K_j(\Psi)}$. Thus, $W_K = W_1 \times W_2$. For the particular case $K_1(\Psi)T \subseteq L \subset H$ the formula of Theorem 3 has a simpler expression. In fact, let Q denote the twisted partition function associated to $\Psi - \Phi(\mathfrak{h},\mathfrak{t})$. Then, the formula of Theorem 3 reads

$$m(\lambda,\mu) = \sum_{w\in W_1,\ s\in W_2} \epsilon(ws)Q(w\mu - s\lambda).$$

4. Partition functions via discrete Heaviside distributions

For $\gamma \in i\mathfrak{u}^\star$ let δ_γ denote Dirac delta function attached to γ. We define

$$y_\gamma = \sum_{n\geq 0} \delta_{\frac{\gamma}{2}+n\gamma} = \delta_{\frac{\gamma}{2}} + \delta_{\frac{\gamma}{2}+\gamma} + \delta_{\frac{\gamma}{2}+2\gamma} + \cdots$$

Fix $S = \{\gamma_1, \cdots, \gamma_q\}$ strict multiset,
Fact: There exists the convolution product

$$y_{\gamma_1} \star \cdots \star y_{\gamma_q}$$

Fact:

$$\sum_{\nu\in i\mathfrak{u}^\star} Q_S(\nu)\delta_\nu = y_{\gamma_1} \star \cdots \star y_{\gamma_q}.$$

For the next statement, to begin with, we assume Ψ and L are so that π_λ restricted to L is admissible. Henceforth, P_U denotes the weight lattice for U. For each irreducible representation of L of infinitesimal character ν, let $n(\lambda,\nu)$ denote its multiplicity in π_λ restricted to L. It is known $n(\lambda,\nu)$ is of polynomial growth in ν, hence

$$\sum_{\nu\in P_U} n(\lambda,\nu)\delta_\nu$$

converges in distribution sense, we extend $n(\lambda,\nu)$, as a function of ν, to the weight lattice of U and to be a skew symmetric function with respect to the action of group W_L. We have

Theorem 4. *Assume condition (C) holds for Ψ and L. Then,*

$$\sum_{\nu\in P_U} n(\lambda,\nu)\delta_\nu$$
$$= \sum_{w\in W_K} \epsilon(w)\,\varpi(w\lambda)$$
$$\delta_{q_\mathfrak{u}(w\lambda)} \star \bigstar_{\beta\in q_\mathfrak{u}(w\Psi_n)\cup(-1)q_\mathfrak{u}(\Psi_K-\Phi_\mathfrak{z})-\Phi_\mathfrak{l}}\; y_\beta$$

A similar formula holds for the distribution attached to multiplicities of the restriction to H. With respect to a proof of the formula we proceed as follows: to begin with, we consider the case L contains T, next we analyze when L is normalized by T (here appears the ϖ factor). Thus, the formula is true for $K_1(\Psi)Z_KU$. To follow we show: if L_1 is a same rank subgroup L and the formula holds for L_1 then it holds for L. Finally, we deduce a formula to compute $H-$multiplicities from $L-$multiplicities.

Next, we precise a way to compute $L-$multiplicities from L_1-multiplicities. Let U a common maximal torus of L_1 and L. Multiplicities are encoded by the distributions

$$\sum_{\nu\in P_U} n_1(\lambda,\nu)\delta_\nu, \qquad \sum_{\nu\in P_U} n(\lambda,\nu)\delta_\nu.$$

For each root α, we define the linear operator

$$d_\alpha(f) = (\delta_{-\alpha/2} - \delta_{\alpha/2})\star f,\;\; f\in\mathcal{D}'(i\mathfrak{u}^\star).$$

Theorem 5. *Assume π_λ restricted to $L_1\subset L$ is admissible, and $\Psi_{L_1}\subset\Psi_L$, then*

$$\bigstar_{\alpha\in\Psi_L-\Psi_{L_1}} d_\alpha\Big(\sum_{\nu\in P_U} n_1(\lambda,\nu)\delta_\nu\Big) = \sum_{\nu\in P_U} n(\lambda,\nu)\delta_\nu$$

Similarly, we have a formula to deduce $H-$multiplicity $m(\lambda,\mu)$ from $L-$multiplicity $n(\lambda,\nu)$.

Theorem 6. *Assume π_λ restricted to L is admissible, let Ψ_H denote a system of positive roots for $\Phi_\mathfrak{h}$ defined by $q_\mathfrak{u}(\lambda)$, then*

$$\prod_{\alpha\in(\Psi_H)_n} d_\alpha\Big[\sum_{\nu\in P_U} n(\lambda,\nu)\,\delta_\nu\Big] = \sum_{\mu\in P_U} m(\lambda,\mu)\delta_\mu$$

5. Push forward of Liouville Measure

We now study multiplicity functions by means of the continuous methods introduced by Heckman [4] and Duflo-Heckamn-Vergne [2]. Let $G, H, K, L, T, U, \lambda, \Psi, q_{\mathfrak{u}}$ have the same meaning as in the previous sections. As usual, we extend λ to be a linear functional on $\mathfrak{g}_{\mathbb{C}}$ which vanishes on each root subspace. For the coadjoint orbit $\Omega := G \cdot \lambda$ let $\beta_{G\cdot\lambda}$ denote the Liouville measure computed with respect to the symplectic form on Ω arising from the Killing form on $\mathfrak{g}$. The Liouville measure of the coadjoint orbit $G \cdot \lambda$ evaluated on a test function φ is given by

$$\beta_{G\cdot\lambda}(\varphi) := A_G(\varphi)(\lambda) = c_G \prod_{\alpha\in\Psi} (\alpha, \lambda) \int_G \varphi(Ad(u) \cdot \nu) du.$$

Here, du is an appropiate choice of Haar measure on G. For the constants involved and proofs of the equalities we refer to [1], Chapter VII. From now on,

we assume condition (C) holds for Ψ, L.

Whence, the restriction map $q_{\mathfrak{h}} : \Omega \to \imath\mathfrak{h}^\star$ as well as $q_{\mathfrak{l}} : \Omega \to \imath\mathfrak{l}^\star$ is proper. In Heckman [4] we find a proof each of the maps $q_{\mathfrak{l}}, q_{\mathfrak{h}}$ is a submersion on an open dense set of Ω whose complement has Lebesguian measure zero. Therefore, there exists a function

$$M_L : \imath\mathfrak{u}^\star \to \mathbb{C}$$

so that the push-forward, $(q_{\mathfrak{l}})_\star(\beta_\Omega)$ of the measure β_Ω by $q_{\mathfrak{l}}$ is given by

$$(q_{\mathfrak{l}})_\star(\beta_\Omega)(\varphi) := (q_{\mathfrak{l}})_\star(\beta_{G\cdot\lambda})(\varphi) = \int_{\imath\mathfrak{u}^\star} M_L(\mu) A_L(\varphi)(\mu) d\mu.$$

Here, $d\mu$ is a convenient Lebesgue measure on $\imath\mathfrak{u}^\star$ and φ a test function on $\imath\mathfrak{u}^\star$. Moreover, M_L is a smooth function on the set of regular values of $q_{\mathfrak{l}}$ and is a W_L-skew-invariant function. For each root $\beta \in \imath\mathfrak{u}^\star$ let Y_β the Heaviside generalized function on $\imath\mathfrak{u}^\star$ associated to β. Thus,

$$Y_\beta(\varphi) = \int_0^\infty \varphi(t\beta) dt.$$

Owing to our hypothesis, for each $w \in W$, the convolution of the Heaviside functions

$$w(Y_n^+) := \bigstar_{\beta\in\Psi_n} Y_{q_{\mathfrak{u}}(w\beta)}, \qquad Y_{\mathfrak{k},\mathfrak{l}}^+ = \bigstar_{\alpha\in q_{\mathfrak{u}}(\Psi_{\mathfrak{k}}-\Psi_{\mathfrak{z}})-\Psi_{\mathfrak{l}}} Y_{-\alpha}.$$

are defined, as well as the product $w(Y_n^+) \star Y_{\mathfrak{k},\mathfrak{l}}$. The next Proposition is a generalization of Theorems shown in [2], [4]. We have,

Proposition 5.

$$M_L = \sum_{w \in W_K} \epsilon(w)\ \varpi(w\lambda)\ \delta_{q_{\mathfrak{u}}(w\lambda)} \star w(Y_n^+)\ \star\ Y_{\mathfrak{k},\mathfrak{l}}^+.$$

For a proof of Proposition 5 we apply a result which is true whenever $q_{\mathfrak{l}}$ is a proper map. The result relates discrete Heaviside measure with continuous Heaviside measure. For each element α of $\imath\mathfrak{u}^\star$ we define

$$t_\alpha(\varphi) := \int_{-\frac{1}{2}}^{\frac{1}{2}} \varphi(t\alpha)dt,$$

It readily follows the identity

$$t_\alpha * y_\alpha = Y_\alpha.$$

Let

$$\Psi_{\mathfrak{g}/\mathfrak{l}} = q_{\mathfrak{u}}(\Psi_n) \cup q_{\mathfrak{u}}(\Psi_K - \Phi_{\mathfrak{z}}) - \Phi_{\mathfrak{l}}.$$

Since each t_α is a compactly supported distribution, the convolution product $r_{\mathfrak{g}/\mathfrak{l}}$ of the family $t_\alpha, \alpha \in \Psi_{\mathfrak{g}/\mathfrak{l}}$ is well defined. We have,

Proposition 6.

$$r_{\mathfrak{g}/\mathfrak{l}} \star (\sum_{\mu \in P_U} n(\lambda, \mu) = M_L.$$

In a work in progress we have found a way to compute the distribution that encodes the multiplicity from the generalized function M_L.

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