DUNKL OPERATORS, CHEREDNIK OPERATORS, AND RADIAL PARTS OF NON-SYMMETRIC ELEMENTS

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1. Introduction

Let G be a non-compact connected Lie group with a semisimple Lie algebra $\mathfrak{g}$. If $\mathfrak{g} = \mathfrak{n} + \mathfrak{a} + \mathfrak{k}$ is a local Iwasawa decomposition of $\mathfrak{g}$, then the analytic subgroups N, A, K, which respectively correspond to $\mathfrak{n}, \mathfrak{a}, \mathfrak{k}$, give a global Iwasawa decomposition $G = NAK$. (In this note K is not assumed to be compact.) Let θ be the Cartan involution for $\mathfrak{k}$ and put $\bar{\mathfrak{n}} = \theta\mathfrak{n}$. The *Harish-Chandra homomorphism* γ is the map from the universal enveloping algebra $U(\mathfrak{g}_{\mathbb{C}})$ of the complexification $\mathfrak{g}_{\mathbb{C}}$ of $\mathfrak{g}$ into the symmetric algebra $S(\mathfrak{a}_{\mathbb{C}})$ of $\mathfrak{a}_{\mathbb{C}}$ defined by

$$\begin{aligned}\gamma : U(\mathfrak{g}_{\mathbb{C}}) &= \bigl(\bar{\mathfrak{n}}_{\mathbb{C}} U(\mathfrak{g}_{\mathbb{C}}) + U(\mathfrak{g}_{\mathbb{C}})\mathfrak{k}_{\mathbb{C}}\bigr) \oplus U(\mathfrak{a}_{\mathbb{C}}) \\ &\xrightarrow{\text{projection}} U(\mathfrak{a}_{\mathbb{C}}) = S(\mathfrak{a}_{\mathbb{C}}) \xrightarrow{\rho\text{-shift}} S(\mathfrak{a}_{\mathbb{C}}),\end{aligned}$$

where $S(\mathfrak{a}_{\mathbb{C}})$ is identified with the algebra of holomorphic polynomial functions on the dual space $\mathfrak{a}_{\mathbb{C}}^*$ of $\mathfrak{a}_{\mathbb{C}}$ and $\rho := \frac{1}{2}\operatorname{Trace}(\operatorname{ad}_{\mathfrak{n}}|_{\mathfrak{a}}) \in \mathfrak{a}_{\mathbb{C}}^*$. We have two different G-action $\ell(\cdot)$, $r(\cdot)$ on $C^\infty(G)$ defined by

$$\ell(g)f(x) = f(g^{-1}x), \qquad r(g)f(x) = f(xg)$$

for $f \in C^\infty(G)$ and $g \in G$. Thus, on the $\ell(G)$-module $C^\infty(G/K)$, $U(\mathfrak{g}_{\mathbb{C}})^K$ (the algebra of K-invariants in $U(\mathfrak{g}_{\mathbb{C}})$) naturally acts by $r(\cdot)$.

Suppose $\lambda \in \mathfrak{a}^*$. A unique function $\phi_\lambda \in C^\infty(G/K)$ satisfying

$$\begin{cases} r(\Delta)\phi_\lambda = \gamma(\Delta)(\lambda)\phi_\lambda \quad \text{for } \Delta \in U(\mathfrak{g}_{\mathbb{C}})^K, \\ \ell(K)\text{-invariant}, \\ \phi_\lambda(eK) = 1 \quad (e \text{ denotes the origin of } G) \end{cases} \tag{1.1}$$

is called a *spherical function*. Let $\gamma_0 : C^\infty(G/K) \to C^\infty(A)$ be the natural restriction map and W the Weyl group for $(\mathfrak{g}, \mathfrak{a})$. Then the second property in (1.1) implies $\gamma_0(\phi_\lambda) \in C^\infty(A)^W$ and ϕ_λ is a unique K-invariant extension of $\gamma_0(\phi_\lambda)$. By [HO] Heckman and Opdam started to develop the theory of the *systems of hypergeometric differential equations*, which are modifications of a system of differential equations for $\gamma_0(\phi_\lambda)$. Although there was at first some difficulty in showing the existence of such modifications, currently we can construct modified systems in a simple way using *Cherednik operator*s introduced by [Ch]. In this context, the most important fact is that a Cherednik operator $\mathscr{T} : S(\mathfrak{a}_{\mathbb{C}}) \to \operatorname{End}_{\mathbb{C}} C^\infty(A)$ with a special parameter (Definition 6.1) satisfies, for $\Delta \in U(\mathfrak{g}_{\mathbb{C}})^K$ and $f \in C^\infty(G/K)^{\ell(K)}$, the *radial part formula*

$$\gamma_0\bigl(r(\Delta)f\bigr) = \mathscr{T}\bigl(\gamma(\Delta)\bigr)\gamma_0(f), \tag{1.2}$$

or equivalently,

$$\gamma_0\bigl(\ell(\Delta)f\bigr) = \mathscr{T}\bigl(\gamma(\Delta)^{\mathrm{opp}}\bigr)\gamma_0(f). \tag{1.3}$$

Here $\cdot^{\mathrm{opp}} : S(\mathfrak{a}_{\mathbb{C}}) \to S(\mathfrak{a}_{\mathbb{C}})$ is the algebra automorphism defined by $H^{\mathrm{opp}} = -H$ for $H \in \mathfrak{a}_{\mathbb{C}}$.

The main purpose of this note is to show that (1.2) and (1.3) hold for more general combinations of Δ and f. That is, (1.2) holds for any $\Delta \in U(\mathfrak{g}_{\mathbb{C}})^K$ and any K-finite $f \in C^\infty(G/K)$ such that all K-types in $U(\mathfrak{k}_{\mathbb{C}})f$ are *single-petaled* (Definition 2.1). Also, (1.3) holds for any $f \in C^\infty(G/K)^{\ell(K)}$ and any $\Delta \in U(\mathfrak{g}_{\mathbb{C}})$ such that all K-types in $\mathrm{ad}(U(\mathfrak{k}_{\mathbb{C}}))\Delta$ are single-petaled. If G/K is of Hermitian type, these results yield further generalizations to the case of homogeneous line bundles on G/K, which we shall study in §7.

In [Op2] Opdam studies the following system of differential-difference equations for $\varphi \in C^\infty(A)$ (the *Cherednik system*):

$$\mathscr{T}(\Delta)\varphi = \Delta(\lambda)\varphi \qquad \forall \Delta \in S(\mathfrak{a}_{\mathbb{C}})^W. \tag{1.4}$$

In view of (1.2) $\gamma_0(\phi_\lambda)$ is a W-invariant solution of (1.4). But by the above result and the generalized Chevalley restriction theorem (see §2), other solutions of (1.4) can also be the restricted parts of some functions in

$$\mathscr{A}(G/K;\lambda) := \bigl\{ f \in C^\infty(G/K);\, r(\Delta)f = \gamma(\Delta)(\lambda)f \quad \text{for } \Delta \in U(\mathfrak{g}_{\mathbb{C}})^K \bigr\}.$$

Now, let $\mathfrak{p}$ the -1-eigenspace of θ in $\mathfrak{g}$. For the *Cartan motion group* $K \ltimes \mathfrak{p}$ and the *rational Dunkl operator*s introduced by [Dun] we have similar results. Since

$$\mathfrak{p} \times K \to G\,;\ (X,k) \mapsto \exp X \cdot k$$

is a diffeomorphism onto, we have a K-module isomorphism

$$C^\infty(G/K) \xrightarrow{\sim} C^\infty(\mathfrak{p})\,; \quad f \longmapsto f(\exp\cdot). \tag{1.5}$$

Using it with a W-module isomorphism

$$C^\infty(A) \xrightarrow{\sim} C^\infty(\mathfrak{a})\,; \quad \varphi \longmapsto \varphi(\exp\cdot), \tag{1.6}$$

we identify γ_0 with the natural restriction map $C^\infty(\mathfrak{p}) \to C^\infty(\mathfrak{a})$. Via the Killing form $B(\cdot,\cdot)$ of $\mathfrak{g}$, $S(\mathfrak{p}_{\mathbb{C}})$ is identified with the algebra $\mathscr{P}(\mathfrak{p})$ of polynomial functions on $\mathfrak{p}$, and $S(\mathfrak{a}_{\mathbb{C}})$ with $\mathscr{P}(\mathfrak{a})$. Thus γ_0 induces a map $S(\mathfrak{p}_{\mathbb{C}}) \to S(\mathfrak{a}_{\mathbb{C}})$, which is also denoted by γ_0. For $X \in \mathfrak{p}$ let $\partial(X)$ denote the X-directional derivative operator acting on $C^\infty(\mathfrak{p})$ and extend $\partial : \mathfrak{p} \to \mathrm{End}_{\mathbb{C}}\, C^\infty(\mathfrak{p})$ to a unique algebra homomorphism of $S(\mathfrak{p}_{\mathbb{C}})$ into $\mathrm{End}_{\mathbb{C}}\, C^\infty(\mathfrak{p})$. In [Je] de Jeu gives a simple proof for the fact that a Dunkl operator $\mathscr{D} : S(\mathfrak{a}_{\mathbb{C}}) \to \mathrm{End}_{\mathbb{C}}\, C^\infty(\mathfrak{a})$ with a special parameter (Definition 3.1) satisfies the radial part formula

$$\gamma_0\bigl(\partial(\Delta)f\bigr) = \mathscr{D}\bigl(\gamma_0(\Delta)\bigr)\gamma_0(f) \tag{1.7}$$

for $\Delta \in S(\mathfrak{p}_{\mathbb{C}})^K$ and $f \in C^\infty(\mathfrak{p})^K$. In §3 we shall show that (1.7) holds for more general combinations of Δ and f (Theorems 3.4, 3.5). Because de Jeu's method is applicable to our case without any change and it is so simple, we shall discuss this case in §3 before the Riemannian symmetric case stated above.

2. The Chevalley restriction theorem

In this section we review a generalization of the Chevalley restriction theorem given in [O]. Let Σ denote the restricted root system for $(\mathfrak{g},\mathfrak{a})$, and Σ^+ the positive system of Σ corresponding to $\mathfrak{n}$. For each $\alpha \in \Sigma$ let $\mathfrak{g}_\alpha$ denote the root space for α and fix $E_\alpha \in \mathfrak{g}_\alpha$ so that $-B(E_\alpha, \theta E_\alpha) = \frac{2}{B(H_\alpha,H_\alpha)}$. Here for $\mu \in \mathfrak{a}_{\mathbb{C}}^*$ we define H_μ to be a unique element in $\mathfrak{a}_{\mathbb{C}}$ such that $\mu(H) = B(H_\mu, H)$ for $H \in \mathfrak{a}$. Let M be the centralizer of A in K. Putting $Z_\alpha = \sqrt{-1}(E_\alpha + \theta E_\alpha) \in \mathfrak{k}_{\mathbb{C}}$, let us introduce some classes of K-types.

Definition 2.1. (i) In this note by the terminology "K-type" we mean a finite-dimensional irreducible representation of K over $\mathbb{C}$ or its equivalence class. The set of K-types is denoted by $\widehat{K}$.
(ii) $(\sigma, V) \in \widehat{K}$ is called *M-spherical* if the M-fixed part V^M of V is not $\{0\}$. The set of M-spherical K-types is denoted by $\widehat{K}_M$.
(iii) For $(\sigma, V) \in \widehat{K}_M$ put

$$V^M_{\text{single}} = \bigl\{v \in V^M;\ \sigma(Z_\alpha)(\sigma(Z_\alpha)^2 - 4)v = 0 \quad \text{for any } \alpha \in \Sigma\bigr\}.$$

Then $(\sigma, V) \in \widehat{K}_M$ is called *quasi-single-petaled* if $V^M_{\text{single}} \neq \{0\}$. The set of quasi-single-petaled K-types is denoted by $\widehat{K}_{\text{qsp}}$.
(iv) $(\sigma, V) \in \widehat{K}_M$ is called single-petaled if $V^M_{\text{single}} = V^M$. The set of single-petaled K-types is denoted by $\widehat{K}_{\text{sp}}$.
(v) The set of scalar K-types (namely, one-dimensional representations of K) is denoted by $\widehat{K}_1$. For $\chi \in \widehat{K}_1$ and $(\sigma, V) \in \widehat{K}$, a K-type $(\chi \otimes \sigma, V_\chi)$ is defined by letting $V_\chi = V$ and $(\chi \otimes \sigma)(k) = \chi(k)\sigma(k) \in \operatorname{End}_{\mathbb{C}} V$ for $k \in K$. When a $v \in V$ is considered as an element of V_χ, it is denoted by $v \otimes v_\chi$.

Remark 2.2. (i) $\widehat{K}_{\text{sp}} \subset \widehat{K}_{\text{qsp}} \subset \widehat{K}_M \subset \widehat{K} \supset \widehat{K}_1$.
(ii) The kernel of the adjoint representation $\operatorname{Ad} : G \to \operatorname{End}_{\mathbb{R}} \mathfrak{g}$ is the center $Z(G)$ of G and $\operatorname{Ad}(G)$ equals the adjoint group of $\mathfrak{g}$. Suppose $(\sigma, V) \in \widehat{K}_M$. Since $M \supset Z(G)$, it is easy to see $Z(G)$ acts on V trivially. Hence (σ, V) is essentially a representation of $\operatorname{Ad}_{\mathfrak{g}}(K)$ and unitarizable.
(iii) If $\mathfrak{k}$ is semisimple, then $\widehat{K}_1 = \{\text{triv}\}$.

For $\chi \in \widehat{K}_1$ put

$$C^\infty(G/K; \chi) := \bigl\{f \in C^\infty(G);\ r(k)f = \chi^{-1}(k)f \quad \text{for any } k \in K\bigr\}. \tag{2.1}$$

It is considered as the space of C^∞-sections of a line bundle on the Riemannian symmetric space G/K, on which G acts by $\ell(\cdot)$. Let us state our generalization of the Chevalley restriction theorem.

Theorem 2.3. *Suppose $\chi \in \widehat{K}_1$ and $(\sigma, V) \in \widehat{K}_M$.*
(i) *The map*

$$\begin{aligned} \Gamma_0^{\chi,\sigma} : \operatorname{Hom}_K(V_\chi, C^\infty(G/K; \chi)) &\longrightarrow \operatorname{Hom}_W(V^M, C^\infty(A))\,; \\ \Phi &\longmapsto \bigl(\varphi : V^M \hookrightarrow V = V_\chi \xrightarrow{\Phi} C^\infty(G/K; \chi) \xrightarrow{\text{restriction}} C^\infty(A)\bigr) \end{aligned}$$

is well-defined. Here we naturally consider V^M and $C^\infty(A)$ as W-modules.
(ii) *$\Gamma_0^{\chi,\sigma}$ is injective.*
(iii) *Define*

$$V^M_{\text{double}} = V^M \cap \sum \bigl\{\sigma(m)\sigma(Z_\alpha)(\sigma(Z_\alpha)^2 - 4)V;\ \alpha \in \Sigma, m \in M\bigr\}.$$

Then $V^M = V^M_{\text{single}} \oplus V^M_{\text{double}}$ is a direct sum decomposition into two W-submodules. A W-submodule $U \subset V^M$ satisfies the condition

$$\operatorname{Im} \Gamma_0^{\chi,\sigma} \supset \bigl\{\varphi \in \operatorname{Hom}_W(V^M, C^\infty(A));\ \operatorname{Ker} \varphi \supset U\bigr\}$$

if and only if $U \supset V^M_{\text{double}}$. *In particular* $\operatorname{Im}\Gamma_0^{\chi,\sigma}$ *contains any* $\varphi \in \operatorname{Hom}_W(V^M, C^\infty(A))$ *such that* $\operatorname{Ker}\varphi \supset V^M_{\text{double}}$.
(iv) $\Gamma_0^{\chi,\sigma}$ *is surjective if and only if* $(\sigma, V) \in \widehat{K}_{\text{sp}}$.

Proof. If $\chi = \text{triv}$ all statements in this case are proved in [O]. From them the general case easily follows. □

Remark 2.4. Suppose $\chi = \text{triv}$. In view of (1.5) and (1.6), $\Gamma_0^\sigma := \Gamma_0^{\text{triv},\sigma}$ is naturally identified with the following map:

$$\begin{aligned} \Gamma_0^\sigma : \operatorname{Hom}_K(V, C^\infty(\mathfrak{p})) &\longrightarrow \operatorname{Hom}_W(V^M, C^\infty(\mathfrak{a}))\,; \\ \Phi &\longmapsto \big(\varphi : V^M \hookrightarrow V \xrightarrow{\Phi} C^\infty(\mathfrak{p}) \xrightarrow{\gamma_0} C^\infty(\mathfrak{a})\big). \end{aligned} \tag{2.2}$$

In fact, [O] originally proved Theorem 2.3 for $\chi = \text{triv}$ in this form.

The following lemma will be used later:

Lemma 2.5. *Suppose* $(\sigma, V) \in \widehat{K}_{\text{qsp}}$ *and let* $p^\sigma : V \to V^M$ *be the orthogonal projection with respect to a* K*-invariant Hermitian inner product of* V. *Suppose* $\alpha \in \Sigma$ *and* $X_\alpha \in \mathfrak{g}_\alpha$.
(i) $\sigma(X_\alpha + \theta X_\alpha)^2 v = |\alpha|^2 B(X_\alpha, \theta X_\alpha)(1 - s_\alpha)v$ *for* $v \in V^M_{\text{single}}$, *where* $|\alpha| = \sqrt{B(H_\alpha, H_\alpha)}$ *and* $s_\alpha \in W$ *is the reflection with respect to* α.
(ii) $p^\sigma\big(\sigma(X_\alpha + \theta X_\alpha)^2 v\big) \in V^M_{\text{double}}$ *for* $v \in V^M_{\text{double}}$.

3. Dunkl operators

Put $R := 2\Sigma$, $R^+ := 2\Sigma^+$, $R_1 := R \setminus 2R$, and $R_1^+ := R_1 \cap R^+$. Then R and R_1 are root systems on which W naturally acts. In [Dun] Dunkl introduces the following operators:

Definition 3.1 (Dunkl operators)**.** Suppose $\mathbf{k} : W\backslash R_1 \to \mathbb{C}$ is a multiplicity function, namely, a $\mathbb{C}$-valued function on R_1 which is constant on each W-orbit. For $\xi \in \mathfrak{a}$ define $\mathscr{D}_{\mathbf{k}}(\xi) \in \operatorname{End}_{\mathbb{C}} C^\infty(\mathfrak{a})$ by

$$\mathscr{D}_{\mathbf{k}}(\xi) = \partial(\xi) + \sum_{\alpha \in R_1^+} \mathbf{k}(\alpha) \frac{\alpha(\xi)}{\alpha}(1 - s_\alpha)$$

where $\partial(\xi)$ is the ξ-directional derivative operator. Moreover, define the multiplicity function $\mathbf{m}_0 : W\backslash R_1 \to \mathbb{C}$ by

$$\mathbf{m}_0(\alpha) = \frac{1}{2}\big(\dim \mathfrak{g}_{\alpha/2} + \dim \mathfrak{g}_\alpha\big). \tag{3.1}$$

If $\mathbf{k} = \mathbf{m}_0$ then we use the brief symbol $\mathscr{D}$ for $\mathscr{D}_{\mathbf{m}_0}$.

Remark 3.2. There is no significant meaning in using R instead of Σ. We do it only for the compatibility with the case of Cherednik operators.

Let $S(\mathfrak{a}_{\mathbb{C}}) \rtimes \mathbb{C}[W]$ be the algebra such that it is, as a linear space, the tensor product $S(\mathfrak{a}_{\mathbb{C}}) \otimes \mathbb{C}[W]$ of $S(\mathfrak{a}_{\mathbb{C}})$ and the group algebra $\mathbb{C}[W]$ of W and such that the multiplication is given by

$$(f_1 \otimes w_1)(f_2 \otimes w_2) = f_1(w_1 f_2) \otimes w_1 w_2 \quad \text{for } f_1, f_2 \in S(\mathfrak{a}_{\mathbb{C}}) \text{ and } w_1, w_2 \in W.$$

The following are well-known properties of Dunkl operators:

Proposition 3.3 ([Dun]). (i) *For* $\xi, \eta \in \mathfrak{a}$, $\mathscr{D}_{\mathbf{k}}(\xi)\mathscr{D}_{\mathbf{k}}(\eta) = \mathscr{D}_{\mathbf{k}}(\eta)\mathscr{D}_{\mathbf{k}}(\xi)$.
(ii) *For* $\xi \in \mathfrak{a}$ *and* $w \in W$, $w\mathscr{D}_{\mathbf{k}}(\xi)w^{-1} = \mathscr{D}_{\mathbf{k}}(w\xi)$. *Hence* $\mathscr{D}_{\mathbf{k}} : \mathfrak{a} \to \operatorname{End}_{\mathbb{C}} C^\infty(\mathfrak{a})$ *uniquely extends to the algebra homomorphism of* $S(\mathfrak{a}_{\mathbb{C}}) \rtimes \mathbb{C}[W]$ *into* $\operatorname{End}_{\mathbb{C}} C^\infty(\mathfrak{a})$.
(iii) *Let* $\{\xi_1, \dots, \xi_r\}$ *be an orthonormal basis of the Euclidean space* $(\mathfrak{a}, B(\cdot,\cdot))$ *and put* $L_{\mathfrak{a}} = \sum_{i=1}^r \xi_i^2 \in S(\mathfrak{a}_{\mathbb{C}})$. *Then*

$$\mathscr{D}_{\mathbf{k}}(L_{\mathfrak{a}}) = \partial(L_{\mathfrak{a}}) + \sum_{\alpha \in R_1^+} \mathbf{k}(\alpha)\Big(\frac{2}{\alpha}\partial(H_\alpha) - \frac{|\alpha|^2}{\alpha^2}(1 - s_\alpha)\Big).$$

Now let us state a generalization of (1.7) for $f \notin C^\infty(\mathfrak{p})^K$.

Theorem 3.4 (radial part formula). *Suppose* $(\sigma, V) \in \widehat{K}_M$, $\Delta \in S(\mathfrak{p}_{\mathbb{C}})^K$, *and* $\Phi \in \operatorname{Hom}_K(V, C^\infty(\mathfrak{p}))$.
(i) $\partial(\Delta) \circ \Phi \in \operatorname{Hom}_K(V, C^\infty(\mathfrak{p}))$, $\mathscr{D}\big(\gamma_0(\Delta)\big) \circ \Gamma_0^\sigma(\Phi) \in \operatorname{Hom}_W(V^M, C^\infty(\mathfrak{a}))$, *and it holds that*

$$\Gamma_0^\sigma\big(\partial(\Delta) \circ \Phi\big)[v] = \mathscr{D}\big(\gamma_0(\Delta)\big) \circ \Gamma_0^\sigma(\Phi)[v] \quad \textit{for } v \in V^M_{\text{single}}. \tag{3.2}$$

(ii) *If* $\operatorname{Ker}\Gamma_0^\sigma(\Phi) \supset V^M_{\text{double}}$, *then* $\operatorname{Ker}\Gamma_0^\sigma\big(\partial(\Delta)\circ\Phi\big) \supset V^M_{\text{double}}$. *Hence for such* Φ *we have*

$$\Gamma_0^\sigma\big(\partial(\Delta) \circ \Phi\big) = \mathscr{D}\big(\gamma_0(\Delta)\big) \circ \Gamma_0^\sigma(\Phi). \tag{3.3}$$

(iii) *If* $(\sigma, V) \in \widehat{K}_{\text{sp}}$ *then* (3.3) *always holds.*

On the other hand, as a generalization of (1.7) for $\Delta \notin S(\mathfrak{p}_{\mathbb{C}})^K$ we have

Theorem 3.5 (radial part formula). *Suppose* $(\sigma, V) \in \widehat{K}_M$, $\Phi \in \operatorname{Hom}_K(V, S(\mathfrak{p}_{\mathbb{C}}))$, *and* $f \in C^\infty(\mathfrak{p})^K$. *Then* $\Gamma_0^\sigma(\Phi) \in \operatorname{Hom}_W(V^M, S(\mathfrak{a}_{\mathbb{C}}))$ *is naturally defined by the identifications* $S(\mathfrak{p}_{\mathbb{C}}) \simeq \mathscr{P}(\mathfrak{p}) \subset C^\infty(\mathfrak{p})$ *and* $S(\mathfrak{a}_{\mathbb{C}}) \simeq \mathscr{P}(\mathfrak{a}) \subset C^\infty(\mathfrak{a})$.
(i) *Let* $\partial(\Phi)f$ *denote the map* $V \ni v \mapsto \partial(\Phi[v])f \in C^\infty(\mathfrak{p})$. *Then* $\partial(\Phi)f \in \operatorname{Hom}_K(V, C^\infty(\mathfrak{p}))$. *Let* $\mathscr{D}\big(\Gamma_0^\sigma(\Phi)\big)\gamma_0(f)$ *denote the map* $V^M \ni v \mapsto \mathscr{D}\big(\gamma_0(\Phi[v])\big)\gamma_0(f) \in C^\infty(\mathfrak{a})$. *Then* $\mathscr{D}\big(\Gamma_0^\sigma(\Phi)\big)\gamma_0(f) \in \operatorname{Hom}_W(V^M, C^\infty(\mathfrak{a}))$ *and it holds that*

$$\Gamma_0^\sigma\big(\partial(\Phi)f\big)[v] = \mathscr{D}\big(\Gamma_0^\sigma(\Phi)\big)\gamma_0(f)[v]. \quad \textit{for } v \in V^M_{\text{single}}. \tag{3.4}$$

(ii) *If* $\operatorname{Ker}\Gamma_0^\sigma(\Phi) \supset V^M_{\text{double}}$, *then* $\operatorname{Ker}\Gamma_0^\sigma\big(\partial(\Phi)f\big) \supset V^M_{\text{double}}$. *Hence for such* Φ *we have*

$$\Gamma_0^\sigma\big(\partial(\Phi)f\big) = \mathscr{D}\big(\Gamma_0^\sigma(\Phi)\big)\gamma_0(f). \tag{3.5}$$

(iii) *If* $(\sigma, V) \in \widehat{K}_{\text{sp}}$ *then* (3.5) *always holds.*

From now on we shall prove Theorem 3.4 by following the method of de Jeu [Je], which uses only three simple lemmas.

Lemma 3.6. *Let* $\{X_1, \dots, X_n\}$ *be an orthonormal basis of the Euclidean space* $(\mathfrak{p}, B(\cdot,\cdot))$ *and put* $L_{\mathfrak{p}} = \sum_{i=1}^n X_i^2 \in S(\mathfrak{p}_{\mathbb{C}})$. *Then* Theorem 3.4 *for* $\Delta = L_{\mathfrak{p}}$ *is true.*

Proof. Suppose $(\sigma, V) \in \widehat{K}_M$ and $\Phi \in \operatorname{Hom}_K(V, C^\infty(\mathfrak{p}))$. First two assertions in Theorem 3.4 (i) are trivial. Although (3.2) for $\Delta = L_{\mathfrak{p}}$ is equivalent to [O, Lemma 3.10], we recall the outline of its proof. For each $\alpha \in \Sigma^+$ choose an orthonormal basis $\big\{X_\alpha^{(1)}, \dots, X_\alpha^{(\dim \mathfrak{g}_\alpha)}\big\}$

of the Euclidean space $(\mathfrak{g}_\alpha, -B(\cdot, \theta\cdot))$. Let $v \in V^M$ and $H \in \mathfrak{a}$. Then a direct calculation (cf. the proof of [O, Lemma 3.10]) leads to

$$\begin{aligned}\partial(L_\mathfrak{p})\Phi[v](H) &= \partial(L_\mathfrak{a})\Phi[v](H) + \sum_{\alpha\in R_1^+} \frac{2\mathbf{m}_0(\alpha)}{\alpha(H)}\partial(H_\alpha)\Phi[v](H) \\ &\quad + \sum_{\alpha\in\Sigma^+}\sum_{i=1}^{\dim\mathfrak{g}_\alpha} \frac{\Phi\Big[p^\sigma\Big(\sigma\big(X_\alpha^{(i)} + \theta X_\alpha^{(i)}\big)^2 v\Big)\Big](H)}{2\alpha(H)^2}.\end{aligned} \tag{3.6}$$

If $v \in V^M_{\text{single}}$, then Lemma 2.5 (i) reduces the right-hand side of (3.6) to

$$\partial(L_\mathfrak{a})\Phi[v](H) + \sum_{\alpha\in R_1^+} \mathbf{m}_0(\alpha)\Big(\frac{2}{\alpha(H)}\partial(H_\alpha)\Phi[v](H) - |\alpha|^2\frac{\Phi[v](H) - \Phi[v](s_\alpha H)}{\alpha(H)^2}\Big),$$

which equals $\mathscr{D}(L_\mathfrak{a})\Phi[v](H)$ by Proposition 3.3 (iii). Since $\gamma_0(L_\mathfrak{p}) = L_\mathfrak{a}$, we get (3.2) for $\Delta = L_\mathfrak{p}$.

To show Theorem 3.4 (ii) for $\Delta = L_\mathfrak{p}$, suppose $\operatorname{Ker}\Gamma_0^\sigma(\Phi) \supset V^M_{\text{double}}$. Then by virtue of Lemma 2.5 (ii) the right-hand side of (3.6) is 0 for $v \in V^M_{\text{double}}$. It means $\operatorname{Ker}\Gamma_0^\sigma(\partial(L_\mathfrak{p})\circ\Phi) \supset V^M_{\text{double}}$, proving Theorem 3.4 (ii) for $\Delta = L_\mathfrak{p}$.

Finally Theorem 3.4 (iii) is immediate from (ii). □

Lemma 3.7 ([He, Je])**.** *Suppose $p \in S(\mathfrak{a}_\mathbb{C})$. Then p is also considered as a polynomial function on $\mathfrak{a}$ by the identification $S(\mathfrak{a}_\mathbb{C}) \simeq \mathscr{P}(\mathfrak{a})$. Let $\tilde{p} \in \operatorname{End}_\mathbb{C} C^\infty(\mathfrak{a})$ denote the multiplication operator by p. Then in the algebra* $\operatorname{End}_\mathbb{C} C^\infty(\mathfrak{a})$ *the following identity holds:*

$$\exp\left(\operatorname{ad}\frac{\mathscr{D}_\mathbf{k}(L_\mathfrak{a})}{2}\right)\tilde{p} := \sum_{d=0}^\infty \frac{1}{d!}\left(\operatorname{ad}\frac{\mathscr{D}_\mathbf{k}(L_\mathfrak{a})}{2}\right)^d\tilde{p} = \mathscr{D}_\mathbf{k}(p).$$

Here the second part is actually a finite sum.

Lemma 3.8. *Suppose $P \in S(\mathfrak{p}_\mathbb{C})$. Then P is also considered as a polynomial function on $\mathfrak{p}$ by the identification $S(\mathfrak{p}_\mathbb{C}) \simeq \mathscr{P}(\mathfrak{p})$. Let $\tilde{P} \in \operatorname{End}_\mathbb{C} C^\infty(\mathfrak{p})$ denote the multiplication operator by P. Then in the algebra* $\operatorname{End}_\mathbb{C} C^\infty(\mathfrak{p})$ *the following identity holds:*

$$\exp\left(\operatorname{ad}\frac{\partial(L_\mathfrak{p})}{2}\right)\tilde{P} := \sum_{d=0}^\infty \frac{1}{d!}\left(\operatorname{ad}\frac{\partial(L_\mathfrak{p})}{2}\right)^d\tilde{P} = \partial(P).$$

Here the second part is actually a finite sum.

Proof of Theorem 3.4. Suppose $(\sigma, V) \in \widehat{K}_M$, $\Delta \in S(\mathfrak{p}_\mathbb{C})^K$, and $\Phi \in \operatorname{Hom}_K(V, C^\infty(\mathfrak{p}))$. By Lemma 3.8 we have for $v \in V$

$$\partial(\Delta)\Phi[v] = \sum_{d=0}^\infty \frac{1}{d!}\left(\operatorname{ad}\left(\frac{\partial(L_\mathfrak{p})}{2}\right)^d\widetilde{\Delta}\right)\Phi[v]. \tag{3.7}$$

Now let $v \in V^M_{\text{single}}$. Then restricting both sides of (3.7) to $\mathfrak{a}$ we have

$$\gamma_0\big(\partial(\Delta)\Phi[v]\big) = \sum_{d=0}^\infty \frac{1}{d!}\left(\operatorname{ad}\left(\frac{\mathscr{D}(L_\mathfrak{a})}{2}\right)^d\widetilde{\gamma_0(\Delta)}\right)\gamma_0\big(\Phi[v]\big) \qquad (\because \text{Lemma 3.6})$$

$$= \mathscr{D}\big(\gamma_0(\Delta)\big)\gamma_0\big(\Phi[v]\big), \qquad (\because \text{Lemma 3.7})$$

which proves (3.2).

In general, if $\operatorname{Ker}\Gamma_0^\sigma(\Phi) \supset V^M_{\text{double}}$ then $\operatorname{Ker}\Gamma_0^\sigma\big(\partial(L_{\mathfrak{p}}) \circ \Phi\big) \supset V^M_{\text{double}}$ by Lemma 3.6 and $\operatorname{Ker}\Gamma_0^\sigma\big(\widetilde{\Delta} \circ \Phi\big) \supset V^M_{\text{double}}$ by an easy observation. Hence it follows from (3.7) $\operatorname{Ker}\Gamma_0^\sigma(\Phi) \supset V^M_{\text{double}}$ implies $\operatorname{Ker}\Gamma_0^\sigma\big(\partial(\Delta)\Phi\big) \supset V^M_{\text{double}}$. We thus get Theorem 3.4 (ii).

Theorem 3.4 (iii) is only an easy corollary of (ii). □

Theorem 3.5 can be similarly proved.

4. Graded Hecke algebras

Let Π be the system of simple roots in R_1 corresponding to the positive system R_1^+. For the study of Cherednik operators we needs the notion of graded Hecke algebras.

Definition 4.1 ([Lu]). Let $\mathbf{k} : W\backslash R_1 \to \mathbb{C}$ be a multiplicity function. Then there exists uniquely (up to equivalence) an algebra $\mathbf{H_k}$ over $\mathbb{C}$ with the following properties:

(i) $\mathbf{H_k} \simeq S(\mathfrak{a}_\mathbb{C}) \otimes \mathbb{C}[W]$ as a $\mathbb{C}$-linear space.
(ii) The maps $S(\mathfrak{a}_\mathbb{C}) \to \mathbf{H_k}, \varphi \mapsto \varphi \otimes 1$ and $\mathbb{C}[W] \to \mathbf{H_k}, w \mapsto 1 \otimes w$ are algebra homomorphisms.
(iii) $(\varphi \otimes 1) \cdot (1 \otimes w) = \varphi \otimes w$ for any $\varphi \in S(\mathfrak{a}_\mathbb{C})$ and $w \in W$.
(iv) $(1 \otimes s_\alpha) \cdot (\xi \otimes 1) = s_\alpha(\xi) \otimes s_\alpha - \mathbf{k}(\alpha)\,\alpha(\xi)$ for any $\alpha \in \Pi$ and $\xi \in \mathfrak{a}_\mathbb{C}$.

We call $\mathbf{H_k}$ the graded Hecke algebra associated to the data $(\mathfrak{a}_\mathbb{C}, \Pi, \mathbf{k})$.

By (ii) we identify $S(\mathfrak{a}_\mathbb{C})$ and $\mathbb{C}[W]$ with subalgebras of $\mathbf{H_k}$. Then (iv) is simply written as

$$(4.1) \qquad s_\alpha \cdot \xi = s_\alpha(\xi) \cdot s_\alpha - \mathbf{k}(\alpha)\,\alpha(\xi) \qquad \forall\alpha \in \Pi\ \forall\xi \in \mathfrak{a}_\mathbb{C}.$$

It is well-known that the center of $\mathbf{H_k}$ equals $S(\mathfrak{a}_\mathbb{C})^W$ (cf. [Lu, Theorem 6.5]).

Definition 4.2. Suppose $\tau = \text{triv}$ or sgn. Here triv and sgn are the trivial and the sign representations of W. Define the left $\mathbf{H_k}$-modules

$$S_{\mathbf{H_k}}(\mathfrak{a}_\mathbb{C};\tau) = \mathbf{H_k} \Big/ \sum_{w\in W} \mathbf{H_k}(w - \tau(w)).$$

Moreover for $\lambda \in \mathfrak{a}_\mathbb{C}^*$ put

$$\mathscr{A}(\mathbf{H_k};\tau,\lambda) = S_{\mathbf{H_k}}(\mathfrak{a}_\mathbb{C};\tau) \Big/ \sum_{\Delta\in S(\mathfrak{a}_\mathbb{C})^W} (\Delta - \Delta(\lambda)) S_{\mathbf{H_k}}(\mathfrak{a}_\mathbb{C};\tau).$$

If $\tau = \text{triv}$ then $S_{\mathbf{H_k}}(\mathfrak{a}_\mathbb{C};\tau)$ and $\mathscr{A}(\mathbf{H_k};\tau,\lambda)$ are also written as $S_{\mathbf{H_k}}(\mathfrak{a}_\mathbb{C})$ and $\mathscr{A}(\mathbf{H_k};\lambda)$.

Owing to Definition 4.1 (i), $S(\mathfrak{a}_\mathbb{C}) \hookrightarrow \mathbf{H_k} \twoheadrightarrow S_{\mathbf{H_k}}(\mathfrak{a}_\mathbb{C};\tau)$ gives a left $S(\mathfrak{a}_\mathbb{C})$-module isomorphism. Hence we identify $S_{\mathbf{H_k}}(\mathfrak{a}_\mathbb{C};\tau)$ with $S(\mathfrak{a}_\mathbb{C})$ as an $S(\mathfrak{a}_\mathbb{C})$-module. Thus we have

$$(4.2) \qquad \mathscr{A}(\mathbf{H_k};\lambda) \simeq \mathscr{A}(\mathbf{H_k};\text{sgn},\lambda) \simeq S(\mathfrak{a}_\mathbb{C}) \Big/ \sum_{\Delta\in S(\mathfrak{a}_\mathbb{C})^W} (\Delta - \Delta(\lambda)) S(\mathfrak{a}_\mathbb{C})$$

as $S(\mathfrak{a}_\mathbb{C})$-modules. On the other hand, $S(\mathfrak{a}_\mathbb{C})$, $S_{\mathbf{H_k}}(\mathfrak{a}_\mathbb{C})$, and $S_{\mathbf{H_k}}(\mathfrak{a}_\mathbb{C};\text{sgn})$ have their own natural W-module structures and the above identification $S(\mathfrak{a}_\mathbb{C}) \simeq S_{\mathbf{H_k}}(\mathfrak{a}_\mathbb{C}) \simeq S_{\mathbf{H_k}}(\mathfrak{a}_\mathbb{C};\text{sgn})$ does not commute with them. Hence if $\varphi \in S(\mathfrak{a}_\mathbb{C})$ and $w \in W$ then we sometimes write the action of w on φ as $w\varphi$, $w \diamond \varphi$, or $w \star \varphi$ according as we consider $\varphi \in S(\mathfrak{a}_{\mathbb{C}})$, $S_{\mathbf{H_k}}(\mathfrak{a}_{\mathbb{C}})$, or $S_{\mathbf{H_k}}(\mathfrak{a}_{\mathbb{C}};\mathrm{sgn})$.

Let $\cdot^{\mathrm{opp}} : \mathbf{H_k} \to \mathbf{H_{-k}}$ be the algebra anti-isomorphism defined by $\xi \mapsto -\xi$ ($\xi \in \mathfrak{a}_{\mathbb{C}}$) and $w \mapsto w^{-1}$ ($w \in W$). If V is any $\mathbf{H_{-k}}$-module, then its dual space V^* is naturally an $\mathbf{H_k}$-module by $hv^* = v^*(h^{\mathrm{opp}}\cdot)$ for $h \in \mathbf{H_k}$ and $v^* \in V^*$. Then we can prove

Proposition 4.3. *Suppose $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$. Then $\mathscr{A}(\mathbf{H_{-k}};-\lambda)^* \simeq \mathscr{A}(\mathbf{H_k};\mathrm{sgn},\lambda)$ as an $\mathbf{H_k}$-module. More precisely, the bilinear form*

$$\begin{aligned} \langle\cdot,\cdot\rangle_\lambda \,:\, S_{\mathbf{H_{-k}}}(\mathfrak{a}_{\mathbb{C}}) \times S_{\mathbf{H_k}}(\mathfrak{a}_{\mathbb{C}};\mathrm{sgn}) &\longrightarrow \mathbb{C}; \\ (\varphi,\psi) &\longmapsto \frac{\sum_{w\in W}(\mathrm{sgn}\,w)\,\varphi(-w\lambda)\,\psi(w\lambda)}{\prod_{\alpha\in R_1^+}\lambda(\alpha^\vee)} \end{aligned} \tag{4.3}$$

is well-defined and satisfies

$$\langle\cdot,h\cdot\rangle_\lambda = \langle h^{\mathrm{opp}}\cdot,\cdot\rangle_\lambda \tag{4.4}$$

for $h \in \mathbf{H_k}$. Moreover it induces a non-degenerate bilinear form on $\mathscr{A}(\mathbf{H_{-k}};-\lambda) \times \mathscr{A}(\mathbf{H_k};\mathrm{sgn},\lambda)$.

5. The Harish-Chandra homomorphism

Before generalizing the radial part formulas (1.2), (1.3), we review two generalizations of the Harish-Chandra homomorphism γ.

Theorem 5.1. [Shi1, Theorem 2.4] *Suppose $\chi \in \widehat{K}_1$. Define the map $\gamma^\chi : U(\mathfrak{g}_{\mathbb{C}}) \to S(\mathfrak{a}_{\mathbb{C}})$ by*

$$\begin{aligned} U(\mathfrak{g}_{\mathbb{C}}) &= \Big(\bar{\mathfrak{n}}_{\mathbb{C}}U(\mathfrak{g}_{\mathbb{C}}) + \sum_{Z\in\mathfrak{k}_{\mathbb{C}}} U(\mathfrak{g}_{\mathbb{C}})(Z+\chi(Z))\Big) \oplus U(\mathfrak{a}_{\mathbb{C}}) \\ &\xrightarrow{\text{projection}} U(\mathfrak{a}_{\mathbb{C}}) = S(\mathfrak{a}_{\mathbb{C}}) \xrightarrow{\rho\text{-shift}} S(\mathfrak{a}_{\mathbb{C}}). \end{aligned}$$

Then γ^χ induces the following exact sequence of algebra homomorphisms:

$$0 \to U(\mathfrak{g}_{\mathbb{C}})^K \cap \sum_{Z\in\mathfrak{k}_{\mathbb{C}}} U(\mathfrak{g}_{\mathbb{C}})(Z+\chi(Z)) \to U(\mathfrak{g}_{\mathbb{C}})^K \xrightarrow{\gamma^\chi} S(\mathfrak{a}_{\mathbb{C}})^W \to 0.$$

Define the multiplicity function $\mathbf{m}_1 : W\backslash R_1 \to \mathbb{C}$ by

$$\mathbf{m}_1(\alpha) = \begin{cases} \frac{1}{2}\dim\mathfrak{g}_{\alpha/2} & \text{if } 2\alpha \notin R, \\ \frac{1}{2}\dim\mathfrak{g}_{\alpha/2} + \dim\mathfrak{g}_\alpha & \text{if } 2\alpha \in R. \end{cases} \tag{5.1}$$

Let $\mathbf{H} = \mathbf{H}_{\mathbf{m}_1}$ and $\mathbf{H}' = \mathbf{H}_{-\mathbf{m}_1}$.

Theorem 5.2 ([O])**.** *For $(\sigma,V) \in \widehat{K}_M$ define the map*

$$\begin{aligned} \Gamma^\sigma : \mathrm{Hom}_K(V,U(\mathfrak{g}_{\mathbb{C}})) &\longrightarrow \mathrm{Hom}_{\mathbb{C}}(V^M, S_{\mathbf{H}'}(\mathfrak{a}_{\mathbb{C}}))\,; \\ \Psi &\longmapsto \big(\psi : V^M \hookrightarrow V \xrightarrow{\Psi} U(\mathfrak{g}_{\mathbb{C}}) \xrightarrow{\gamma} S(\mathfrak{a}_{\mathbb{C}}) = S_{\mathbf{H}'}(\mathfrak{a}_{\mathbb{C}})\big). \end{aligned}$$

Then we have the following:
(i) *Γ^σ is injective.*
(ii) *For any $\Psi \in \mathrm{Hom}_K(V,U(\mathfrak{g}_{\mathbb{C}}))$, $v \in V^M_{\mathrm{single}}$, and $w \in W$,*

$$\Gamma^\sigma(\Psi)[wv] = w \diamond \Gamma^\sigma(\Psi)[v]. \tag{5.2}$$

Furthermore, $\operatorname{Im}\Gamma^\sigma \subset \operatorname{Hom}_W(V^M, S_{\mathbf{H}'}(\mathfrak{a}_{\mathbb{C}}))$ *if and only if* $(\sigma, V) \in \widehat{K}_{\mathrm{sp}}$.
(iii) *A* W*-submodule* $U \subset V^M$ *satisfies the condition*

$$\operatorname{Im}\Gamma^\sigma \supset \{\psi \in \operatorname{Hom}_W(V^M, S_{\mathbf{H}'}(\mathfrak{a}_{\mathbb{C}})); \operatorname{Ker}\psi \supset U\}$$

if and only if $U \supset V^M_{\mathrm{double}}$. *In particular* $\operatorname{Im}\Gamma^\sigma$ *contains any* $\psi \in \operatorname{Hom}_W(V^M, S_{\mathbf{H}'}(\mathfrak{a}_{\mathbb{C}}))$ *such that* $\operatorname{Ker}\psi \supset V^M_{\mathrm{double}}$.
(iv) *If* $(\sigma, V) \in \widehat{K}_{\mathrm{sp}}$ *then we have* $\operatorname{Im}\Gamma^\sigma = \operatorname{Hom}_W(V^M, S_{\mathbf{H}'}(\mathfrak{a}_{\mathbb{C}}))$ *and the following exact sequence:*

$$0 \to \operatorname{Hom}_K(V, U(\mathfrak{g}_{\mathbb{C}})\mathfrak{k}_{\mathbb{C}}) \to \operatorname{Hom}_K(V, U(\mathfrak{g}_{\mathbb{C}})) \xrightarrow{\Gamma^\sigma} \operatorname{Hom}_W(V^M, S_{\mathbf{H}'}(\mathfrak{a}_{\mathbb{C}})) \to 0.$$

6. Cherednik operators

Recall $R = 2\Sigma$, $R^+ = 2\Sigma^+$, and $\Pi \subset R_1^+$ is the set of simple roots of R. By (1.6) we identify $C^\infty(\mathfrak{a})$ with $C^\infty(A)$, so that $\operatorname{End}_{\mathbb{C}} C^\infty(A) \ni \partial(\xi)$ $(\xi \in \mathfrak{a}_{\mathbb{C}})$. For $\mu \in \mathfrak{a}_{\mathbb{C}}$ let $e^\mu \in C^\infty(A)$ denote the function $a \mapsto \exp\mu(\log a)$.

Definition 6.1 (Cherednik operators [Ch])**.** Suppose $\mathbf{k} : W\backslash R \to \mathbb{C}$ is a multiplicity function and put $\rho_{\mathbf{k}} = \frac{1}{2}\sum_{\alpha\in R^+}\mathbf{k}(\alpha)\alpha$. For $\xi \in \mathfrak{a}$ define $\mathscr{T}_{\mathbf{k}}(\xi) \in \operatorname{End}_{\mathbb{C}} C^\infty(A)$ by

$$\mathscr{T}_{\mathbf{k}}(\xi) = \partial(\xi) + \sum_{\alpha\in R^+}\mathbf{k}(\alpha)\frac{\alpha(\xi)}{1-e^{-\alpha}}(1-s_\alpha) - \rho_{\mathbf{k}}(\xi).$$

If $\mathbf{k}$ equals the multiplicity function $\mathbf{m} : W\backslash R \to \mathbb{C}$ defined by

$$\mathbf{m}(\alpha) = \frac{1}{2}\dim\mathfrak{g}_{\alpha/2}, \tag{6.1}$$

then we use the brief symbol $\mathscr{T}$ for $\mathscr{T}_{\mathbf{m}}$.

Proposition 6.2. (i) *For* $\xi, \eta \in \mathfrak{a}$, $\mathscr{T}_{\mathbf{k}}(\xi)\mathscr{T}_{\mathbf{k}}(\eta) = \mathscr{T}_{\mathbf{k}}(\eta)\mathscr{T}_{\mathbf{k}}(\xi)$.
(ii) *Let* $\mathbf{k}_1 : W\backslash R_1 \to \mathbb{C}$ *be the multiplicity function defined by*

$$\mathbf{k}_1(\alpha) = \begin{cases}\mathbf{k}(\alpha) & \text{if } 2\alpha \notin R,\\ \mathbf{k}(\alpha) + 2\mathbf{k}(2\alpha) & \text{if } 2\alpha \in R.\end{cases}$$

Then $s_\alpha\mathscr{T}_{\mathbf{k}}(\xi) = \mathscr{T}_{\mathbf{k}_1}(s_\alpha\xi)s_\alpha - \mathbf{k}_1(\alpha)\alpha(\xi)$ *for* $\xi \in \mathfrak{a}$ *and* $\alpha \in \Pi$. *Hence* $\mathscr{T}_{\mathbf{k}} : \mathfrak{a} \to \operatorname{End}_{\mathbb{C}} C^\infty(A)$ *uniquely extends to the algebra homomorphism of* $\mathbf{H}_{\mathbf{k}_1}$ *into* $\operatorname{End}_{\mathbb{C}} C^\infty(A)$.
(iii) *Let* $L_{\mathfrak{a}} \in S(\mathfrak{a}_{\mathbb{C}})$ *be as in* Proposition 3.3 (iii). *Then*

$$\mathscr{T}_{\mathbf{k}}(L_{\mathfrak{a}}) = \partial(L_{\mathfrak{a}}) + \sum_{\alpha\in R_1^+}\mathbf{k}(\alpha)\Big(\coth\frac{\alpha}{2}\partial(H_\alpha) - \frac{|\alpha|^2}{4\sinh^2\frac{\alpha}{2}}(1-s_\alpha)\Big) + B(H_{\rho_{\mathbf{k}}}, H_{\rho_{\mathbf{k}}}). \tag{6.2}$$

Remark 6.3. Proposition 6.2 (i), (ii) are given in [Ch] (see also [Op2]). On the other hand, (iii) is calculated in [Sha]. If $\mathbf{k} = \mathbf{m}$ then $\mathbf{H}_{\mathbf{k}_1} = \mathbf{H}$ and $\mathbf{H}_{-\mathbf{k}_1} = \mathbf{H}'$.

Theorem 6.4 (radial part formula)**.** *Suppose* $(\sigma, V) \in \widehat{K}_M$, $\Delta \in U(\mathfrak{p}_{\mathbb{C}})^K$, *and* $\Phi \in \operatorname{Hom}_K(V, C^\infty(G/K))$.
(i) $r(\Delta)\circ\Phi \in \operatorname{Hom}_K(V, C^\infty(G/K))$, $\mathscr{T}(\gamma(\Delta))\circ\Gamma_0^\sigma(\Phi) \in \operatorname{Hom}_W(V^M, C^\infty(A))$, *and it holds that*

$$\Gamma_0^\sigma\big(r(\Delta)\circ\Phi\big)[v] = \mathscr{T}\big(\gamma(\Delta)\big)\circ\Gamma_0^\sigma(\Phi)[v] \quad \text{for } v \in V^M_{\mathrm{single}}. \tag{6.3}$$

(ii) *If* $\operatorname{Ker}\Gamma_0^\sigma(\Phi) \supset V^M_{\text{double}}$, *then* $\operatorname{Ker}\Gamma_0^\sigma\bigl(r(\Delta)\circ\Phi\bigr) \supset V^M_{\text{double}}$. *Hence for such* Φ *we have*

$$\Gamma_0^\sigma\bigl(r(\Delta)\circ\Phi\bigr) = \mathscr{T}\bigl(\gamma(\Delta)\bigr)\circ\Gamma_0^\sigma(\Phi). \tag{6.4}$$

(iii) *If* $(\sigma, V) \in \widehat{K}_{\text{sp}}$ *then* (6.4) *always holds.*

Let $\mathcal{S}ol_\lambda$ be the solution space of (1.4). Then

$$\mathscr{A}(\mathbf{H}'; -\lambda) \times \mathcal{S}ol_\lambda \ni (h, \varphi) \mapsto \mathscr{T}(h^{\text{opp}})\varphi(e) \in \mathbb{C}$$

defines a non-degenerate bilinear form with the obvious invariance property. Hence by Proposition 4.3 we get a natural $\mathbf{H}$-module isomorphism $\mathcal{S}ol_\lambda \simeq \mathscr{A}(\mathbf{H}; \operatorname{sgn}, \lambda)$.

Corollary 6.5. *Suppose* $\lambda \in \mathfrak{a}_{\mathbb{C}}$ *and* $(\sigma, V) \in \widehat{K}_{\text{sp}}$. *Then* Γ_0^σ *induces the isomorphism*

$$\Gamma_0^\sigma : \operatorname{Hom}_K(V, \mathscr{A}(G/K; \lambda)) \xrightarrow{\sim} \operatorname{Hom}_W(V^M, \mathcal{S}ol_\lambda) \simeq \operatorname{Hom}_W(V^M, \mathscr{A}(\mathbf{H}; \operatorname{sgn}, \lambda)).$$

Theorem 6.6 (radial part formula)**.** *Suppose* $(\sigma, V) \in \widehat{K}_M$, $\Psi \in \operatorname{Hom}_K(V, U(\mathfrak{g}_{\mathbb{C}}))$, *and* $f \in C^\infty(G/K)^{\ell(K)}$.
(i) *Let* $\ell(\Psi)f$ *be the map* $V \ni v \mapsto \ell(\Psi[v])f \in C^\infty(G/K)$. *Then* $\ell(\Psi)f \in \operatorname{Hom}_K(V, C^\infty(G/K))$. *Let* $\mathscr{T}\bigl(\Gamma^\sigma(\Psi)^{\text{opp}}\bigr)\gamma_0(f)$ *denote the map* $V^M \ni v \mapsto \mathscr{T}\bigl(\gamma(\Psi[v])^{\text{opp}}\bigr)\gamma_0(f) \in C^\infty(A)$. *Then it holds that*

$$\Gamma_0^\sigma\bigl(\ell(\Psi)f\bigr)[v] = \mathscr{T}\bigl(\Gamma^\sigma(\Psi)^{\text{opp}}\bigr)\gamma_0(f)[v]. \quad \textit{for } v \in V^M_{\text{single}}. \tag{6.5}$$

(ii) *If* $\operatorname{Ker}\Gamma^\sigma(\Psi) \supset V^M_{\text{double}}$, *then* $\operatorname{Ker}\Gamma_0^\sigma\bigl(\ell(\Psi)f\bigr) \supset V^M_{\text{double}}$. *Hence for such* Ψ *we have*

$$\Gamma_0^\sigma\bigl(\ell(\Psi)f\bigr) = \mathscr{T}\bigl(\Gamma^\sigma(\Psi)^{\text{opp}}\bigr)\gamma_0(f). \tag{6.6}$$

(iii) *If* $(\sigma, V) \in \widehat{K}_{\text{sp}}$ *then* (6.6) *always holds.*

7. Hermitian cases

Assume G/K is an irreducible Hermitian symmetric space. Thus $\mathfrak{g}$ is simple and $\mathfrak{k}$ has a non-trivial center $\mathfrak{z}$. Choose $Z_0 \in \mathfrak{z}$ so that $\operatorname{ad}_{\mathfrak{p}}(Z_0)^2 = -1$ and put $\mathfrak{p}_\pm := \bigl\{X \in \mathfrak{p}_{\mathbb{C}};\ [Z_0, X] = \pm\sqrt{-1}\bigr\}$. Let $\{\alpha_1, \dots, \alpha_r\}$ be the orthogonal basis of $\mathfrak{a}^*$ such that

$$\begin{aligned}
&|\alpha_1| = \cdots = |\alpha_r|,\\
&\Sigma^+ \subset \widetilde{\Sigma}^+ := \bigl\{\alpha_i/2, \alpha_i, (\alpha_j \pm \alpha_k)/2\,;\ 1 \le i \le r, 1 \le j < k \le r\bigr\},\\
&\Sigma^+ \supset \bigl\{\alpha_i, (\alpha_j \pm \alpha_k)/2\,;\ 1 \le i \le r, 1 \le j < k \le r\bigr\}.
\end{aligned}$$

Following the convention in [Sch, Shi1, Shi2], put $Z = -\dfrac{2}{|\alpha_1|^2 B(Z_0, Z_0)} Z_0$. Let $\mathfrak{m} = \operatorname{Lie} M$.

Lemma 7.1. *For any* $\alpha \in \Sigma \setminus \{\pm\alpha_i\}$ *and* $X_\alpha \in \mathfrak{g}_\alpha$, $X_\alpha + \theta X_\alpha \in [\mathfrak{k}, \mathfrak{k}]$. *On the other hand,* $\dim \mathfrak{g}_{\alpha_i} = 1$ $(i = 1, \dots, r)$ *and we can uniquely choose* $E_i \in \mathfrak{g}_{\alpha_i}$ *and* $Y \in \mathfrak{m}$ *so that*

$$-B(E_i, \theta E_i) = \frac{2}{|\alpha_i|^2} \quad \textit{and} \quad Z_0 = \frac{1}{2}\sum_{i=1}^r (E_i + \theta E_i) + Y.$$

Furthermore, for any i *and* $w \in N_K(\mathfrak{a})$ (*the normalizer of* $\mathfrak{a}$ *in* K) *there exists some* j *such that* $\operatorname{Ad}(w)(E_i + \theta E_i) = (E_j + \theta E_j)$.

Theorem 7.2. *Let* $(\sigma, V) \in \widehat{K}_{\mathrm{qsp}}$. *Then for at least either* $\epsilon = +$ *or* $\epsilon = -$, *it holds that*

$$\big(\sigma(E_i + \theta E_i) - \epsilon\sqrt{-1}(1 - s_{\alpha_i})\big)V^M_{\mathrm{single}} = 0 \qquad \textit{for } i = 1, \dots, r. \tag{7.1}$$

We say a quasi-single-petaled K-type (σ, V) is *holomorphic* if it satisfies (7.1) for $\epsilon = +$ and *anti-holomorphic* if it satisfies (7.1) for $\epsilon = -$. The set of holomorphic (resp., anti-holomorphic) quasi-single-petaled K-types is denoted by $\widehat{K}^+_{\mathrm{qsp}}$ (resp., $\widehat{K}^-_{\mathrm{qsp}}$). For example, $(\mathrm{Ad}, \mathfrak{p}_+) \in \widehat{K}^+_{\mathrm{qsp}}$, $(\mathrm{Ad}, \mathfrak{p}_-) \in \widehat{K}^-_{\mathrm{qsp}}$, and $(\mathrm{triv}, \mathbb{C}) \in \widehat{K}^+_{\mathrm{qsp}} \cap \widehat{K}^-_{\mathrm{qsp}}$.

For $s \in \mathbb{C}$ there exists a unique $\chi_s \in \widehat{K}_1$ such that $\chi_s(Z) = \sqrt{-1}s$ and any K-type in $\widehat{K}_1$ is obtained in this way. By a calculation in the proof of [Shi2, Lemma 2.4], we have

Lemma 7.3. *For* $i = 1, \dots, r$,

$$\chi_s(E_i + \theta E_i) = \sqrt{-1}s.$$

Let $L_{\mathfrak{g}} \in U(\mathfrak{g}_{\mathbb{C}})$ be the Casimir element of $\mathfrak{g}$. Put $\mathfrak{a}_{\mathrm{reg}} = \{H \in \mathfrak{a};\, \alpha(H) \neq 0 \text{ for } \alpha \in \Sigma\}$. Suppose $s \in \mathbb{C}$ and $\epsilon = \pm$. The next lemma is the key to our radial part formulas.

Lemma 7.4. *Suppose* $(\sigma, V) \in \widehat{K}^\epsilon_{\mathrm{qsp}}$, $\Phi \in \mathrm{Hom}_K(V_{\chi_s}, C^\infty(G/K; \chi_s))$, *and* $v \in V^M_{\mathrm{single}}$. *Then the following equality holds on* $\exp \mathfrak{a}_{\mathrm{reg}}$*:*

$$\begin{aligned}\gamma_0\big(\ell(L_{\mathfrak{g}})\Phi[v \otimes v_{\chi_s}]\big) = \Bigg(&\partial(L_{\mathfrak{a}}) + \chi_s(L_{\mathfrak{m}}) \\ &+ \sum_{\alpha \in \Sigma^+} \dim \mathfrak{g}_\alpha \Big(\coth \alpha\, \partial(H_\alpha) - \frac{|\alpha|^2}{2\sinh^2 \alpha}(1 - s_\alpha)\Big) \\ &+ \sum_{i=1}^r \frac{s|\alpha_i|^2}{4\cosh^2(\alpha_i/2)}(s + \epsilon(1 - s_{\alpha_i}))\Bigg)\gamma_0(\Phi[v \otimes v_{\chi_s}])\end{aligned} \tag{7.2}$$

where $L_{\mathfrak{m}} \in U(\mathfrak{m}_{\mathbb{C}})$ *is the Casimir element of* $\mathfrak{m}$ *with respect to* $B(\cdot, \cdot)$.

Put $\widetilde{R} = \pm 2\widetilde{\Sigma}^+$ and define the multiplicity function $\widetilde{\mathbf{m}}_{\epsilon s} : W\backslash\widetilde{R} \to \mathbb{C}$ by

$$\widetilde{\mathbf{m}}_{\epsilon s}(\alpha_i) = \frac{1}{2}\dim \mathfrak{g}_{\alpha_i/2} - \epsilon s, \qquad \widetilde{\mathbf{m}}_{\epsilon s}(\pm\alpha_j \pm \alpha_k) = \frac{1}{2}\dim \mathfrak{g}_{(\pm\alpha_j \pm \alpha_k)/2}, \qquad \widetilde{\mathbf{m}}_{\epsilon s}(2\alpha_i) = \frac{1}{2} + \epsilon s$$

for $i, j, k = 1, \dots, r$ with $j \neq k$. Put $\widetilde{\mathbf{H}}_{\epsilon s} = \mathbf{H}_{(\widetilde{\mathbf{m}}_{\epsilon s})_1}$ Define the following algebra homomorphism :

$$\widetilde{\mathscr{T}_{\epsilon s}} : \widetilde{\mathbf{H}}_{\epsilon s} \ni h \longmapsto \prod_{i=1}^r \Big(\cosh \frac{\alpha_i}{2}\Big)^{\epsilon s} \circ \mathscr{T}_{\widetilde{\mathbf{m}}_{\epsilon s}}(h) \circ \prod_{i=1}^r \Big(\cosh \frac{\alpha_i}{2}\Big)^{-\epsilon s} \in \mathrm{End}_{\mathbb{C}}\, C^\infty(A).$$

Here $\mathscr{T}_{\widetilde{\mathbf{m}}_{\epsilon s}}$ is a Cherednik operator for the root system $\widetilde{R}$.

Theorem 7.5 (radial part formula)**.** *Suppose* $(\sigma, V) \in \widehat{K}^\epsilon_{\mathrm{qsp}}$, $\Phi \in \mathrm{Hom}_K(V_{\chi_s}, C^\infty(G/K; \chi_s))$, *and* $\Delta \in U(\mathfrak{g}_{\mathbb{C}})^K$. *Use the symbols* $\Gamma^{s,\sigma}_0, \gamma^s$ *instead of* $\Gamma^{\chi_s,\sigma}_0, \gamma^{\chi_s}$.
(i) $r(\Delta) \circ \Phi \in \mathrm{Hom}_K(V_{\chi_s}, C^\infty(G/K; \chi_s))$, $\widetilde{\mathscr{T}_{\epsilon s}}\big(\gamma^s(\Delta)\big) \circ \Gamma^{s,\sigma}_0(\Phi) \in \mathrm{Hom}_W(V^M, C^\infty(A))$, *and it holds that*

$$\Gamma^{s,\sigma}_0\big(r(\Delta) \circ \Phi\big)[v] = \widetilde{\mathscr{T}_{\epsilon s}}\big(\gamma^s(\Delta)\big) \circ \Gamma^{s,\sigma}_0(\Phi)[v] \quad \textit{for } v \in V^M_{\mathrm{single}}. \tag{7.3}$$

(ii) *If* $(\sigma, V) \in \widehat{K}^{\epsilon}_{\mathrm{qsp}} \cap \widehat{K}_{\mathrm{sp}}$ *then we have*

$$\Gamma_0^{s,\sigma}\big(r(\Delta) \circ \Phi\big) = \widetilde{\mathscr{T}_{\epsilon s}}\big(\gamma^s(\Delta)\big) \circ \Gamma_0^{s,\sigma}(\Phi). \tag{7.4}$$

Theorem 7.6 (radial part formula)**.** *Suppose* $(\sigma, V) \in \widehat{K}^{\epsilon}_{\mathrm{qsp}}$, $\Psi \in \mathrm{Hom}_K(V, U(\mathfrak{g}_{\mathbb{C}}))$. *Suppose* $f \in C^\infty(G/K; \chi_s)$ *satisfies* $\ell(k)f = \chi_s(k)f$ *for* $k \in K$.
(i) *Let* $\ell(\Psi)f$ *be the map* $V_{\chi_s} \ni v \otimes v_{\chi_s} \mapsto \ell(\Psi[v])f \in C^\infty(G/K; \chi_s)$. *Then* $\ell(\Psi)f \in \mathrm{Hom}_K(V_{\chi_s}, C^\infty(G/K; \chi_s))$. *Let* $\widetilde{\mathscr{T}_{\epsilon s}}\big(\Gamma^{-s,\sigma}(\Psi)^{\mathrm{opp}}\big)\gamma_0(f)$ *denote the map*

$$V^M \ni v \mapsto \widetilde{\mathscr{T}_{\epsilon s}}\big(\gamma^{-s}(\Psi[v])^{\mathrm{opp}}\big)\gamma_0(f) \in C^\infty(A).$$

Then it holds that

$$\Gamma_0^{s,\sigma}\big(\ell(\Psi)f\big)[v] = \widetilde{\mathscr{T}_{\epsilon s}}\big(\Gamma^{-s,\sigma}(\Psi)^{\mathrm{opp}}\big)\gamma_0(f)[v]. \quad \textit{for } v \in V^M_{\mathrm{single}}. \tag{7.5}$$

(ii) *If* $(\sigma, V) \in \widehat{K}^{\epsilon}_{\mathrm{qsp}} \cap \widehat{K}_{\mathrm{sp}}$ *then we have*

$$\Gamma_0^{s,\sigma}\big(\ell(\Psi)f\big) = \widetilde{\mathscr{T}_{\epsilon s}}\big(\Gamma^{-s,\sigma}(\Psi)^{\mathrm{opp}}\big)\gamma_0(f). \tag{7.6}$$

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