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The isomorphism problem of Coxeter groups and related topics

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A group W is called a *Coxeter group* if there is a generating set S of W (*Coxeter generating set*), consisting of elements of order 2, such that W admits the following group presentation

$$W = \langle S \mid (st)^{o(st)} = 1 \text{ for all } s, t \in S \text{ with } o(st) < \infty \rangle$$

where $o(st)$ denotes the order of $st \in W$. (In this talk S is not assumed to be a finite set.) As you know well, Coxeter groups arise from representation theory quite often, e.g. as Weyl groups, where the Coxeter generating set is usually given naturally by the context and so the 'type' of the Coxeter group is uniquely determined.

However, it is *not* true that the type of a Coxeter group is always uniquely determined by the group itself; namely, when regarding as abstract groups, two Coxeter groups of different types may be isomorphic with each other. In other words, a choice of a Coxeter generating set of a Coxeter group has certain variations in general. A typical example is the Weyl group G_2 (or the Coxeter group of type $I_2(6)$) which is isomorphic to the direct product $S_2 \times S_3$ of two symmetric groups (so it also admits a Coxeter generating set of type $A_1 \times A_2$). This example also shows that, even if a Coxeter group is irreducible with respect to a given Coxeter generating set, it may be reducible with respect to another Coxeter generating set.

The isomorphism problem of Coxeter groups is, roughly speaking, the problem of deciding when such phenomena occur. Although the theory of Coxeter groups has a long history, almost all of the results on the isomorphism problem (except those on the finite Coxeter groups) were obtained very recently, namely in this decade. The aim of this talk is to introduce those recent results on this problem and related topics, e.g. the following results of the speaker:

Theorem. Let W be a Coxeter group with Coxeter generating set S.

1. If W is irreducible (with respect to S) and $|W| = \infty$, then W is irreducible with respect to every Coxeter generating set. More strongly, W is directly indecomposable as an abstract group.

2. The product of all finite irreducible component of W (with respect to S) is independent on the choice of S.

3. Suppose that $|W| = \infty$ and either "$o(st) < \infty$ for all $s, t \in S$" or "all elements of S are conjugate in W". Then the set of reflections in W is independent on the choice of S.

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