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# Some topics for unitary representations of solvable Lie groups

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## §0. Introduction

In this talk I shall explain some topics for unitary representations of solvable Lie groups, their present state and problems for futur development. At the beginning of 1970's Auslander-Kostant succeeded in the framework of the orbit method to construct the unitary dual for a connected and simply connected type I solvable Lie group, and then their results were extended to non type I solvable Lie groups by Pukanzsky. These works are landmarks in the representation theory of solvable Lie groups. If, however, we try to study the holomorphically induced representation and its application in detail, it remains until now to be difficult.

Concerning induced representations or restricted representations, we would like to decompose them, construct intertwining operators or study some related algebra of invariant differential operators. Then, we know little even for exponential Lie groups. We have more tools in hand only for nilpotent Lie groups. The theory of representations is developed in a rather different fashion between semi-simple and solvable Lie groups. The algebraic structure of semi-simple Lie groups is so rich that it offers us many ingredients. As for solvable Lie groups, the poor structure obliges us to use the main method of induction. In any way it's incontestable that the orbit method is very fruitful in the unitary representation theory of solvable Lie groups. The innovatory idea of Kirillov to associate a coadjoint orbit to an irreducible unitary representation seems to be proud of its worthy results. It's a nice application of Mackey's theory to solvable Lie groups. Once this frame is opted for, we can study many objects in analysis by means of algebraic and geometric properties of coadjoint orbits.

The aim of this talk is to invite young people into the research of this domain, where many problems are waiting them.

## §1. Orbite method

1.1. Induced representations

Let's start by defining the induced representation and mention its some properties. Let $G$ be a Lie group with Lie algebra $\mathfrak{g}$ and $H$ a closed subgroup of $G$ with Lie algebra $\mathfrak{h}$. we note $\Delta_G$ (resp. $\Delta_H$) the modular function of $G$ (resp. H) and put

$$\chi(h) = \Delta_{H,G}(h) = \frac{\Delta_H(h)}{\Delta_G(h)}$$

for $h \in H$. As

$$\Delta_G(g) = |\det \mathrm{Ad}(g)|^{-1} \ (g \in G),$$

we have

$$\chi(\exp X) = e^{\mathrm{tr}_{\mathfrak{g}/\mathfrak{h}} \mathrm{ad} X} (X \in \mathfrak{h}).$$

Let's designate by $\mathcal{K}(G, H)$ the space of continuous functions $\phi$ on $G$ with values in $\mathbb{C}$, satisfying the covariance relations

$$\phi(gh) = \chi(h)\phi(g) \ (g \in G, h \in H),$$

and having compact support modulo $H$. Then $G$ acts in $\mathcal{K}(G, H)$ by left translation, and there exists up to scalar multiple a $G$-invariant positive linear form on $\mathcal{K}(G, H)$. It's denoted by $\mu_{G,H}$ and we write, for $\phi \in \mathcal{K}(G, H)$,

$$\mu_{G,H}(\phi) = \oint_{G/H} \phi(g) d\mu_{G,H}(g).$$

In fact, let $\mathcal{K}(G)$ be the space of continuous functions with compact support on $G$. The map $F \mapsto F^{\chi}$ of $\mathcal{K}(G)$ into $\mathcal{K}(G, H)$ defined by

$$F^{\chi}(g) = \int_H F(gh)\chi(h)^{-1} d\mu_H(h),$$

$\mu_H$ being a left Haar measure on $H$, turns out to be surjective. Moreover, if $F \in \mathcal{K}(G)$ satisfies $F^{\chi} = 0$, then $\mu_G(F) = 0$. Therefore the left Haar measure $\mu_G$ gives $\mu_{G,H}$ by passing to the quotient. So,

$$\int_G F(g) d\mu_G(g) = \oint_{G/H} d\mu_{G,H}(g) \int_H F(gh)\chi^{-1}(h) d\mu_H(h)$$

for any $F \in \mathcal{K}(G)$.

Let's be given now a unitary representation $\sigma$ of $H$ in a Hilbert space $\mathcal{H}_\sigma$. We designate by $\mathcal{K}(G, \sigma)$ the space of continuous functions $F$ on $G$ with values in $\mathcal{H}_\sigma$, satisfying the covariance relations

$$F(gh) = \chi(h)^{1/2} \sigma(h)^{-1} (F(g)) \ \ (g \in G, h \in H),$$

and having compact support modulo $H$. Since

$$\|F(gh)\|^2_{\mathcal{H}_\sigma} = \chi(h) \|F(g)\|^2_{\mathcal{H}_\sigma},$$

the function

$$\|F\|^2_{\mathcal{H}_\sigma} : g \mapsto \|F(g)\|^2_{\mathcal{H}_\sigma}$$

belongs to the space $\mathcal{K}(G, H)$. We equip $\mathcal{K}(G, \sigma)$ with the norm

$$\|F\| = \left(\mu_{G,H}(\|F\|_{\mathcal{H}^2_\sigma})\right)^{1/2},$$

and consider its completion for this norm to get a Hilbert space $\mathcal{H}$. It's well known that $\mathcal{H} \neq \{0\}$, and $G$ acts in $\mathcal{H}$ by left translation. This is our realization of the induced representation $\pi = \mathrm{ind}_H^G \sigma$, i.e.

$$\left(\pi(g)F\right)(x) = F(g^{-1}x) \ (g, x \in G, F \in \mathcal{H}).$$

This procedure is frequently utilized to construct unitary representations starting from those of subgroups. In particular, a unitary representation of $G$ induced up by a unitary character of a closed subgroup is said to be monomial. We say $G$ is monomial if every irreducible unitary representation is equivalent to monomial one. It's known ([7], [46]) that exponential Lie groups we'll introduce later are monomial, but it's no longer true in general for solvable Lie groups.

We'll constantly need a property of induced representations known as "induction by stages". The equivalence relation is denoted by the symbol $\simeq$.

Theorem 1.1.1. ([8]) Let $G$ be a locally compact topological group, $H_1, H_2$ two closed subgroups of $G$ such that $H_1 \subset H_2$, and $U$ a unitary representation of $H_1$. Then,

$$\operatorname{ind}_{H_1}^G U \simeq \operatorname{ind}_{H_2}^G \left(\operatorname{ind}_{H_1}^{H_2} U\right).$$

The theorem of imprimitivity of Mackey [41] is very important for the theory of unitary representations of solvable Lie groups. We always assume that $G$ is separable and uniquely consider unitary representations in separable Hilbert spaces. Let $A$ be a closed invariant abelian subgroup of $G$. $G$ acts in $\hat{A}$, the set of unitary characters of $A$ : pour $g \in G, \chi \in \hat{A}$, put $(g \cdot \chi)(a) = \chi(g^{-1}ag)$ $(a \in A)$.

Theorem 1.1.2. (1) Let $G(\chi)$ be the stabilizer of $\chi \in \hat{A}$ in $G$.

(i) Let $\rho$ be an irreducible unitary representation of $G(\chi)$ such that $\rho|_A$ is a multiple of $\chi$, i.e. $\rho|_A = m\chi$ with a certain $m \in \mathbb{N} \cup \{\infty\}$. Then $\operatorname{ind}_{G(\chi)}^G \rho$ is irreducible.

(ii) Let $\rho_1, \rho_2$ be two irreducible unitary representations of $G(\chi)$ such that $\rho_1|_A, \rho_2|_A$ are multiples of $\chi$. Then $\rho_1 \simeq \rho_2$ if and only if $\operatorname{ind}_{G(\chi)}^G \rho_1 \simeq \operatorname{ind}_{G(\chi)}^G \rho_2$.

(2) Suppose that for every $\chi \in \hat{A}$ the orbit $G \cdot \chi$ is locally closed in $\hat{A}$. Then for any irreducible unitary representation $\pi$ of $G$, we see $\pi \simeq \operatorname{ind}_{G(\chi)}^G \rho$ with a certain irreducible unitary representation $\rho$ of $G(\chi)$ such that $\rho|_A$ is a multiple of $\chi$.

We finish this section by a transitivity property of the form $\mu_{G,H}$ (cf. [7], Chap. V). We consider a closed subgroup $K$ of $H$, equipped with a left Haar measure $\mu_K$. Put $\eta = \Delta_{K,G}$. For any $\psi \in \mathcal{K}^\eta(G)$ and any $g \in G$, the function $h \mapsto \psi(gh)\Delta_{H,G}(h)^{-1}$ belongs to $\mathcal{K}^{\Delta_{K,H}}(H)$. We can hence define the function

$$g \mapsto \oint_{H/K} \psi(gh)\Delta_{H,G}^{-1}(h)d\mu_{H,K}(h).$$

This is an element of $\mathcal{K}^\chi(G)$. Returning to the definition of the linear forms $\mu_{G,H}, \mu_{G,K}$ and $\mu_{H,K}$, we prove the following formula :

$$\oint_{G/K} \psi d\mu_{G,K} = \oint_{G/H} d\mu_{G,H}(g) \oint_{H/K} \psi(gh)\chi(h)^{-1}d\mu_{H,K}(h)$$

for every $\psi \in \mathcal{K}^\eta(G)$.

1.2. Theory of Auslander-Kostant

Generalizing the orbit method, Auslander-Kostant [2] developed their theory for solvable Lie groups of type I. Let's first define ingredients of the theory. $\mathfrak{g}^*$ denotes the dual vector space of $\mathfrak{g}$. $G$ acts in $\mathfrak{g}$ by means of the adjoint action and in $\mathfrak{g}^*$ by means of its contra-gradient action :

$$(g{\cdot}f)(X) = (\mathrm{Ad}^*(g){\cdot}f)(X) = f(\mathrm{Ad}(g^{-1})X) \ (g \in G, f \in \mathfrak{g}^*, X \in \mathfrak{g}).$$

The representation of $G$ defined in this manner is called coadjoint representation of $G$. Let $G(f)$ the stabilizer of $f \in \mathfrak{g}^*$ in $G$. Thus the Lie algebra of $G(f)$ is

$$\mathfrak{g}(f) = \{X \in \mathfrak{g}; f([X,Y]) = 0, \forall\, Y \in \mathfrak{g}\}.$$

We define the alternating bilinear form $B_f$ on $\mathfrak{g} \times \mathfrak{g}$ by $B_f(X,Y) = f([X,Y])$. For a vector subspace $\mathfrak{a}$ of $\mathfrak{g}$, we note $f|_{\mathfrak{a}}$ the restriction of $f$ to $\mathfrak{a}$ and set

$$\mathfrak{a}^{\perp,\ \mathfrak{g}^*} = \{f \in \mathfrak{g}^*, f|_{\mathfrak{a}} = 0\},$$
$$\mathfrak{a}^f = \{X \in \mathfrak{g}; B_f(X,Y) = 0, \forall\, Y \in \mathfrak{a}\}.$$

If it doesn't give a confusion, we simply write $\mathfrak{a}^{\perp}$ instead of $\mathfrak{a}^{\perp,\ \mathfrak{g}^*}$. If $\mathfrak{a} \subset \mathfrak{a}^f$, $\mathfrak{a}$ is said to be isotropic (for the form $B_f$). It follows : $\mathfrak{a}$ is a maximal isotropic subspace $\Leftrightarrow \mathfrak{a} = \mathfrak{a}^f \Leftrightarrow \mathfrak{a} \subset \mathfrak{a}^f$ and $\dim \mathfrak{a} = \frac{1}{2}(\dim \mathfrak{g} + \dim \mathfrak{g}(f))$. We'll note $S(f,\mathfrak{g})$ (resp. $M(f,\mathfrak{g})$) the set of subalgebras $\mathfrak{h}$ of $\mathfrak{g}$ such that $\mathfrak{h} \subset \mathfrak{h}^f$ (resp. $\mathfrak{h} = \mathfrak{h}^f$).

Let $\mathfrak{g}_{\mathbb{C}}$ the complexification of $\mathfrak{g}$. We extend by linearity $f, B_f$ on $\mathfrak{g}_{\mathbb{C}}$.

Definition 1.2.1. Let $\mathfrak{p}$ a complex subalgebra of $\mathfrak{g}_{\mathbb{C}}$. We say $\mathfrak{p}$ is a polarization of $G$ at $f \in \mathfrak{g}^*$, if $\mathfrak{p}$ verifies the following conditions :

1) $\mathfrak{p}$ is maximal isotropic subspace for the form $B_f$ ;
2) $\mathfrak{p} + \overline{\mathfrak{p}}$ is a subalgebra of $\mathfrak{g}_{\mathbb{C}}$ ;
3) $\mathfrak{p}$ is stable under $\mathrm{Ad}G(f)$.

When we say $\mathfrak{p}$ is a polarization of $\mathfrak{g}$, it means that $\mathfrak{p}$ is a polarization of the connected and simply connected Lie group corresponding to $\mathfrak{g}$. We note $P(f,G)$ the set of polarizations of $G$ at $f \in \mathfrak{g}^*$.

Definition 1.2.2. $\mathfrak{p} \in P(f,G)$ being given, we consider the real subalgebras $\mathfrak{d}, \mathfrak{e}$ of $\mathfrak{g}$ defined by

$$\mathfrak{d} = \mathfrak{p} \cap \mathfrak{g},\ \ \mathfrak{e} = (\mathfrak{p} + \bar{\mathfrak{p}}) \cap \mathfrak{g}.$$

We easily see $\mathfrak{d}_{\mathbb{C}} = \mathfrak{p} \cap \bar{\mathfrak{p}}$, $\mathfrak{e}_{\mathbb{C}} = \mathfrak{p} + \bar{\mathfrak{p}}$ and $\mathfrak{d} = \mathfrak{e}^f$. It follows that $B_f$ induces the non-degenerate bilinear form $\hat{B}_f$ on the quotient vector space $\mathfrak{e}/\mathfrak{d}$. Moreover,

$$(\mathfrak{e}/\mathfrak{d})_{\mathbb{C}} \simeq \mathfrak{e}_{\mathbb{C}}/\mathfrak{d}_{\mathbb{C}} = (\mathfrak{p} + \bar{\mathfrak{p}})\,/\mathfrak{p} \cap \bar{\mathfrak{p}} = \mathfrak{p}/\mathfrak{d}_{\mathbb{C}} \oplus \bar{\mathfrak{p}}/\mathfrak{d}_{\mathbb{C}}.$$

Definition 1.2.3. We define the linear operator $J$ on $(\mathfrak{e}/\mathfrak{d})_{\mathbb{C}}$ as follows : $J(X) = -iX, i = \sqrt{-1}$, if $X \in \mathfrak{p}/\mathfrak{d}_{\mathbb{C}}$ and $J(X) = iX$ if $X \in \bar{\mathfrak{p}}/\mathfrak{d}_{\mathbb{C}}$.

Then $J$ defines a real endomorphism such that $J^2 = -1$, namely a canonical complex structure on $\mathfrak{e}/\mathfrak{d}$. For $u \in \mathfrak{e}/\mathfrak{d}$,

$$u + iJu \in \mathfrak{p}/\mathfrak{d}_{\mathbb{C}},\ \ u - iJu \in \bar{\mathfrak{p}}/\mathfrak{d}_{\mathbb{C}}.$$

Definition 1.2.4. We define the bilinear form $S_f$ on $\mathfrak{e}/\mathfrak{d}$ by

$$S_f(u,v) = \hat{B}_f(u,Jv),\ u,v \in \mathfrak{e}/\mathfrak{d}.$$

Proposition 1.2.5. $S_f$ is a non-degenerate symmetric bilinear form on $\mathfrak{e}/\mathfrak{d}$, and $J$ keeps $\hat{B}_f, S_f$ invariant, i.e.

$$\hat{B}_f(Ju,Jv) = \hat{B}_f(u,v),\ S_f(Ju,Jv) = S_f(u,v).$$

Proof. It's evident that $\mathfrak{p}/\mathfrak{d}_{\mathbb{C}}$ is isotropic for the form $\hat{B}_f$, and we have for $u,v \in \mathfrak{e}/\mathfrak{d}$

$$0 = \hat{B}_f(u+iJu, v+iJv) = \hat{B}_f(u,v) - \hat{B}_f(Ju,Jv) + i\left(\hat{B}_f(Ju,v) + \hat{B}_f(u,Jv)\right).$$

The imaginary part of this equation gives us

$$\hat{B}_f(u,Jv) = -\hat{B}_f(Ju,v) = \hat{B}_f(v,Ju). \tag{1.2.1}$$

Hence $S_f(u,v) = S_f(v,u)$, namely that $S_f$ is symmetric. Since $\hat{B}_f, J$ are non-degenerate, $S_f$ is also non-degenerate. Using (1.2.1) and $J^2 = -1$,

$$\hat{B}_f(Ju,Jv) = -\hat{B}_f(u,J(Jv)) = \hat{B}_f(u,v),$$

$$\begin{aligned} S_f(Ju,Jv) &= \hat{B}_f(Ju,J(Jv)) = -\hat{B}_f(Ju,v) \\ &= \hat{B}_f(v,Ju) = S_f(v,u) = S_f(u,v). \end{aligned}$$

c.q.f.d.

Definition 1.2.6. We say that $\mathfrak{p} \in P(f,G)$ is positive if the symmetric form $S_f$ is positive definite or if $\mathfrak{e}/\mathfrak{d} = \{0\}$. In particular, $\mathfrak{p}$ is said to be real if $\mathfrak{p} = \bar{\mathfrak{p}}$.

We designate by $P^+(f,G)$ the set of positive polarizations of $G$ at $f$. We take $\mathfrak{p} \in P(f,G)$ and define the subalgebras $\mathfrak{d}, \mathfrak{e}$ as before. Let $D_0$ (resp. $E_0$) be connected Lie subgroup of $G$ corresponding to $\mathfrak{d}$ (resp. $\mathfrak{e}$). Because $\mathfrak{p}$ is stable by $\mathrm{Ad}G(f)$,

$$D = G(f)D_0,\ E = G(f)E_0$$

are two subgroups of $G$.

Proposition 1.2.7. $D$, $D_0$ are closed in $G$. Moreover, $D_0$ is the connected component of the unit element of $D$ and $\mathfrak{d}$ is the Lie algebra of $D$.

Proof. $\mathfrak{d}$ and $\mathfrak{e}$ being mutually the orthogonal complement of each other with respect to $B_f$, we have for $X \in \mathfrak{g}$ :

$$X \in \mathfrak{d} \Leftrightarrow X{\cdot}f(Y) = B_f(Y,X) = 0\ (\forall\ Y \in \mathfrak{e}).$$

Taking the image of the exponential mapping, we have for $a \in D_0$ :

$$(a{\cdot}f - f)(Y) = 0\ (\forall\ Y \in \mathfrak{e}). \tag{1.2.2}$$

This means that the equation (1.2.2) remains valid for all $a \in \overline{D_0}$. But this fact implies in turn that every element $X$ of the Lie algebra of $\overline{D_0}$ verifies

$$X{\cdot}f(Y) = 0\ (\forall\ Y \in \mathfrak{e}).$$

Thus, $X \in \mathfrak{d}$, namely that $D_0 = \overline{D_0}$.

Let's repeat the same argument to prove the rest of the proposition. Let $D_1$ be the connected component of the unit element of $\overline{D} = \overline{G(f)D_0}$. If $a \in D_1$,

$$(a{\cdot}f - f)(Y) = 0 \ (\forall\, Y \in \mathfrak{e}).$$

This implies that the Lie algebra $\mathfrak{d}_1$ of $D_1$ is contained in $\mathfrak{d}$. On the other hand, $D_0 \subset D$ carries $\mathfrak{d} \subset \mathfrak{d}_1$. Consequently, $\mathfrak{d} = \mathfrak{d}_1$ and $D_0 = D_1$. Further, $D_0 \subset D \subset \overline{D}$. In this manner $D$ is a closed subgroup and $D_0$ is the connected component of the unit element of $D$. c.q.f.d.

Now we consider $D$-orbit $D{\cdot}f$ dans $\mathfrak{g}^*$.

Proposition 1.2.8. $D{\cdot}f$ is open in the affine plane $f + \mathfrak{e}^\perp$. Evidently, $D{\cdot}f = D_0{\cdot}f$.

Proof. Let's begin by seeing that $f + \mathfrak{e}^\perp$ is stable under the action of $D$. As $\mathfrak{e}$ is $D$-stable, the same for $\mathfrak{e}^\perp$. Taking $D = D_0G(f)$ into account, we notice that $D{\cdot}f = D_0G(f){\cdot}f = D_0{\cdot}f$. Hence, for $d \in D$ and $\ell \in \mathfrak{e}^\perp$ there exists $a \in D_0$ verifying

$$d{\cdot}(f + \ell) - f = a{\cdot}f - f + d{\cdot}\ell.$$

This means by the relation (1.2.2) that $d{\cdot}(f + \ell) - f \in \mathfrak{e}^\perp$. So, $f + \mathfrak{e}^\perp$ is $D$-stable and $\mathfrak{d}{\cdot}f \subset \mathfrak{e}^\perp$. On the other hand, $\mathfrak{d}{\cdot}f \cong \mathfrak{d}/\mathfrak{g}(f)$ and $\mathfrak{e} = \mathfrak{d}^f$ give

$$\dim \mathfrak{d} + \dim \mathfrak{e} = \dim \mathfrak{g} + \dim \mathfrak{g}(f).$$

We have thus

$$\dim \mathfrak{d}{\cdot}f = \dim(\mathfrak{d}/\mathfrak{g}(f)) = \dim \mathfrak{d} - \dim \mathfrak{g}(f) = \dim \mathfrak{g} - \dim \mathfrak{e} = \dim \mathfrak{e}^\perp.$$

In conclusion, $\mathfrak{d}{\cdot}f = \mathfrak{e}^\perp$. As $\mathfrak{d}{\cdot}f$ is the tangent space at $f$ to $D_0{\cdot}f \subset f + \mathfrak{e}^\perp$, the implicit function theorem assures us that $D{\cdot}f$ is open in $f + \mathfrak{e}^\perp$. c.q.f.d.

Next we'll examine $E$-orbit $E{\cdot}f = E_0{\cdot}f$ in $\mathfrak{g}^*$.

Definition 1.2.9. We say that $\mathfrak{p} \in P(f, G)$ satisfies the strong Pukanszky condition if $E{\cdot}f$ is closed in $\mathfrak{g}^*$.

Remark. When $\mathfrak{p}$ is real, it comes from the proposition 1.2.8 that

$$\mathfrak{p} \text{ satisfies the strong Pukanszky condition} \Leftrightarrow D{\cdot}f = f + \mathfrak{e}^\perp$$

since $D = E$.

Lemma 1.2.10. If $\mathfrak{p} \in P(f, G)$ satisfies the strong Pukanszky condition, $E_0, E$ are closed in $G$ and $E_0$ is the connected component of the unit element of $E$.

Proof. Let $\psi : G \to \mathfrak{g}^*$ be the mapping defined by $\psi(g) = g{\cdot}f$. It's clear that $E = \psi^{-1}(E{\cdot}f) = \psi^{-1}(E_0{\cdot}f)$, from which $E$ is closed in $G$. Let $E_1$ be the connected component of the unit element of $E$. Trivially, $E_0 \subset E_1$. On the other hand, $\psi(E_0) = \psi(E_1)$ and $G(f)_0 \subset E_0$, $G(f)_0$ denoting the connected component of the unit element of $G(f)$. Comparing the dimension, we conclude that $E_0 = E_1$. c.q.f.d.

Proposition 1.2.11. If $\mathfrak{p} \in P(f, G)$ satisfies the strong Pukanszky condition, then $D{\cdot}f = f + \mathfrak{e}^\perp$.

Proof. Let's put $K = \{a \in E_0; a\cdot f \in f + \mathfrak{e}^\perp\}$. It's clear that $K$ is closed in $E_0$, and hence in $G$ because of the lemma 1.2.10. Furthermore, $\mathfrak{e}^\perp$ being stable by $E_0$, $K$ is a subgroup of $E_0$ and $f + \mathfrak{e}^\perp$ is $K$-stable. It's immediate that $D_0 \subset K$. It follows from the proposition 1.2.8 that $D_0\cdot f$ is an open set of $f + \mathfrak{e}^\perp$. Dividing $K$ into the classes by $D_0$, we see that $K\cdot f$ is an open set of $f + \mathfrak{e}^\perp$. On the other hand, the strong Pukanszky condition implies that $K\cdot f = (E_0)\cdot f \cap (f + \mathfrak{e}^\perp)$ is closed in $f + \mathfrak{e}^\perp$. Hence $K\cdot f = f + \mathfrak{e}^\perp$.

Let now $\mathfrak{k}$ be the Lie algebra of $K$. It follows from what has been seen that $B_f(\mathfrak{k}, \mathfrak{e}) = \{0\}$, hence $\mathfrak{k} \subset \mathfrak{d}$. The inclusion $\mathfrak{d} \subset \mathfrak{k}$ being trivial, we have $\mathfrak{k} = \mathfrak{d}$ and $D_0$ is nothing but the connected component of the unit element of $K$. In particular, $D_0$ is invariant in $K$. For $\ell \in \mathfrak{e}^\perp$, let's write $f + \ell = k\cdot f$ with $k \in K$. Then,

$$D_0\cdot(f + \ell) = D_0\cdot(k\cdot f) = k\cdot(D_0\cdot f).$$

According to the proposition 1.2.8 $D_0\cdot f$ being an open set of $f + \mathfrak{e}^\perp$, each orbit $D_0\cdot(f + \ell)$ turns out to be open in $f + \mathfrak{e}^\perp$. Thus, $D_0\cdot f = f + \mathfrak{e}^\perp$, and finally $D\cdot f = f + \mathfrak{e}^\perp$ because $D = D_0 G(f)$. c.q.f.d.

Lemma 1.2.12. Suppose that $\mathfrak{p} \in P(f, G)$ satisfies the strong Pukanszky condition. Let's designate by $G(f)_0$ the connected component of the unit element of $G(f)$. Then $D_0 \cap G(f) = G(f)_0$. Let $D_1$ be the simply connected covering group of $D_0$ and $\tau : D_1 \to D_0$ the canonical projection. Then, $\tau^{-1}(G(f)_0) = G(f)_1$ is connected.

Proof. Taking $D\cdot f = D_0\cdot f$ into account, we have $D\cdot f \simeq D_0/(D_0 \cap G(f))$. Because $G(f)_0 \subset D_0$, $G(f)_0$ is the connected component of the unit element of $D_0 \cap G(f)$. $D_0\cdot f$ being simply connected by the proposition 1.2.11, $D_0 \cap G(f)$ turns out to be connected. Hence, $D_0 \cap G(f) = G(f)_0$ and

$$D_1/G(f)_1 \simeq D_0/G(f)_0 = D_0/D_0 \cap G(f) = D\cdot f = D_0\cdot f.$$

From the fact that $D_0\cdot f$ is simply connected, $G(f)_1 = \tau^{-1}(G(f)_0)$ is connected. c.q.f.d.

The definition of $\mathfrak{g}(f)$ implies that the restriction $f|_{\mathfrak{g}(f)}$ of $f$ to $\mathfrak{g}(f)$ supplies a homomorphism of Lie algebras $\mathfrak{g}(f) \to \mathbb{R}$.

Definition 1.2.13. We say that $f \in \mathfrak{g}^*$ is integral if there exists a homomorphism $\eta_f : G(f) \to \mathbb{T}$ such that $d\eta_f = if|_{\mathfrak{g}(f)}$.

We assume hereafter that $f \in \mathfrak{g}^*$ is integral and note $\eta_f$ the associated character of $G(f)$. It follows from the relation $f([\mathfrak{d}, \mathfrak{e}]) = \{0\}$ that $f|_{\mathfrak{d}}$ supplies a homomorphism $\mathfrak{d} \to \mathbb{R}$.

Proposition 1.2.14. When $\mathfrak{p} \in P(f, G)$ satisfies the strong Pukanszky condition, $\eta_f$ uniquely extends into a character $\chi_f : D \to \mathbb{T}$ such that $d\chi_f = if|_{\mathfrak{d}}$.

Proof. Let's borrow the notations from the lemma 1.2.12. Since $f([\mathfrak{d}, \mathfrak{d}]) = \{0\}$, there exists the unique character $\chi_f^1 : D_1 \to \mathbb{T}$ such that $d\chi_f^1 = if|_{\mathfrak{d}}$. When $\mathfrak{p} \in P(f, G)$ satisfies the strong Pukanszky condition, the lemma 1.2.12 signifies that $G(f)_1$ is connected and that we have

$$\chi_f^1|_{G(f)_1} = \left(\eta_f|_{G(f)_0}\right) \circ \tau.$$

The kernel $K$ of the homomorphism $\tau : D_1 \to D_0$ is contained in $G(f)_1 = \tau^{-1}(G(f)_0)$ and $\chi_f^1|_K$ is trivial. The result is that there exists the unique homomorphism $\chi_f^0 : D_0 \to \mathbb{T}$ such that $\chi_f^1 = \chi_f^0 \circ \tau$. Evidently, $d\chi_f^0 = if|_{\mathfrak{d}}$. Keeping $D_0$ invariant, $G(f)$ acts on the group of the unitary characters. Now, the unitary character of a connected Lie group is determined by its differential. Taking $G(f)\cdot f = f$ into account, it follows that

$$\chi_f^0(udu^{-1}) = \chi_f^0(d) \ (u \in G(f), d \in D_0).$$

Let now $A$ be the semi-direct product of $D_0$ by $G(f)$, and let's define the mapping $\mu_f : A \to \mathbb{T}$ by

$$\mu_f(d, u) = \chi_f^0(d)\eta_f(u) \ (d \in D_0, u \in G(f)).$$

Then, $\mu_f$ is a unitary character of $A$. Next we consider the homomorphism $\sigma$ of $A$ onto $D$ defined by $\sigma(d, u) = du$. It follows from the lemma 1.2.12 that

$$\ker\sigma = \{(u, u^{-1}); u \in G(f) \cap D_0 = G(f)_0\}.$$

As $\chi_f^0$ coincides on $G(f)_0$ with $\eta_f$, the homomorphism $\mu_f$ is trivial on $\ker\sigma$ and induces a unitary character $\chi_f$ of $D$. It's clear that $\chi_f$ possesses the required properties. The uniqueness of $\chi_f$ follows from $D = D_0G(f)$, because it coincides on $G(f)$ with $\eta_f$ and it's determined on $D_0$ by its differential. c.q.f.d.

We now intend to construct a unitary representation of $G$ starting from $\mathfrak{p} \in P(f, G)$ satisfying the strong Pukanszky condition. Since $E = E_0D$, $X = E/D$ is connected. Moreover, the alternating bilinear form $\hat{B}_f$ on $\mathfrak{e}/\mathfrak{d}$ being non-degenerate and $D$-invariant, it induces on $X$ a measure $\mu_X$, invariant under the action of $E$. Let $M(E, \chi_f)$ denote the space of measurable numeric functions $\phi$ verifying the conditions of covariance

$$\phi(ab) = \chi_f(b)^{-1}\phi(a) \ (a \in E, b \in D).$$

We consider the space of functions $\phi \in M(E, \chi_f)$ such that

$$\int_X |\phi|^2 d\mu_X < \infty$$

and its completion $\mathcal{H}(E, \chi_f)$, which is a Hilbert space. In fact, $\mathcal{H}(E, \chi_f)$ is the Hilbert space of the induced representation $\mathrm{ind}_D^E\chi_f$.

Let $C^\infty(E)$ be the space of numeric $C^\infty$ functions on $E$. For $z = x + iy$ $(x, y \in \mathfrak{e})$ and $\psi \in C^\infty(E)$, put $\psi\cdot z = \psi\cdot x + i\psi\cdot y$, where

$$(\psi\cdot x)(a) = \frac{d}{dt}\psi(a\exp(tX))|_{t=0} \ (a \in E).$$

We further set

$$C^\infty(E, f, \mathfrak{p}) = \{\psi \in C^\infty(E); \psi\cdot z = -if(z)\psi, \ z \in \mathfrak{p}\},$$

$$\mathcal{L} = C^\infty(E, f, \mathfrak{p}) \cap M(E, \chi_f), \mathcal{H}(f, \eta_f, \mathfrak{p}, E) = \mathcal{L} \cap \mathcal{H}(E, \chi_f).$$

Proposition 1.2.15. ([2]) The space $\mathcal{H}(f, \eta_f, \mathfrak{p}, E)$ is a closed subspace of the Hilbert space $\mathcal{H}(E, \chi_f)$.

Since $\mathcal{H}(f, \eta_f, \mathfrak{p}, E)$ is stable under the action of $\operatorname{ind}_D^E \chi_f$, it supplies us a sub-representation, written $\operatorname{ind}_D^E(\eta_f, \mathfrak{p})$, of $\operatorname{ind}_D^E \chi_f$. We finally put

$$\rho(f, \eta_f, \mathfrak{p}, G) = \operatorname{ind}_E^G \left(\operatorname{ind}_D^E(\eta_f, \mathfrak{p})\right),$$

which is a sub-representation of $\operatorname{ind}_D^G \chi_f$, and call it holomorphically induced representation. We'll note $\mathcal{H}(f, \mathfrak{p}, G)$ the Hilbert space of $\operatorname{ind}_D^G \chi_f$, and $\mathcal{H}(f, \eta_f, \mathfrak{p}, G)$ its closed subspace corresponding to $\rho(f, \eta_f, \mathfrak{p}, G)$.

Let's consider an exact sequence of Lie groups :

$$1 \to N \to G \xrightarrow{p} \tilde{G} \to 1.$$

Let $\mathfrak{n}$ (resp. $\mathfrak{g}, \tilde{\mathfrak{g}}$) be the Lie algebra of $N$ (resp. $G, \tilde{G}$) and $dp : \mathfrak{g} \to \tilde{\mathfrak{g}}$ the differential of $p$. We designate by the same notation the linear extension of $dp$ on $\mathfrak{g}_{\mathbb{C}}$.

Proposition 1.2.16. We assume $\tilde{f} \in \tilde{\mathfrak{g}}^*$ integral and note $\eta_{\tilde{f}}$ the associated unitary character of $\tilde{G}(\tilde{f})$. Let's suppose that $\tilde{\mathfrak{p}} \in P(\tilde{f}, \tilde{G})$ satisfies the strong Pukanszky condition, and set $f = \tilde{f} \circ dp \in \mathfrak{g}^*, \mathfrak{p} = p^{-1}(\tilde{\mathfrak{p}})$. Then $f$ is integral, $G(f) = p^{-1}\left(\tilde{G}(\tilde{f})\right)$, and the character $\eta_f$ of $G(f)$ defined by $\eta_f = \eta_{\tilde{f}} \circ p$ is the one which corresponds to $f$. Moreover, $\mathfrak{p}$ is a polarization of $G$ at $f$ satisfying the strong Pukanszky condition and

$$\rho(f, \eta_f, \mathfrak{p}, G) \simeq \rho(\tilde{f}, \eta_{\tilde{f}}, \tilde{\mathfrak{p}}, \tilde{G}) \circ p.$$

Proof. It suffices to check various definitions remarking the following facts. First, $\tilde{\mathfrak{g}}^*$ is identified to $\mathfrak{n}^{\perp} \subset \mathfrak{g}^*$ and $\mathfrak{p}$ satisfies the strong Pukanszky condition. Next, $D = p^{-1}(\tilde{D}), \chi_f = \chi_{\tilde{f}} \circ p, E = p^{-1}(\tilde{E})$, and $p$ induces an isomorphism between $E/D$ (resp. $G/D, G/E$) and $\tilde{E}/\tilde{D}$ (resp. $\tilde{G}/\tilde{D}, \tilde{G}/\tilde{E}$). Finally, we have

$$\mathcal{H}(\tilde{f}, \eta_{\tilde{f}}, \tilde{\mathfrak{p}}, \tilde{G}) \circ p = \mathcal{H}(f, \eta_f, \mathfrak{p}, G).$$

c.q.f.d.

We mention some importants results obtained in Auslander-Kostant [2]. Let $G$ be a connected and simply connected solvable Lie group with Lie algebra $\mathfrak{g}$. Let $\mathfrak{n}$ be an nilpotent ideal of $\mathfrak{g}$ which contains $[\mathfrak{g}, \mathfrak{g}]$, and $N$ the connected Lie subgroup of $G$ corresponding to $\mathfrak{n}$. Let $f \in \mathfrak{g}^*$ and $f_0 = f|_{\mathfrak{n}}$. As $\mathfrak{n}$ is stable by $\operatorname{Ad}(G)$, $G$ acts in $\mathfrak{n}^*$. We note $G(f_0)$ the stabilizer of $f_0$ in $G$.

Definition 1.2.17. Let $\mathfrak{p} \in P(f, G)$. We say that $\mathfrak{p}$ is $\mathfrak{n}$-admissible if $\mathfrak{p} \cap \mathfrak{n}_{\mathbb{C}} \in P(f_0, N)$. If further $\mathfrak{p} \cap \mathfrak{n}_{\mathbb{C}}$ is stable by $\operatorname{Ad}(G(f_0))$, we say that $\mathfrak{p}$ is strongly $\mathfrak{n}$-admissible.

Remark. If $\mathfrak{p} \cap \mathfrak{n}_{\mathbb{C}}$ is a maximal isotropic subspace for $B_{f_0}$, then $\mathfrak{p}$ becomes $\mathfrak{n}$-admissible.

We obtain in these circumstances the following two theorems.

Theorem 1.2.18. For all $f \in \mathfrak{g}^*$ there exists $\mathfrak{p} \in P^+(f, G)$ which is strongly $\mathfrak{n}$-admissible. Moreover, if $\mathfrak{p} \in P(f, G)$ is $\mathfrak{n}$-admissible, $\mathfrak{p}$ satisfies the strong Pukanszky condition.

Theorem 1.2.19. Suppose that $f \in \mathfrak{g}^*$ is integral and that $\mathfrak{p} \in P^+(f, G)$ is strongly $\mathfrak{n}$-admissible. Then,

$$\mathcal{H}(f, \eta_f, \mathfrak{p}, G) \neq \{0\}$$

and $\rho(f, \eta_f, \mathfrak{p}, G)$ gives an irreducible unitary representation of $G$ whose equivalence class does not depend on $\mathfrak{p}$ nor on $\mathfrak{n}$.

1.3. Exponential group

In this section we'll give the definition of exponential groups and explain the orbit method for them in its general lines. When we simply say a Lie algebra, it means a real Lie algebra of finite dimension. Let $\mathfrak{g}$ be a solvable Lie algebra which acts on a real vector space $V$ of dimension $n$. As $\mathfrak{g}_{\mathbb{C}}$-module, $V_{\mathbb{C}}$ possesses a Jordan-Hölder sequence :

$$\{0\} = V_0 \subset V_1 \subset \cdots \subset V_{n-1} \subset V_n = V_{\mathbb{C}},\ \dim_{\mathbb{C}} V_j = j\ (0 \leq j \leq n).$$

The action of $\mathfrak{g}_{\mathbb{C}}$ on $V_j/V_{j-1}$ $(1 \leq j \leq n)$ yields a linear form $\lambda_j$ on $\mathfrak{g}_{\mathbb{C}}$, and except their order these linear forms don't depend on the choice of the Jordan-Hölder sequence.

Definition 1.3.1 The restriction of $\lambda_j$ $(1 \leq j \leq n)$ on $\mathfrak{g}$ is called weight of $\mathfrak{g}$ in $V$, and the weight of adjoint representation is called root of $\mathfrak{g}$.

Definition 1.3.2. Let $G$ be a connected and simply connected Lie group with Lie algebra $\mathfrak{g}$. When the exponential map $\exp : \mathfrak{g} \to G$ is surjective, we call $G$ exponential group.

Theorem 1.3.3. ([15]) Let $G$ be a connected and simply connected Lie group with Lie algebra $\mathfrak{g}$. The following assertions are equivalent :

(1) $G$ is an exponential group;
(2) the exponential map is injective;
(3) the exponential map is a diffeomorphism;
(4) each root of $\mathfrak{g}$ is written as $X \to \lambda(X)(1 + i\alpha)$ where $\lambda \in \mathfrak{g}^*$ and $\alpha \in \mathbb{R}$;
(5) $\mathfrak{g}$ doesn't possess any root which admits a non zero purely imaginary value.

Definition 1.3.4. When a Lie algebra $\mathfrak{g}$ satisfies (5) of the theorem 1.3.3, $\mathfrak{g}$ is called exponential Lie algebra.

Example 1.3.5.

(i) A connected and simply connected nilpotent Lie group is an exponential group.

(ii) Let $\mathfrak{g}$ be a Lie algebra such that $\dim \mathfrak{g} = n$. When there exists a sequence of ideals

$$\{0\} = \mathfrak{g}_0 \subset \mathfrak{g}_1 \subset \cdots \subset \mathfrak{g}_{n-1} \subset \mathfrak{g}_n = \mathfrak{g},\ \dim \mathfrak{g}_j = j\ (0 \leq j \leq n),$$

we say that $\mathfrak{g}$ is completely solvable. A completely solvable Lie algebra is exponential.

(iii) Let $\mathfrak{g}_3 = \langle X, P, Q \rangle_{\mathbb{R}}; [X, P] = -Q, [X, Q] = P$. $\mathfrak{g}_3$ is not exponential.

(iv) Let $\mathfrak{g}_4 = \langle X, P, Q, E \rangle_{\mathbb{R}}; [X, P] = -Q, [X, Q] = P, [P, Q] = E$. $\mathfrak{g}_4$ is not exponential.

An exponential group $G = \exp \mathfrak{g}$ enjoys the following property : let $\mathfrak{h}$ be a Lie subalgebra of $\mathfrak{g}$. There exists a basis $\{X_1, \ldots, X_p\}$ of a supplementary vector subspace to $\mathfrak{h}$ in $\mathfrak{g}$ such that, if we put $g_j(t) = \exp(tX_j)$ $(t \in \mathbb{R})$, the mapping

$$(t_1, \ldots, t_p, X) \to g_1(t_1)\cdots g_p(t_p) \exp X$$

is a diffeomorphism from $\mathbb{R}^p \times \mathfrak{h}$ onto $G$. Such a basis will be said to be coexponential to $\mathfrak{h}$ in $\mathfrak{g}$, and constructed as follows. We first remark that if $\mathfrak{k}$ is a subalgebra containing $\mathfrak{h}$, the reunion of a coexponential basis to $\mathfrak{k}$ in $\mathfrak{g}$ and a coexponential basis to $\mathfrak{h}$ in $\mathfrak{k}$ make together a coexponential basis to $\mathfrak{h}$ in $\mathfrak{g}$. It suffices hence to examine the following cases :

(1) $\mathfrak{h}$ is an ideal of codimension 1 in $\mathfrak{g}$ ;

(2) $\mathfrak{h}$ isn't an ideal, and $\mathfrak{g}/\mathfrak{h}$ is an irreducible $\mathfrak{h}$-module.

In the case (1), any element of $\mathfrak{g}$ not belonging to $\mathfrak{h}$ forms a coexponential basis. It follows that if $\mathfrak{g}$ is nilpotent, we construct a coexponential basis to any subalgebra by iteration of the case (1), because every subalgebra of codimension 1 is an ideal.

We proceed to the case (2). Let $\mathfrak{n}$ be the maximal nilpotent ideal of $\mathfrak{g}$. As $\mathfrak{n} \not\subset \mathfrak{h}$, $\mathfrak{n}/(\mathfrak{h} \cap \mathfrak{n})$ is identified to a non trivial sub-$\mathfrak{h}$-module of $\mathfrak{g}/\mathfrak{h}$, hence to $\mathfrak{g}/\mathfrak{h}$. A subspace containing $\mathfrak{n}$ is an ideal of $\mathfrak{g}$ hence we construct a coexponential basis to $\mathfrak{n}$ in $\mathfrak{g}$ by iterating (1). We can suppose that it is formed by elements of $\mathfrak{h}$. Applying the case (1), we construct a coexponential basis $\{X\}$ ou $\{X_1, X_2\}$, in accordance with the dimension, to $\mathfrak{h} \cap \mathfrak{n}$ in $\mathfrak{n}$. It is then clear that this is also a coexponential basis to $\mathfrak{h}$ in $\mathfrak{g}$.

Definition 1.3.6. Let $G$ be a Lie group with Lie algebra $\mathfrak{g}$ and $V$ a $G$-module or $\mathfrak{g}$-module. We say that $V$ is of exponential type if every weight of $\mathfrak{g}$ in $V$ is written as

$$X \mapsto \lambda(X)(1 + i\alpha)$$

with $\alpha \in \mathbb{R}, \lambda \in \mathfrak{g}^*$.

Theorem 1.3.7. Let $G = \exp \mathfrak{g}$ be an exponential group (with Lie algebra $\mathfrak{g}$) and $V$ a $G$-module of exponential type. Then the stabilizer in $G$ of any point in $V$ is connected.

Proof. Let's designate by $\rho$ the action of $G$ in $V$. Let $X \in \mathfrak{g}, v \in V$ such that $\rho(\exp X)v = v$. The set of $t \in \mathbb{R}$ verifying $\rho(\exp(tX))v = v$ is a closed subgroup of $\mathbb{R}$. If it is discrete, let $t_0$ be its smallest positive element. We have :

$$\rho\left(\exp(\frac{t_0}{2}X)\right)\left(\rho\left(\exp(\frac{t_0}{2}X)\right)v - v\right) = -\left(\rho\left(\exp(\frac{t_0}{2}X)\right)v - v\right) \neq 0,$$

from which $d\rho(\frac{t_0}{2}X)$ has an eigen value $in\pi$ with a non zero integer $n$. c.q.f.d.

We can parametrize the orbits in this situation to find the following lemma.

Lemma 1.3.8. Let $G$ be an exponential group and $V$ a $G$-module of exponential type. We note $G(v)$ le stabilizer in $G$ of $v \in V$ and equip the orbit $G{\cdot}v$ with the induced topology from that of $V$. Then $G{\cdot}v$ is homeomorphic to the homogeneous space $G/G(v)$. They are homeomorphic to $\mathbb{R}^d$ for a certain non negative integer $d$.

Hereafter in this section we designate by $G = \exp \mathfrak{g}$ an exponential group with Lie algebra $\mathfrak{g}$. Let $f \in \mathfrak{g}^*$. We note as before $S(f, \mathfrak{g})$ (resp. $M(f, \mathfrak{g})$) the set of subalgebras of $\mathfrak{g}$ which are isotropic (resp. maximal isotropic) subspaces for $B_f$. Now being given $\mathfrak{h} \in S(f, \mathfrak{g})$, we define a unitary character $\chi_f$ of $H = \exp \mathfrak{h}$ by

$$\chi_f(\exp X) = e^{if(X)} \ (\forall\ X \in \mathfrak{h}).$$

Next, put

$$\hat{\rho}(f, \mathfrak{h}, G) = \mathrm{ind}_H^G \chi_f$$

and note $\hat{\mathcal{H}}(f, \mathfrak{h}, G)$ the Hilbert space of $\hat{\rho}(f, \mathfrak{h}, G)$. We finally designate by $I(f, G)$ the set of $\mathfrak{h} \in S(f, \mathfrak{g})$ such that the induced representation $\hat{\rho}(f, \mathfrak{h}, G)$ is irreducible.

Remark.

(i) $G(f)$ being connected by the theorem 1.3.7, it is simply connected. Every $f \in \mathfrak{g}^*$ is integral and the unitary character $\eta_f$ is uniquely determined.

(ii) For $\mathfrak{p} \in P(f, G)$ satisfying the strong Pukanszky condition, $\rho(f, \eta_f, \mathfrak{p}, G)$ (resp. $\mathcal{H}(f, \eta_f, \mathfrak{p}, G)$) defined in the section 1.2 is simply written $\rho(f, \mathfrak{p}, G)$ (resp. $\mathcal{H}(f, \mathfrak{p}, G)$).

(iii) If $\mathfrak{h} \in M(f, G)$, it's trivial that $\mathfrak{h}$ contains $\mathfrak{g}(f)$ and that $\mathfrak{h}_{\mathbb{C}}$ is stable by $\mathrm{Ad}\,(G(f))$. Namely that $\mathfrak{h}_{\mathbb{C}}$ is a real polarization of $G$ at $f$.

(iv) Let $\mathfrak{h} \in M(f, \mathfrak{g})$. When $\mathfrak{h}_{\mathbb{C}} \in P^+(f, G)$ satisfies the strong Pukanszky condition, $\rho(f, \mathfrak{p}_{\mathbb{C}}, G)$ coincides with $\hat{\rho}(f, \mathfrak{h}, G)$.

Theorem 1.3.9. ([7], Chap. VI) Let $G = \exp \mathfrak{g}$ be an exponential group with Lie algebra $\mathfrak{g}$ and $f \in \mathfrak{g}^*$. Then :
(1) $I(f, \mathfrak{g}) \neq \emptyset$ ;
(2) $I(f, \mathfrak{g}) \subset M(f, \mathfrak{g})$ ;
(3) for $\mathfrak{h}_1, \mathfrak{h}_2 \in I(f, \mathfrak{g})$, we have $\hat{\rho}(f, \mathfrak{h}_1, G) \simeq \hat{\rho}(f, \mathfrak{h}_2, G)$.

The assertion (3) of the theorem 1.3.9 permits that the notation $\hat{\rho}(f, \mathfrak{h}, G)$ is simplified to $\hat{\rho}(f)$ ; we often confuse a unitary representation with its equivalence class. Thus, the mapping $f \mapsto \hat{\rho}(f)$ gives a mapping from $\mathfrak{g}^*$ into the unitary dual $\hat{G}$ of $G$, the set of equivalence classes of irreducible unitary representations of $G$. Furthermore, for $\mathfrak{h} \in S(f, \mathfrak{g}), g \in G$, we see $g{\cdot}\mathfrak{h} \in S(g{\cdot}f, \mathfrak{g})$ and $\hat{\rho}(g{\cdot}f, g{\cdot}\mathfrak{h}, G) \simeq \hat{\rho}(f, \mathfrak{h}, G)$. It follows that the mapping described above induces the mapping, noted also $\hat{\rho} = \hat{\rho}_G$ from the space of coadjoint orbits $\mathfrak{g}^*/G$ of $G$ into $\hat{G}$.

Theorem 1.3.10. ([7]) The map $\hat{\rho}$ is a bijection of $\mathfrak{g}^*/G$ onto $\hat{G}$.

Let's make this result more precise. We equip $\hat{G}$ with the Fell topology ([18], [19]). Let $\pi \in \hat{G}$, whose Hilbert space is noted $\mathcal{H}_\pi$. We define in $\hat{G}$ a neighborhood of $\pi$ as follows. We consider finite vectors $v_1, \ldots, v_k$ in $\mathcal{H}_\pi$, a compact subset $C$ of $G$ and a positive number $\epsilon > 0$. The neighborhood $\mathcal{U}(v_1, \ldots, v_k; C; \epsilon)$ of $\pi$ is constituted by $\rho \in \hat{G}$ such that there exists $w_1, \ldots, w_k$ in its Hilbert space $\mathcal{H}_\rho$ verifying

$$|(\pi(g)v_i, v_j) - (\rho(g)w_i, w_j)| < \epsilon,\ g \in C,\ 1 \leq i, j \leq k.$$

Theorem 1.3.11. ([37]) We equip the space of coadjoint orbits $\mathfrak{g}^*/G$ with the quotient topology and the unitary dual $\hat{G}$ with the Fell topology. The Kirillov-Bernat map $\hat{\rho}$ is then homeomorphism.

We simply note $dg$ a left Haar measure on $G$ and introduce the space $L^1(G)$ with respect to $dg$. We define for $\varphi \in L^1(G)$ the operator $\pi(\varphi)$ in $\mathcal{H}_\pi$ by

$$\pi(\varphi) = \int_G \varphi(g)\pi(g)dg.$$

When $G = \exp \mathfrak{g}$ is nilpotent, the bijection $\hat{\rho}$ is obtained via the character formula of Kirillov [36]. $dX$ denoting the Lebesgue measure on $\mathfrak{g}$ and $\mathcal{D}(G)$ the space of $C^\infty$ functions with compact support on $G$, we set, for $\varphi \in \mathcal{D}(G)$,

$$\hat{\varphi}(\ell) = \int_{\mathfrak{g}} \varphi(\exp X)e^{i\ell(X)}dX \ (\ell \in \mathfrak{g}^*).$$

Theorem 1.3.12. Suppose that $G = \exp \mathfrak{g}$ is nilpotent. Let $\pi \in \hat{G}$ and $\Omega(\pi)$ its associated orbit in $\mathfrak{g}^*$. If $\varphi \in \mathcal{D}(G)$, the operator $\pi(\varphi)$ is of trace class. We can normalize the $G$-invariant measure on $\Omega(\pi)$ so that we have the formula

$$\mathrm{Tr}(\pi(\varphi)) = \int_{\Omega(\pi)} \hat{\varphi}(\ell)dv(\ell)$$

for all $\varphi \in \mathcal{D}(G)$.

Contrary to the nilpotent cas, it happens in the exponential case that $I(f, \mathfrak{g}) \neq M(f, \mathfrak{g})$. The set $I(f, \mathfrak{g})$ is characterized by the following theorem.

Theorem 1.3.13. ([7]) Let $G = \exp \mathfrak{g}, f \in \mathfrak{g}^*, \mathfrak{h} \in S(f, \mathfrak{g})$ and $H = \exp \mathfrak{h}$. The following assertions are equivalent :

(1) $H{\cdot}f = f + \mathfrak{h}^\perp$ ;
(2) $f + \mathfrak{h}^\perp \subset G{\cdot}f$ et $\mathfrak{h} \in M(f, \mathfrak{g})$ ;
(3) $\mathfrak{h} \in M(f + \lambda, \mathfrak{g})$ for every $\lambda \in \mathfrak{h}^\perp$ ;
(4) $\mathfrak{h} \in I(f, \mathfrak{g})$.

Definition 1.3.14. When $\mathfrak{h} \in S(f, \mathfrak{g})$ satisfies the assertion (1) of the theorem 1.3.13, we say that $\mathfrak{h}$ verifies the Pukanszky condition.

Remark. $\mathfrak{h}$ verifies the Pukanszky condition if and only if $\mathfrak{h}_\mathbb{C} \in P^+(f, G)$ satisfies the strong Pukanszky condition.

It happens that $M(f, g) = \emptyset$ if $\mathfrak{g}$ is not exponential, what shows the necessity to introduce (complex) polarizations.

Example 1.3.15. Let $\mathfrak{g} = \mathfrak{g}_4 = \langle X, P, Q, E \rangle_\mathbb{R}; [X, P] = -Q, [X, Q] = P, [P, Q] = E$. Take $f = E^* \in \mathfrak{g}^*$. Then $\mathfrak{g}(f) = \mathbb{R}X + \mathbb{R}E$. Since there doesn't exist any subalgebra of $\mathfrak{g}$ having the dimension 3 and containing $\mathfrak{g}(f)$, we conclude that $M(f, \mathfrak{g}) = \emptyset$.

We know well a standard process owing to M. Vergne to construct an element of $I(f,\mathfrak{g})$. Let's consider a good sequence of subalgebras of $\mathfrak{g}$, namely a sequence of subalgebras :

$$\{0\} = \mathfrak{g}_0 \subset \mathfrak{g}_1 \subset \cdots \subset \mathfrak{g}_{n-1} \subset \mathfrak{g}_n = \mathfrak{g},\ \dim(\mathfrak{g}_j/\mathfrak{g}_{j-1}) = 1\ (1 \leq j \leq n)$$

such that, if $\mathfrak{g}_j$ is not an ideal of $\mathfrak{g}$, $\mathfrak{g}_{j-1}$ and $\mathfrak{g}_{j+1}$ are both ideals of $\mathfrak{g}$ and the action of $\mathfrak{g}$ on $\mathfrak{g}_{j+1}/\mathfrak{g}_{j-1}$ is irreducible. Let $f_j = f|_{\mathfrak{g}_j}$ for $1 \leq j \leq n$.

Theorem 1.3.16. ([7], Chap. IV) $\mathfrak{h} = \sum_{j=1}^{n} \mathfrak{g}_j(f_j)$ belongs to $I(f,\mathfrak{g})$.

We call polarizations of Vergne those elements of $I(f,\mathfrak{g})$ constructed by this process.

Concerning the result (3) of the theorem 1.3.9, the explicit construction of an intertwining operator between two monomial representations $\hat{\rho}(f,\mathfrak{h}_1,G)$ and $\hat{\rho}(f,\mathfrak{h}_2,G)$, where $\mathfrak{h}_1,\mathfrak{h}_2 \in I(f,\mathfrak{g})$, appears as a natural question. For every $X \in \mathfrak{h}_1 \cap \mathfrak{h}_2$, we find :

$$\mathrm{Tr}\ \mathrm{ad}_{\mathfrak{h}_1/(\mathfrak{h}_1\cap\mathfrak{h}_2)} X + \mathrm{Tr}\ \mathrm{ad}_{\mathfrak{h}_2/(\mathfrak{h}_1\cap\mathfrak{h}_2)} X = 0,$$

what leads to :

$$\Delta_{H_1,G}(h) = \Delta_{H_2,G}(h)\Delta^2_{H_1\cap H_2,H_2}(h)\ (h \in H_1 \cap H_2).$$

Then, for $\phi \in \mathcal{H}(f,\mathfrak{h}_1,G)$ et $g \in G$, the function $\Phi_g$ on $H_2$ defined by

$$\Phi_g(h) = \phi(gh)\chi_f(h)\Delta^{-1/2}_{H_2,G}(h)$$

verifies la relation

$$\Phi_g(hx) = \Delta_{H_1\cap H_2,H_2}(x)\Phi_g(h)\ (h \in H_2, x \in H_1 \cap H_2).$$

We are thus able to consider the integral :

$$(T_{\mathfrak{h}_2\mathfrak{h}_1}\phi)(g) = \oint_{H_2/H_1\cap H_2} \phi(gh)\chi_f(h)\Delta^{-1/2}_{H_2,G}(h)d\nu(h)\ (g \in G). \qquad (1.3.1),$$

where $\nu = \mu_{H_2,H_1\cap H_2}$.

At least on formal level, it's clear that the function $T_{\mathfrak{h}_2\mathfrak{h}_1}\phi$ verifies the covariance condition to belong to $\mathcal{H}(f,\mathfrak{h}_2,G)$ and that the operator $T_{\mathfrak{h}_2\mathfrak{h}_1}$ commutes with the action of $G$ by left translation. Moreover, we have recently proved :

Theorem 1.3.17. Let $G$ be an exponential group, $f \in \mathfrak{g}^*$, $\mathfrak{h}_j \in I(f,\mathfrak{g})$ and $H_j = \exp\mathfrak{h}_j$ $(j = 1,2)$. Then the product set $H_2H_1$ is closed in $G$.

Thus the integrale (1.3.1) is convergent for all continuous functions $\phi$ with compact support modulo $H_1$. We are working to show that $T_{\mathfrak{h}_2\mathfrak{h}_1}$ supplies a true intertwining operator between $\hat{\rho}(f,\mathfrak{h}_1,G)$ and $\hat{\rho}(f,\mathfrak{h}_2,G)$.

Before getting the theorem 1.3.17, we already got the following. If $\mathfrak{h}_1$ or $\mathfrak{h}_2$ is a polarization of Vergne, we verify by taking a convenient coexponential basis to $\mathfrak{h}_1$ in $\mathfrak{g}$ that $H_2H_1$ is closed in $G$, and using the transitivity of forms $\mu_{\cdot,\cdot}$ that the operator $T_{\mathfrak{h}_2\mathfrak{h}_1}$ supplies a true intertwining operator.

Let always $f \in \mathfrak{g}^*$. We consider three lagrangian subspaces $W_j (1 \leq j \leq 3)$ of $\mathfrak{g}$ for the bilinear form $B_f$ and define, as Kashiwara, a quadratic form $Q$ on $W_1 \oplus W_2 \oplus W_3$ by the formula :

$$Q(X_1, X_2, X_3) = f([X_1, X_2]) + f([X_2, X_3]) + f([X_3, X_1]).$$

The index of the quadratic form $Q$ is called Maslov index of the spaces $W_j$ and is noted $\tau(W_1, W_2, W_3)$. Here are the principal properties of this index.

Lemma 1.3.18. ([38], [39]) Let's write $\tau_{ijk}$ instead of $\tau(W_i, W_j, W_k)$.

(a) $\tau_{123} = -\tau_{213} = -\tau_{132}$.

(b) $\tau_{234} - \tau_{134} + \tau_{124} - \tau_{123} = 0$.

(c) If $\mathfrak{p}$ is an isotropic subspace for $B_f$ of $\mathfrak{g}$ containing $\mathfrak{g}(f)$ and if $W$ is a lagrangian subspace of $\mathfrak{g}$, then $W^{\mathfrak{p}} = (W \cap \mathfrak{p}^f) + \mathfrak{p}$ is lagrangian. Moreover, if $\mathfrak{p}$ is contained in $W_1 \cap W_2 + W_2 \cap W_3 + W_3 \cap W_1$, we have $\tau_{123} = \tau(W_1^{\mathfrak{p}}, W_2^{\mathfrak{p}}, W_3^{\mathfrak{p}})$.

Making intervene a polarization of Vergne $\mathfrak{h}_0$ at $f \in \mathfrak{g}^*$, we set

$$T'_{\mathfrak{h}_2\mathfrak{h}_1} = e^{\frac{i\pi}{4}\tau(\mathfrak{h}_1,\mathfrak{h}_0,\mathfrak{h}_2)} T_{\mathfrak{h}_2\mathfrak{h}_0} \circ T_{\mathfrak{h}_0\mathfrak{h}_1}.$$

Theorem 1.3.19. ([1]) The intertwining operator $T'_{\mathfrak{h}_2\mathfrak{h}_1}$ doesn't depend on the choice of $\mathfrak{h}_0$ and verifies the composition formula :

$$T'_{\mathfrak{h}_1\mathfrak{h}_3} \circ T'_{\mathfrak{h}_3\mathfrak{h}_2} \circ T'_{\mathfrak{h}_2\mathfrak{h}_1} = e^{\frac{i\pi}{4}\tau(\mathfrak{h}_3,\mathfrak{h}_2,\mathfrak{h}_1)}.$$

Moreover, $T'_{\mathfrak{h}_2\mathfrak{h}_1}$ coincides with $T_{\mathfrak{h}_2\mathfrak{h}_1}$ if at least $\mathfrak{h}_1$ or $\mathfrak{h}_2$ is of Vergne.

Now the theorem 1.3.17 in hand, hopefully we don't need to make intervene the third polarization which is of Vergne.

## §2. Disintegration

Let always $G = \exp \mathfrak{g}$ be an exponential group with Lie algebra $\mathfrak{g}$. It's well known that there exists a strong duality between the induction and the restriction of representations. In this chapter we'll study their disintegration into irreducibles in order to establish the Frobenius reciprocity.

### 2.1. Monomial representations

Let's start by a simple lemma.

Lemma 2.1.1. ([45]) Let $V$ be a real vector space of finite dimension, where $G$ acts by a representation of exponential type. Let $v$ be a non zero vector of $V$ such that we have $g{\cdot}v = v$ for any $g \in G$. We consider, for arbitrarily fixed $x \in V$, the line $L_x = x + \mathbb{R}v$. Then, there happens two possibilities :

$$\text{either } L_x \cap G{\cdot}x = \{x\} \text{ or } L_x \cap G{\cdot}x = L_x.$$

In other words, the line passing $x$ and having the direction of the invariant vector $v$ encounters the orbit $G{\cdot}x$ at only one point, unless completely contained in the orbit.

Proof. Note $V_0$ the subspace of $V$ generated by $v$, $\bar{V}$ the quotient space $V/V_0$ and $p : V \to \bar{V}$ the projection. The representation of $G$ on $\bar{V}$, obtained by passing to the quotient, is evidently exponential type. Then, in order that we have $g{\cdot}x \in L_x$ $(g \in G)$, it's necessary and sufficient that $g$ belongs to $G(p(x))$. On the other hand, writing $g{\cdot}x = x + \lambda(g)v$, we immediately see that $\lambda$ define a homomorphism of $G(p(x))$ into $\mathbb{R}$. Since $G(p(x))$ is connected by the theorem 1.3.7, we have either $\lambda \equiv 0$ or that the image of $\lambda$ coincides with whole $\mathbb{R}$. From this it suffices to observe that the common part of $L_x$ and $G{\cdot}x$ is nothing but the set $\{g{\cdot}x; g \in G(p(x))\}$. c.q.f.d.

Definition 2.1.2. In the situation of the above lemma, the orbit $G{\cdot}x$ is said to be saturated in the direction of $v$ if $L_x \subset G{\cdot}x$, in the other case $G{\cdot}x$ is said to be non-saturated.

When there exists an ideal $\mathfrak{g}_0$ of $\mathfrak{g}$ such that $\dim \mathfrak{g}/\mathfrak{g}_0 = 1$, a linear form $\ell \in \mathfrak{g}^*$ verifying $\ell|_{\mathfrak{g}_0} = 0$ is an invariant vector for the coadjoint representation of $G$. Let $\mathfrak{g} = \mathbb{R}X + \mathfrak{g}_0$, $p$ the projection of $\mathfrak{g}^*$ onto $\mathfrak{g}_0^*$ and $G_0 = \exp \mathfrak{g}_0$. The following lemma is easily seen from the definition of the radical of an alternating bilinear form.

Lemma 2.1.3. Let $\ell \in \mathfrak{g}^*$ and $\ell_0 = p(\ell)$. If $\mathfrak{g}(\ell) \subset \mathfrak{g}_0$, then $\mathfrak{g}(\ell) \subset \mathfrak{g}_0(\ell_0)$ and $\dim \mathfrak{g}_0(\ell_0) = \dim \mathfrak{g}(\ell) + 1$. If $\mathfrak{g}(\ell) \not\subset \mathfrak{g}_0$, then $\mathfrak{g}_0(\ell_0) \subset \mathfrak{g}(\ell)$ and $\dim \mathfrak{g}(\ell) = \dim \mathfrak{g}_0(\ell_0) + 1$.

Lemma 2.1.4. Let $\ell \in \mathfrak{g}^*, \ell_0 = p(\ell)$ and $\Omega = G{\cdot}\ell$.

(1) If the orbit $\Omega$ is saturated in the direction $\mathfrak{g}_0^\perp$, there exists a family $\{\omega_s\}_{s\in\mathbb{R}}$ of $G_0$-orbits in $\mathfrak{g}_0^*$ such that $p(\Omega) = \cup_{s\in\mathbb{R}}\omega_s$, and $\exp(tX)\cdot\omega_s = \omega_{s+t}$. Moreover, $G(\ell_0) \subset G_0$.

(2) If the orbit $\Omega$ is non-saturated in the direction $\mathfrak{g}_0^\perp, p(\Omega) = G_0{\cdot}\ell_0$.

Proof. (1) We have $G = \exp(\mathbb{R}X){\cdot}G_0 = G_0{\cdot}\exp(\mathbb{R}X)$. Put $\omega_0 = G_0{\cdot}\ell_0$. Then $\omega_0 \subset p(\Omega)$ and we immediately see that, for every $t \in \mathbb{R}$, $\exp(tX){\cdot}\omega_0$ is a $G_0$-orbit which is contained in $p(\Omega)$. Put $\omega_t = \exp(tX){\cdot}\omega_0$. Because $p(\Omega) = p(G{\cdot}\ell) = G{\cdot}\ell_0 = \exp(\mathbb{R}X){\cdot}\omega_0$, the reunion of $\{\omega_t\}_{t\in\mathbb{R}}$ is equal to $p(\Omega)$. By definition of $\omega_s$ we have $\exp(tX){\cdot}\omega_s = \omega_{s+t}$. Further $\omega_s = \omega_t$ if and only if $s = t$. In fact, if $\exp(sX){\cdot}\ell_0 \in \omega_t$ we have $\exp(sX){\cdot}\ell_0 = \exp(tX){\cdot}g_0{\cdot}\ell_0$ where $g_0 \in G_0$. Hence we have $(\exp(t-s)X)\cdot g_0 \in G(\ell_0)$. If we show that $G(\ell_0) \subset G_0$, we would have $\exp((t-s)X) \in G_0$ hence $s = t$. Let's verify hence that $G(\ell_0) \subset G_0$. It suffices for this to show that $\mathfrak{g}(\ell_0) \subset \mathfrak{g}_0$. The lemma 2.1.3 assures that there exists $X_1 \in \mathfrak{g}_0(\ell_0)\backslash\mathfrak{g}(\ell)$, from which $\lambda = X_1{\cdot}\ell \neq 0$ belongs to $\mathfrak{g}_0^\perp$. Let $Y$ be an arbitrary element of $\mathfrak{g}(\ell_0)$. Then $Y{\cdot}\ell \in \mathfrak{g}_0^\perp$ hence $Y{\cdot}\ell = (tX_1){\cdot}\ell$ for a certain $t \in \mathbb{R}$. It follows that $Y - tX_1 \in \mathfrak{g}(\ell) \subset \mathfrak{g}_0$ hence $Y \in \mathfrak{g}_0$ since $X_1 \in \mathfrak{g}_0$.

(2) As $\mathfrak{g}(\ell) \not\subset \mathfrak{g}_0$, we have $G = G_0{\cdot}G(\ell)$. The orbit $G{\cdot}\ell$ is hence equal to $G_0{\cdot}\ell$. From this we immediately deduce that $p(G{\cdot}\ell) = p(G_0{\cdot}\ell) = G_0{\cdot}\ell_0$. c.q.f.d.

Let's write simply $\hat{\rho}_0$ instead of $\hat{\rho}_{G_0}$.

Proposition 2.1.5. Let $\pi_0 \in \hat{G}_0$. We suppose that $\pi_0 \simeq \hat{\rho}_0(\ell_0)$ where $\ell_0 \in \mathfrak{g}_0^*$. Let $\ell$ be an extension of $\ell_0$ to $\mathfrak{g}$ and $\Omega = G{\cdot}\ell$.

(1) If $\Omega$ is saturated in the direction $\mathfrak{g}_0^\perp$, then $\mathrm{ind}_{G_0}^G \pi_0 \simeq \hat{\rho}(\ell)$.

(2) If $\Omega$ is non-saturated in the direction $\mathfrak{g}_0^\perp$, then $\mathrm{ind}_{G_0}^G \pi_0 \simeq \int_{\mathbb{R}}^{\oplus} \hat{\rho}(\ell^\nu)d\nu$, where $\ell^\nu \in \mathfrak{g}*$ is defined by $\ell^\nu|_{\mathfrak{g}_0} = \ell_0$ et $g^\nu(X) = -2\pi\nu$ where $X$ is a fixed element of $\mathfrak{g}(\ell)\backslash\,(\mathfrak{g}(\ell) \cap \mathfrak{g}_0)$.

We now suppose that $\mathfrak{h} \in S(f, \mathfrak{g})$ is given, and propose to study the monomial representation $\tau = \hat{\rho}(f, \mathfrak{h}, G) = \mathrm{ind}_H^G \chi_f$. The affine subspace $\Gamma_\tau = f + \mathfrak{h}^\perp$ of $\mathfrak{g}^*$ plays a principal role to study $\tau$. Here we are interested in its canonical central decomposition. Note $\mathfrak{z}$ the center of $\mathfrak{g}$ and $\mathfrak{a}$ a non-central minimal ideal of $\mathfrak{g}$, i.e. minimal among non-central ideals. Evidently, $\dim \mathfrak{a}/(\mathfrak{a} \cap \mathfrak{z}) \leq 2$.

As everybody notices, when it's a matter of exponential groups, the main tool of proofs would be the induction. We make use of the induction on $\dim \mathfrak{g}, \dim \mathfrak{h}, \dim \mathfrak{g}/\mathfrak{h}$ and $\dim \mathfrak{g} + \dim \mathfrak{g}/\mathfrak{h}$. It's often without problem to pass from $\mathfrak{h}$ to $\mathfrak{h} + \mathfrak{z}$, what brings us to the case where $\mathfrak{h}$ contains $\mathfrak{z}$. If $f \in \mathfrak{g}^*$ vanishes on an non zero ideal of $\mathfrak{g}$, we are able to go down to the quotient by this ideal. After these observations we stand in the case where $\dim \mathfrak{z} \leq 1, \dim \mathfrak{a} \leq 3$. If $\mathfrak{g} \neq \mathfrak{h} + [\mathfrak{g}, \mathfrak{g}]$, there exists an ideal $\mathfrak{g}_0$ of $\mathfrak{g}$ containing $\mathfrak{h}$ such that $\dim \mathfrak{g}/\mathfrak{g}_0 = 1$. We are now ready to combine the proposition 2.1.5 with the induction hypothesis applied to $G_0 = \exp \mathfrak{g}_0$. If $\mathfrak{g} = \mathfrak{h} + [\mathfrak{g}, \mathfrak{g}]$, let $\mathfrak{k} = \mathfrak{h} + \mathfrak{a}$ and $K = \exp \mathfrak{k}$. Considering the theorem of induction by stages, our first affair is to analyze the monomial representation $\mathrm{ind}_H^K \chi_f$.

Exemple 2.1.6. (1) Let $G = \exp \mathfrak{g}_2$ with $\mathfrak{g}_2 = \langle X, Y \rangle_{\mathbb{R}} = \mathbb{R}X + \mathbb{R}Y : [X, Y] = Y$. Let $f \in \mathfrak{g}_2^*, \mathfrak{h} = \mathbb{R}X$ et $H = \exp \mathfrak{h}$. Then $\mathrm{ind}_H^G \chi_f \simeq \mathrm{ind}_{H'}^G \chi_{Y^*} \oplus \mathrm{ind}_{H'}^G \chi_{-Y^*}$ with $H' = \exp \mathfrak{h}', \mathfrak{h}' = \mathbb{R}Y$.

(2) Let $G = \exp(\mathfrak{g}_3(\alpha))$ with $\mathfrak{g}_3(\alpha) = \langle T, Y_1, Y_2 \rangle_{\mathbb{R}} : [T, Y_1] = Y_1 - \alpha Y_2, [T, Y_2] = Y_2 + \alpha Y_1$ $(0 \neq \alpha \in \mathbb{R})$. Let $f \in \mathfrak{g}_3(\alpha)^*, \mathfrak{h} = \mathbb{R}T$ and $H = \exp \mathfrak{h}$. Then

$$\mathrm{ind}_H^G \chi_f \simeq \int_{[0,2\pi]}^{\oplus} \mathrm{ind}_{H'}^G \chi_{\hat{\theta}} d\theta$$

with $H' = \exp(\mathbb{R}Y_1 + \mathbb{R}Y_2), \hat{\theta} = (\cos\theta)Y_1^* + (\sin\theta)Y_2^* \in \mathfrak{g}_3(\alpha)^*$.

(3) Let $G = \exp \mathfrak{g}_4$ with $\mathfrak{g}_4 = \langle T, X, Y, Z \rangle_{\mathbb{R}} : [T, X] = -X, [T, Y] = Y, [X, Y] = Z$. Let $f = \alpha T^* + \beta Z^* \in \mathfrak{g}_4^*$ $(\beta \neq 0), \mathfrak{h} = \langle T, X, Z \rangle_{\mathbb{R}}$ and $H = \exp \mathfrak{h}$. Then $\mathrm{ind}_H^G \chi_f \simeq \mathrm{ind}_{H'}^G \chi_f$ with $H' = \exp \mathfrak{h}', \mathfrak{h}' = \langle T, Y, Z \rangle_{\mathbb{R}}$.

(4) Let $G = \exp \mathfrak{g}_6$ with $\mathfrak{g}_6 = \langle T, X_1, X_2, Y_1, Y_2, Z \rangle_{\mathbb{R}} : [T, X_1] = -X_1 - \alpha X_2, [T, X_2] = -X_2 + \alpha X_1, [T, Y_1] = Y_1 - \alpha Y_2, [T, Y_2] = Y_2 + \alpha Y_1, [X_i, Y_j] = \delta_{ij} Z$ $(0 \neq \alpha \in \mathbb{R})$. Let $f = \beta T^* + \gamma Z^*$ $(\gamma \neq 0), \mathfrak{h} = \langle T, X_1, X_2, Z \rangle_{\mathbb{R}}$ and $H = \exp \mathfrak{h}$. Then $\mathrm{ind}_H^G \chi_f \simeq \mathrm{ind}_{H'}^G \chi_f$ with $H' = \exp \mathfrak{h}', \mathfrak{h}' = \langle T, Y_1, Y_2, Z \rangle_{\mathbb{R}}$.

This way of reasoning brings us to the following result. We take on $\Gamma_\tau$ a finite measure $\tilde{\mu}$ equivalent to the Lebesgue measure and regard it as a measure on $\mathfrak{g}^*$. Put $\mu = \hat{\rho}_*(\tilde{\mu})$, the image of $\tilde{\mu}$ by the Kirillov-Bernat map $\hat{\rho} : \mathfrak{g}^* \to \hat{G}$. For $\pi \in \hat{G}$, $\Omega(\pi) = \Omega_G(\pi)$ denotes the coadjoint orbit of $G$ associated to $\pi$ and $m(\pi)$ the number of $H$-orbits contained in $\Gamma_\tau \cap \Omega(\pi)$.

Theorem 2.1.7. ([11], [23])

$$\tau \simeq \int_{\hat{G}}^{\oplus} m(\pi) \pi d\mu(\pi). \tag{2.1.1}$$

Let's generalize a little this result. We consider a subgroup $K = \exp \mathfrak{k}$ and $\sigma \in \hat{K}$ to study the induced representation $\operatorname{ind}_K^G \sigma$. We designate by $\omega(\sigma)$ the coadjoint orbit $\hat{\rho}_K^{-1}(\sigma) \subset \mathfrak{k}^*$ of $K$ associated to $\sigma$, and by $p : \mathfrak{g}^* \to \mathfrak{k}^*$ the restriction map. A $K$-invariant measure on $\omega(\sigma)$ and the Lebesgue measure on $\mathfrak{k}^\perp$ determine a measure $\hat{\mu}$ on the subvariety $p^{-1}(\omega(\sigma))$ of $\mathfrak{g}^*$. We take a finite measure $\tilde{\mu}$ on $\mathfrak{g}^*$ equivalent to $\hat{\mu}$ and put $\mu = (\hat{\rho}_G)_*(\tilde{\mu})$. Next, for $\pi \in \hat{G}$ we note $n_\pi(\sigma)$ the number of $K$-orbits contained in $\Omega(\pi) \cap p^{-1}(\omega(\sigma))$. Since $\sigma$ is monomial, the theorem 2.1.7 is generalized :

Theorem 2.1.8. ([11], [25])

$$\operatorname{ind}_K^G \sigma \simeq \int_{\hat{G}}^{\oplus} n_\pi(\sigma) \pi d\mu(\pi).$$

2.2. Restriction of unitary representations

We stand in the situation described at the beginning of the preceding section. Keep the notations used in the proposition 2.1.5. Let $\pi \in \hat{G}$, and we study the restriction $\pi|_{G_0}$ de $\pi$ à $G_0$.

Proposition 2.2.1. ([33]) Let $\pi = \hat{\rho}(\ell)$ with $\ell \in \mathfrak{g}^*$.

(1) Suppose that the orbit $G{\cdot}\ell$ is saturated in the direction $\mathfrak{g}_0^\perp$. Let $X \in \mathfrak{g} \backslash \mathfrak{g}_0$. We have

$$\pi|_{G_0} \simeq \int_{\mathbb{R}}^{\oplus} \hat{\rho}_0(\ell_s) ds,$$

où $\ell_s = \exp(sX){\cdot}\ell_0$.

(2) Suppose that the orbit $G{\cdot}\ell$ is non-saturated in the direction $\mathfrak{g}_0^\perp$. Then $\pi|_{G_0} \simeq \hat{\rho}_0(\ell_0)$.

Proof. (1) We make use of the theorem of subgroups of Mackey ([40]). Let $\mathfrak{h}$ be a polarization of Vergne of $\mathfrak{g}$ at $\ell$ constructed from a good sequence of subalgebras which pass through $\mathfrak{g}_0$, and $H = \exp \mathfrak{h}$. Then $\mathfrak{h} \in I(\ell, \mathfrak{g})$ and $\mathfrak{h} \subset \mathfrak{g}_0$. Let's verify that $H$ and $G_0$ are regularly related. Since $H \subset G_0$, the double classes $HgG_0 = HG_0g = G_0g$ are simply the classes modulo $G_0$. Hence, the space of double classes is the group $G/G_0$. It's thus countably separated and the subgroups $K$ and $G_0$ are regularly related. The group $G/G_0$ being identified to $\mathbb{R}$ by the map $s \mapsto \exp(sX)G_0$, we can choose as admissible measure ([40]) the Haar measure on $G/G_0$, namely the Lebesgue measure on $\mathbb{R}$, and we have

$$\pi|_{G_0} \simeq \int_{\mathbb{R}}^{\oplus} V_t dt,$$

where $V_t$ is the representation of $G_0$ induced by the representation

$$\sigma_t : g_0 \mapsto \chi_\ell(\exp(tX) g_0 \exp(-tX))$$

of a subgroup $H_t = G_0 \cap (\exp(-tX) H \exp(tX)) = \exp(-tX) H \exp(tX)$ of $G_0$. Here $\sigma_t$ is nothing but $\exp(-tX){\cdot}\chi_\ell = \chi_{\exp(-tX)\cdot\ell}$. Hence,

$$V_t \simeq \operatorname{ind}_{H_t}^{G_0} \chi_{\exp(-tX)\cdot\ell} \simeq \exp(-tX) \cdot \left(\operatorname{ind}_H^{G_0} \chi_\ell\right) \simeq \exp(-tX) \cdot \hat{\rho}_0(\ell_0).$$

So, $V_t$ is irreducible and the Lie algebra of $H_t$ belongs to $I(\exp(-tX)\cdot\ell_0,\mathfrak{g}_0)$. From this, $V_t \simeq \hat{\rho}_0(\exp(-tX)\cdot\ell_0)$. If we set $\ell_s = \exp(sX)\cdot\ell_0$, we obtain the desired disintegration :

$$\pi|_{G_0} \simeq \int_{\mathbb{R}}^{\oplus} \hat{\rho}_0(\ell_{-s})ds \simeq \int_{\mathbb{R}}^{\oplus} \hat{\rho}_0(\ell_s)ds.$$

(2) If we construct a polarization of Vergne $\mathfrak{h} \in I(\ell,\mathfrak{g})$ as above, it follows that $\mathfrak{h}_0 = \mathfrak{h}\cap\mathfrak{g}_0 \in I(\ell_0,\mathfrak{g}_0)$ and $\mathfrak{h} = \mathfrak{h}_0 + \mathfrak{g}(\ell)$. $\mathfrak{h}_0$ being an ideal of $\mathfrak{h}$, $H = H_0\exp(\mathbb{R}X)$ with $H_0 = \exp\mathfrak{h}_0$. Let's apply again the theorem of subgroups of Mackey to the pair $(H, G_0)$. We remark that there exists only one double class because $HG_0 = H_0(\exp(\mathbb{R}X)G_0) = G$, what proves that $H$ and $G_0$ are regularly related. We have hence

$$\pi|_{G_0} \simeq \mathrm{ind}_{G_0\cap H}^{G_0}\chi_{\ell_0}.$$

But $G_0\cap H = H_0$ and as $\mathfrak{h}_0 \in I(\ell_0,\mathfrak{g}_0)$, we have $\mathrm{ind}_{G_0\cap H}^{G_0}\chi_{\ell_0} \simeq \hat{\rho}_0(\ell_0)$. Thus, $\pi|_{G_0} \simeq \hat{\rho}_0(\ell_0)$, what proves the assertion. c.q.f.d.

Now let $K = \exp\mathfrak{k}$ be a subgroup of $G$ and $p : \mathfrak{g}^* \to \mathfrak{k}^*$ the restriction map. Let $\pi \in \hat{G}$. We take a finite measure $\tilde{\nu} = \tilde{\nu}_\pi$ on $\mathfrak{g}^*$ equivalent to the $G$-invariant measure on the orbit $\Omega(\pi)$ and put $\nu = (\hat{\rho}_K\circ p)_*(\tilde{\nu})$. Utilizing the measure $\nu$ on $\hat{K}$ obtained in this fashion and the same multiplicity $n_\pi(\sigma)$ as in the theorem 2.1.8, we have the canonical central decomposition of the restriction $\pi|_K$ of $\pi$ à $K$.

Theorem 2.2.2. ([12], [25])

$$\pi|_K \simeq \int_{\hat{K}}^{\oplus} n_\pi(\sigma)\sigma d\nu(\sigma).$$

Corollary 2.2.3. The Frobenius reciprocity establishes in these circumstances.

Let $\pi_j$ $(j = 1, 2)$ be two irreducible unitary representations of $G$. The direct product of $\pi_1$ and $\pi_2$, noted $\pi_1\times\pi_2$, corresponds to the orbit $\Omega_{G\times G}(\pi_1\times\pi_2) = (\Omega(\pi_1),\Omega(\pi_2)) \subset \mathfrak{g}^*\oplus\mathfrak{g}^*$. We identify $G$ to the subgroup de $G\times G$ constituted by the diagonal elements.

Corollary 2.2.4. ([25]) Let $p : \mathfrak{g}^*\oplus\mathfrak{g}^* \to \mathfrak{g}^*$ the restriction map. Then

$$\pi_1\otimes\pi_2 \simeq \int_{\hat{G}}^{\oplus} m(\pi)\pi d\nu(\pi),$$

where $\nu = (\hat{\rho}_G\circ p)_*(\tilde{\nu}_{\pi_1\times\pi_2})$ and where the multiplicity $m(\pi)$ is obtained by the number of $G$-orbits included in $(\Omega(\pi_1),\Omega(\pi_2))\cap p^{-1}(\Omega_G(\pi))$.

## §3. $e-$central elements

In order to proceed into more detailed analysis of monomial representations, we assume in this chapter that $G = \exp\mathfrak{g}$ is a connected and simply connected nilpotent Lie group with Lie algebra $\mathfrak{g}$. Let's introduce $e$-central elements owing to Corwin-Greenleaf [14]. Let

$$\{0\} = \mathfrak{g}_0 \subset \mathfrak{g}_1 \subset \cdots \subset \mathfrak{g}_{n-1} \subset \mathfrak{g}_n = \mathfrak{g},\ \dim\mathfrak{g}_k = k\ (0 \le k \le n) \tag{3.1}$$

be a flag of ideals of $\mathfrak{g}$, $\{X_j\}_{1\leq j\leq n}$ a Malcev basis associated to this flag, i.e. $X_j \in \mathfrak{g}_j\backslash\mathfrak{g}_{j-1}$ $(1 \leq j \leq n)$ and $\{X_j^*\}_{1\leq j\leq n}$ the dual basis of $\mathfrak{g}^*$. We note $(\ell_1,\ldots,\ell_n), \ell_j = \ell(X_j)$, the coordinates of $\ell \in \mathfrak{g}^*$. We have $\mathfrak{g}_j^\perp = \langle X_{j+1}^*,\ldots,X_n^*\rangle_{\mathbb{R}} \subset \mathfrak{g}^*, \mathfrak{g}_j^* \cong \mathfrak{g}^*/\mathfrak{g}_j^\perp$ and the projection $p_j : \mathfrak{g}^* \to \mathfrak{g}_j^*$ intertwines the actions of $G$ on $\mathfrak{g}^*$ and $\mathfrak{g}_j^*$. For $\ell \in \mathfrak{g}^*$ we define $e_j(\ell) = \dim(G\cdot p_j(\ell))\,, e(\ell) = (e_1(\ell),\ldots,e_n(\ell))$ and put $\mathcal{E} = \{e(\ell); \ell \in \mathfrak{g}^*\}$. Let $e \in \mathcal{E}$. We define $G$-invariant layer $U_e = \{\ell \in \mathfrak{g}^*; e(\ell) = e\}$ and, putting $e_0 = 0$, the set of jump indices $S(e) = \{1 \leq j \leq n; e_j = e_{j-1}+1\}$ and that of non-jump indices $T(e) = \{1 \leq j \leq n; e_j = e_{j-1}\}$. Now, let $\mathcal{U}(\mathfrak{g})$ be the enveloping algebra of $\mathfrak{g}_{\mathbb{C}}$ and we say that $A \in \mathcal{U}(\mathfrak{g})$ is $e$-central if $\pi_\ell(A)$, où $\pi_\ell = \hat{\rho}_G(\ell)$, is a scalar operator for any $\ell \in U_e$.

Let's describe the fundamental results of Corwin-Greenleaf. There exists a Zariski open set $\mathcal{Z}$ of $\mathfrak{g}^*$ such that $\mathcal{Z}\cap U_e$ is non empty and $G$-invariant, and $A_j \in \mathcal{U}(\mathfrak{g}_j)$ for each $j \in T(e)$ with following properties.

1. Each $A_j$ is $e$-central on $\mathcal{Z}\cap U_e$, i.e. $\pi_\ell(A_j)$ is a scalar operator for $\ell \in \mathcal{Z}\cap U_e$, having the form $A_j = P_jX_j + Q_j$, where

   (i) $P_j$ is a polynomial of $A_k$ $(k \in T(e), k < j)$, in particular $P_j \in \mathcal{U}(\mathfrak{g}_{j-1})$;

   (ii) $P_j$ is $e$-central on $\mathcal{Z}\cap U_e$;

   (iii) $Q_j \in \mathcal{U}(\mathfrak{g}_{j-1})$, in particular $P_1, Q_1 \in \mathbb{C}1$.

2. $\pi_\ell(P_j) \neq 0$ for $\forall \ell \in \mathcal{Z}\cap U_e$.
3. $\pi_\ell(A_j) = \varphi_j(\ell)Id$, où $\varphi_j(\ell) = \tilde{p_j}(\ell')\ell_j + \tilde{q_j}(\ell')$ with two rational functions $\tilde{p_j}, \tilde{q_j}$ on $\mathcal{Z}\cap U_e$ which depend only on $\ell' = (\ell_1,\ldots,\ell_{j-1})$.
4. $\tilde{p_j}(\ell')$ is $G$-invariant and $\tilde{p_j}(\ell') \neq 0$ for $\forall \ell \in \mathcal{Z}\cap U_e$.

Remark 3.1. The construction of these elements $A_j$ can be repeated on $U_e\backslash(\mathcal{Z}\cap U_e)$.

Returning to the monomial representation $\tau = \operatorname{ind}_H^G \chi_f$, where $H = \exp\mathfrak{h}$ with $\mathfrak{h} \in S(f,\mathfrak{g})$, we consider the algebra $D_\tau(G/H)$ of $G$-invariant differential operators on the line bundle over $G/H$ associated to the data $(H,\chi_f)$. We take a basis $\{Y_s\}_{1\leq s\leq d}$ of $\mathfrak{h}$ and define a vector subspace

$$\mathfrak{a}_\tau = \sum_{s=1}^{d} \mathbb{C}\,(Y_s + if(Y_s))$$

in $\mathcal{U}(\mathfrak{g})$. Let $\mathcal{U}(\mathfrak{g})\mathfrak{a}_\tau$ be the left ideal of $\mathcal{U}(\mathfrak{g})$ generated by $\mathfrak{a}_\tau$, and

$$\mathcal{U}(\mathfrak{g},\tau) = \{A \in \mathcal{U}(\mathfrak{g}); [A,Y] \in \mathcal{U}(\mathfrak{g})\mathfrak{a}_\tau,\ \forall Y \in \mathfrak{h}\}.$$

The elements of $\mathcal{U}(\mathfrak{g})$ acting as left $G$-invariant differential operators : for $X \in \mathfrak{g}$ and $\psi \in C^\infty(G)$,

$$(R(X)\psi)\,(g) = \frac{d}{dt}\psi(g\exp\,(tX))|_{t=0}\ (\forall\ g \in G),$$

it turns out that the algebra $D_\tau(G/H)$ is the image of the map $R : \mathcal{U}(\mathfrak{g},\tau) \ni A \mapsto R(A)$, whose kernel is $\mathcal{U}(\mathfrak{g})\mathfrak{a}_\tau$. Thus, it's isomorphic to

$$\mathcal{U}(\mathfrak{g},\tau)/\mathcal{U}(\mathfrak{g})\mathfrak{a}_\tau \cong \left(\mathcal{U}(\mathfrak{g})/\mathcal{U}(\mathfrak{g})\mathfrak{a}_\tau\right)^H,$$

where the last expression represents the set of $H$-invariant elements.

Corwin and Greenleaf [14] presented two conjectures concerning the algebra $D_\tau(G/H)$.

Commutativity conjecture. The algebra $D_\tau(G/H)$ is commutative if and only if $\tau$ is of finite multiplicity, namely that in the theorem 2.1.7 $m(\pi) < \infty$ almost everywhere for $\mu$.

Polynomial conjecture. When $\tau$ is of finite multiplicity, $D_\tau(G/H)$ is isomorphic to the algebra $\mathbb{C}[\Gamma_\tau]^H$ of $H$-invariant polynomial functions on $\Gamma_\tau$.

Remark 3.2. The commutativity conjecture had previously been presented by M. Duflo [17] in a much more general frame.

There exists one and only one $e \in \mathcal{E}$ such that $\Gamma_\tau \cap U_e$ is a Zariski open set of $\Gamma_\tau$.

Theorem 3.3. ([30]) Let $j \in T(e)$, and take the $e$-central element $A_j$. Then $\pi_\ell(A_j) = \varphi_j(\ell)Id$ for $\ell \in \Gamma_\tau$ and $\varphi_j(\ell)$ ia a polynomial function on $\Gamma_\tau$.

Recall the flag of ideals (3.1) of $\mathfrak{g}$. Let

$$\mathcal{I} = \{i_1 < i_2 < \cdots < i_d\} = \{1 \leq i \leq n; \mathfrak{h} \cap \mathfrak{g}_i \neq \mathfrak{h} \cap \mathfrak{g}_{i-1}\}$$

and $\mathcal{J} = \{j_1 < j_2 < \cdots < j_q\} = \{1, 2, \ldots, n\} \backslash \mathcal{I}$, where $d = \dim \mathfrak{h}$ and $q = \dim \mathfrak{g}/\mathfrak{h}$. On the one hand, putting $\mathfrak{k}_0 = \mathfrak{h}$, $\mathfrak{k}_r = \mathfrak{h} + \mathfrak{g}_{j_r}$ $(1 \leq r \leq q)$, we have a sequence of subalgebras

$$\mathfrak{h} = \mathfrak{k}_0 \subset \mathfrak{k}_1 \subset \cdots \subset \mathfrak{k}_{q-1} \subset \mathfrak{k}_q = \mathfrak{g}, \ \dim \mathfrak{k}_r = d + r, \tag{3.2}$$

and on the other hand, putting $\mathfrak{h}_0 = \{0\}, \mathfrak{h}_s = \mathfrak{h} \cap \mathfrak{g}_{i_s}$ $(1 \leq s \leq d)$, we obtain a sequence of ideals of $\mathfrak{h}$ :

$$\{0\} = \mathfrak{h}_0 \subset \mathfrak{h}_1 \subset \cdots \subset \mathfrak{h}_{d-1} \subset \mathfrak{h}_d = \mathfrak{h}, \ \dim \mathfrak{h}_s = s. \tag{3.3}$$

Let's choose the basis $\{Y_s\}_{1 \leq s \leq d}$ of $\mathfrak{h}$ in such a way that $Y_s \in \mathfrak{h}_s \setminus \mathfrak{h}_{s-1}$ $(1 \leq s \leq d)$. Next, for $1 \leq s \leq d$, we set

$$\mathfrak{a}_s = \sum_{j=1}^{s} \mathbb{C}(Y_j + if(Y_j)).$$

We designate by $T(e_H)$ the set of indices $i_s \in \mathcal{I}$ such that $\mathfrak{h}_s \subset \mathfrak{h}_{s-1} + \mathfrak{g}(\ell)$ for $\tilde{\mu}$-almost all $\ell \in \Gamma_\tau$. As $T(e_H) \subset T(e)$, let $U(e) = T(e) \backslash T(e_H)$. We note $\Diamond$ the principal anti-automorphism of $\mathcal{U}(\mathfrak{g})$. Let $i_s \in T(e_H)$ and $T(e) \cap \{1, 2, \ldots, i_s\} = \{m_1 < m_2 < \cdots < m_k = i_s\}$. The $e$-central elements $A_{m_j}$ $(1 \leq j \leq k)$ of Corwin-Greenleaf are denoted by $\sigma_j$ for simplicity. The following lemma will be very useful for us.

Lemma 3.4. Modulo $\mathcal{U}(\mathfrak{g}_{i_s})\mathfrak{a}_s$, $\Diamond(\sigma_k)$ is algebraic on $\{\Diamond(\sigma_1), \ldots, \ \Diamond(\sigma_{k-1})\}$.

Let $\mathcal{F}$ be the algebra of functions $\zeta$ on $G \cdot \Gamma_\tau$ such that there exists $W \in \mathcal{U}(\mathfrak{g})$ verifying $\pi_\ell(W) = \zeta(\ell)Id$ for all $\ell \in \Gamma_\tau$. Making use of the lemma 3.4, we find :

Theorem 3.5. ([30]) $\{\varphi_j; j \in U(e)\}$ is a transcendental basis of $\mathcal{F}$.

## §4. Frobenius reciprocity

Let $G$ be a Lie group, which we suppose a reunion of countable compact sets, with Lie algebra $\mathfrak{g}$. We uniquely consider unitary representations $\pi$ whose Hilbert space $\mathcal{H}_\pi$

is separable. Let $v \in \mathcal{H}_\pi$. When the function $G \ni g \mapsto \pi(g)v \in \mathcal{H}_\pi$ is $C^\infty$, we call $v$ a $C^\infty$-vector. We note $\mathcal{H}_\pi^\infty$ the space of $C^\infty$-vectors of $\pi$. $\mathcal{H}_\pi^\infty$ is a dense subspace of $\mathcal{H}_\pi$, on which $\mathfrak{g}$ acts by the differential $d\pi$ of $\pi$ :

$$d\pi(X)v = \frac{d}{dt}\pi(\exp(tX))v|_{t=0} \ (X \in \mathfrak{g}, v \in \mathcal{H}_\pi^\infty).$$

The differential representation $d\pi$ uniquely extends as a representation of the enveloping algebra $\mathcal{U}(\mathfrak{g})$. $\{X_1, \ldots, X_n\}$ being a basis of $\mathfrak{g}$, $\mathcal{H}_\pi^\infty$ becomes a Fréchet space for the semi-norms

$$\rho_d(v) = \sum_{1 \leq i_k \leq n} \|d\pi(X_{i_1} \cdots X_{i_d})v\| \ (d \in \mathbb{N}).$$

We designate by $\mathcal{H}_\pi^{-\infty}$ the anti-dual of $\mathcal{H}_\pi^\infty$, i.e. the vector space of continuous anti-linear forms of $\mathcal{H}_\pi^\infty$ into $\mathbb{C}$. The elements of $\mathcal{H}_\pi^{-\infty}$ are called generalized vectors of $\pi$. We provide $\mathcal{H}_\pi^{-\infty}$ with the strong dual topology of $\mathcal{H}_\pi^\infty$. The anti-dual of $\mathcal{H}_\pi^{-\infty}$ is identified with $\mathcal{H}_\pi^\infty$. For $a \in \mathcal{H}_\pi^{\pm\infty}$ and $b \in \mathcal{H}_\pi^{\mp\infty}$, we note $\langle a, b\rangle$ the image of $b$ by $a$ and hence $\langle a, b\rangle = \overline{\langle b, a\rangle}$. The actions of $G$ and of $\mathfrak{g}$ continuously extend on $\mathcal{H}_\pi^{-\infty}$ by duality. Remark that

$$\pi(\varphi)\left(\mathcal{H}_\pi^{-\infty}\right) \subset \mathcal{H}_\pi^\infty$$

if $\varphi \in \mathcal{D}(G)$. Being given a closed subgroup $K$ and its character $\chi : K \to \mathbb{C}^*$, set

$$\left(\mathcal{H}_\pi^{-\infty}\right)^{K,\chi} = \{a \in \mathcal{H}_\pi^{-\infty}; \pi(k)a = \chi(k)a, \ \forall k \in K\}.$$

Theorem 4.1. ([22], [35]) Let $G = \exp \mathfrak{g}$ be an exponential group, $f \in \mathfrak{g}^*, \mathfrak{h} \in I(f, \mathfrak{g})$. We define as before the character $\chi_f$ of $H = \exp \mathfrak{h}$ by $\chi_f(\exp X) = e^{if(X)}$ $(X \in \mathfrak{h})$ and $\tau = \operatorname{ind}_H^G \chi_f \in \hat{G}$. Then, for $\pi \in \hat{G}$,

$$\dim \left(\mathcal{H}_\pi^{-\infty}\right)^{H,\chi_f \Delta_{H,G}^{1/2}} = \begin{cases} 1, & \pi \simeq \tau, \\ 0, & \pi \not\simeq \tau. \end{cases}$$

We noticed in the corollary 2.2.3 that the Frobenius reciprocity established. Further, the theorem 4.1 also announces a kind of Frobenius reciprocity in a very special case. We ask if the reciprocity of this type remains valid in the general situation :

Question 4.2. In the formula (2.1.1) of the disintegration of a monomial representation, is it true that :

$$m(\pi) = \dim \left(\mathcal{H}_\pi^{-\infty}\right)^{H,\chi_f \Delta_{H,G}^{1/2}}$$

for $\mu$-almost all $\pi \in \hat{G}$?

We are going to examine in detail this question for nilpotent case. Suppose in the sequel of this chapiter that $G = \exp \mathfrak{g}$ is a connected and simply connected nilpotent Lie group with Lie algebra $\mathfrak{g}$. Because R. Penney [43] showed the inequality

$$m(\pi) \leq \dim \left(\mathcal{H}_\pi^{-\infty}\right)^{H,\chi_f \Delta_{H,G}^{1/2}}$$

for $\mu$-almost all $\pi \in \hat{G}$, we are interested in the inverse inequality.

Let $\ell \in \mathfrak{g}^*$, $\mathfrak{b} \in M(\ell, \mathfrak{g}) = I(\ell, \mathfrak{g})$ and $B = \exp \mathfrak{b}$. By means of coexponential basis to $\mathfrak{b}$ in $\mathfrak{g}$, the irreducible unitary representation $\pi = \mathrm{ind}_B^G \chi_\ell = \hat{\rho}(\ell)$ is realized in $L^2(\mathbb{R}^m)$ ($m = \dim \mathfrak{g}/\mathfrak{b}$). In this situation the following theorem will be very useful for us.

Theorem 4.3. The Fréchet space $\mathcal{H}_\pi^\infty$ coincides with the Schwartz space $\mathcal{S}(\mathbb{R}^m)$.

Here are some comments on the formula (2.1.1) :

(1) We are in the following two alternatives : either there exists a uniform bound for multiplicities $m(\pi)$ for $\mu$-almost all $\pi$, or $m(\pi) = \infty$ for $\mu$-almost all $\pi$. According to these two eventualities, we say that $\tau$ is of finite multiplicity or infinite multiplicity.

(2) $\tau$ is of finite multiplicity if and only if $\mathfrak{h} + \mathfrak{g}(\ell)$ is $\tilde{\mu}$-almost everywhere a lagrangian subspace, i.e. maximal isotropic, with respect to the bilinear form $B_\ell$.

(3) When $\tau$ is of finite multiplicities, for $\mu$-almost all $\pi \in \hat{G}$, each connected component of $\hat{\rho}^{-1}(\pi) \cap \Gamma_\tau$ is a $H$-orbit of dimension equal to $\frac{1}{2} \dim \hat{\rho}^{-1}(\pi)$. The multiplicity $m(\pi)$ is hence computed by the number of connected components of $\hat{\rho}^{-1}(\pi) \cap \Gamma_\tau$.

Suppose now that $\tau = \mathrm{ind}_H^G \chi_f$ is of finite multiplicity. For $\pi \in \hat{G}$, we write $\Omega(\pi)$ in stead of $\hat{\rho}^{-1}(\pi)$. Up to a $\tilde{\mu}$-negligible subset of $\Gamma_\tau$, let $C_k$ ($1 \leq k \leq m(\pi)$) the connected components of $\Omega(\pi) \cap \Gamma_\tau$. Each of them is a $H$-orbit. We fix $\ell \in \Omega(\pi)$ and $\mathfrak{b} \in M(\ell, \mathfrak{g})$, in other words a realization of $\pi = \mathrm{ind}_B^G \chi_\ell$ with $B = \exp \mathfrak{b}$. For $1 \leq k \leq m(\pi)$, take $g_k \in G$ such that $g_k \cdot \ell \in C_k$ and an invariant measure $d\dot{h}$ on the homogeneous space $H/(H \cap g_k B g_k^{-1})$.

Proposition 4.4. ([21]) We can produce linearly independent elements $a_\pi^k$ ($1 \leq k \leq m(\pi)$) in $(\mathcal{H}_\pi^{-\infty})^{H,\chi_f}$ by the following formula : for $\phi \in \mathcal{H}_\pi^\infty$,

$$\langle a_\pi^k, \phi \rangle = \int_{H/(H \cap g_k B g_k^{-1})} \overline{\phi(hg_k)\chi_f(h)} d\dot{h}. \tag{4.1}$$

Proof. Let's see first the integral in the right hand is well defined. In fact, $h' \in H \cap g_k B g_k^{-1}$ being arbitrary,

$$\begin{aligned} \phi(hh'g_k)\chi_f(hh') &= \phi(hg_k g_k^{-1} h' g_k)\chi_f(h)\chi_f(h') \\ &= \chi_\ell(g_k^{-1} h'^{-1} g_k)\phi(hg_k)\chi_f(h)\chi_f(h') \\ &= \chi_{g_k \cdot \ell}(h'^{-1})\chi_f(h')\phi(hg_k)\chi_f(h) = \phi(hg_k)\chi_f(h). \end{aligned}$$

Next, there exists a coexponential basis to $\mathfrak{h} \cap g_k \cdot \mathfrak{b}$ in $\mathfrak{h}$ which makes a part of coexponential basis to $g_k \cdot \mathfrak{b}$ in $\mathfrak{g}$. In view of the theorem 4.3, the translated space of $\mathcal{H}_\pi^\infty$ by $g_k$ from right is identified by using this basis to the Schwartz space $\mathcal{S}(\mathbb{R}^m)$, $m = \dim(G/B)$, and $d\dot{h}$ to a Lebesgue measure on $\mathbb{R}^p \subset \mathbb{R}^m$, $p = \dim(H/(H \cap g_k B g_k^{-1}))$, from where the continuity of $a_\pi^k$. In reality this right translation by $g_k$ is nothing but an intertwining operator between two realizations of $\pi$ at points $\ell$ and $g_k \cdot \ell$. A direct calculation assures the necessary semi-invariance of $a_\pi^k$.

Finally, we choose a Haar measure $db$ on $B$ and define, for $\psi \in \mathcal{D}(G)$, a function $\tilde{\psi}$ on $G$ by

$$\tilde{\psi}(g) = \int_B \psi(gb)\chi_\ell(b)db.$$

It's clear that $\tilde{\psi} \in \mathcal{H}_\pi^\infty$ and that the generalized vector $a_\pi^k$ gives a distribution $\widetilde{a_\pi^k}$ on $G$ by the formula $\widetilde{a_\pi^k}(\psi) = \langle \tilde{\psi}, a_\pi^k \rangle$. Then the support of $\widetilde{a_\pi^k}$ coincides with the closed double class $Hg_kB$. This being observed, in order to see that $\{a_\pi^k\}_{1 \leq k \leq m(\pi)}$ are linearly independent, it suffices to verify that $Hg_jB \neq Hg_kB$ if $j \neq k$. Otherwise, the connected set $g_j{\cdot}(\ell + \mathfrak{b}^\perp) \cap \Gamma_\tau$ de $\Omega(\pi) \cap \Gamma_\tau$ crosses at the same time $C_j$ and $C_k$, what is absurd. c.q.f.d.

Remark 4.5. The generalized vector $a_\pi^k$ does not depend on the choice of $g_k \in G$ up to a scalar multiplication, same for the choice of $\mathfrak{b} \in M(\ell, \mathfrak{g})$.

We are able to reply affirmatively to the question 4.2 when $G$ is nilpotent.

Theoreme 4.6. Let $G = \exp \mathfrak{g}$ be a nilpotent Lie group, $f \in \mathfrak{g}^*, \mathfrak{h} \in S(f, \mathfrak{g})$ and $\tau = \operatorname{ind}_H^G \chi_f$. Let

$$\tau \simeq \int_{\hat{G}}^{\oplus} m(\pi)\pi d\mu(\pi)$$

be the canonical central decomposition of $\tau$ as in the theorem 2.1.7. Then we have a kind of Frobenius reciprocity :

$$m(\pi) = \dim \left(\mathcal{H}_\pi^{-\infty}\right)^{H,\chi_f}$$

for $\mu$-almost all $\pi \in \hat{G}$. In particular, if $\tau$ is of finite multiplicity,

$$\left(\mathcal{H}_\pi^{-\infty}\right)^{H,\chi_f} = \sum_{k=1}^{m(\pi)} \mathbb{C}a_\pi^k$$

for $\mu$-almost all $\pi \in \hat{G}$.

Proof. Here we merely mention some guide lines of a proof. We employ the induction on $\dim \mathfrak{g} + \dim(\mathfrak{g}/\mathfrak{h})$. We can assume that $\mathfrak{h}$ contains the center $\mathfrak{z}$ of $\mathfrak{g}$ and that $f$ does not vanish on any non-zero ideal. This leads us to the case where $\dim \mathfrak{z} = 1, f|_\mathfrak{z} \neq 0$. Take as usual a Heisenberg triple $\{X, Y, Z\}$ such that $\mathfrak{z} = \mathbb{R}Z, f(Z) = 1, [X, Y] = Z, \mathfrak{g} = \mathfrak{g}_0 + \mathbb{R}X$ where $\mathfrak{g}_0$ denotes the centralizer of $Y$ in $\mathfrak{g}$. Let $\ell \in \Omega(\pi)$, and we realize $\pi$ using a polarization $\mathfrak{b}$ at $\ell$ of $\mathfrak{g}$ contained in $\mathfrak{g}_0$. In accordance with the decomposition $G = \exp(\mathbb{R}X)G_0$, $G_0 = \exp\ \mathfrak{g}_0$, the space $\mathcal{H}_\pi^\infty$ turns into $\mathcal{S}(\mathbb{R}^m) \cong \mathcal{S}(\mathbb{R})\widehat{\oplus}\mathcal{S}(\mathbb{R}^{m-1})$, where $\mathcal{S}(\mathbb{R}^{m-1})$ represents the space $\mathcal{H}_{\pi_0}^\infty$ with $\pi_0 = \operatorname{ind}_B^{G_0} \chi_\ell \in \widehat{G_0}$. Every $g \in G$ is uniquely written as $g = \exp(xX)g_0$ with $x \in \mathbb{R}, g_0 \in G_0$. We would like to descend on the subgroup $G_0$ deleting the first coordinate $x$.

Let $a \in (\mathcal{H}_\pi^{-\infty})^{H,\chi_f}$. If $\mathfrak{h} \not\subset \mathfrak{g}_0$, we take $X$ in $\mathfrak{h} \cap \ker f$. The semi-invariance of $a$ requires that with a certain $a_0 \in \left(\mathcal{H}_{\pi_0}^{-\infty}\right)^{H,\chi_f}$,

$$\langle a, \phi(x)\psi(g_0)\rangle = \left(\int_{\mathbb{R}} \overline{\phi(x)} dx\right) \langle a_0, \psi(g_0)\rangle,$$

where $\phi \in \mathcal{S}(\mathbb{R}), x \in \mathbb{R}, \psi \in \mathcal{H}_{\pi_0}^{\infty}, g_0 \in G_0$. We descend in this way on $G_0$.

Suppose now that $\mathfrak{h} \subset \mathfrak{g}_0$. It suffices for us to treat the case where $\tau$ is of finite multiplicity. Then, we deduce from the lemma 3.4 that there exists $e$-central elements of Corwin-Greenleaf $\{\sigma_1, \ldots, \sigma_\kappa\}$ such that we have a polynomial relation :

$$P\left(\overline{(\lozenge(\sigma_1))}, \ldots, \overline{(\lozenge(\sigma_\kappa))}, Y\right) \equiv 0$$

modulo $\mathcal{U}(\mathfrak{g})\overline{\mathfrak{a}_\tau}$, where $P$ designates a polynomial of $\kappa+1$ variables and where $Y$ appears effectively. If we apply this relation to our generalized vector $a$, we find that there exists a non constant polynomial $F(x)$ such that $F(x)a = 0$. Let $\{\alpha_j\}_{1\leq j\leq r}$ the real roots of the equation $F(x) = 0$. We see that the support of the distribution $\tilde{a}$ is contained in the disjoint reunion of sub-varieties $M_j = \exp(\alpha_j X)G_0$ $(1 \leq j \leq r)$ of $G$. Consequently, $a$ is written in a neighborhood of $M_j$ as

$$a = \sum_{k=0}^{u} \frac{\partial^k}{\partial x^k} D_k$$

with certains distributions $D_k$ $(1 \leq k \leq u)$ on $G_0$. It only remains for us to show that $u = 0$, what can be done by a suitable choice of the polynomial $P$ used above and by the hypothesis that $\tau$ is of finite multiplicity. c.q.f.d.

Remark 4.7. When $G = \exp \mathfrak{g}$ is exponential and the monomial representation $\tau$ is of finite multiplicity, is it possible to produce elements of $(\mathcal{H}_\pi^{-\infty})^{H,\chi_f \Delta_{H,G}^{1/2}}$ by the similar formula : for $\phi \in \mathcal{H}_\pi^\infty$,

$$\langle a_\pi^k, \phi\rangle = \oint_{H/H\cap g_k B g_k^{-1}} \overline{\phi(hg_k)\chi_f(h)} \Delta_{H,G}^{-1/2}(h) d\nu(h) \ (g \in G), \tag{4.2}$$

where $\nu = \mu_{H,H\cap g_k B g_k^{-1}}$. Even on the formation de this value we already encounter two questions : possibility to consider this integral and its convergence just as in our study of intertwining operators. Finally, the elements $a_\pi^k$ $(1 \leq k \leq m(\pi))$ do they supply generalized vectors of $\pi$ ? It's difficult to determine the space $\mathcal{H}_\pi^\infty$ in the exponential case, but maybe there would be some chance to exploit by taking a polarization of Vergne $\mathfrak{b}$.

## §5. Plancherel formula

Let as before $G = \exp \mathfrak{g}$ be an exponential group with Lie algebra $\mathfrak{g}, f \in \mathfrak{g}^*, \mathfrak{h} \in S(f, \mathfrak{g})$ and $\tau = \hat{\rho}(f, \mathfrak{h}, G) = \mathrm{ind}_H^G \chi_f$ with $H = \exp \mathfrak{h}, \chi_f(\exp X) = e^{if(X)}$ $(X \in \mathfrak{h})$. We are interested in this chapter in the abstract Plancherel formula, due to Penney [43] and

Bonnet [9], applied to the cyclic representation $(\tau, \delta_\tau)$, where $\delta_\tau \in (\mathcal{H}_\tau^{-\infty})^{H,\chi_f \Delta_{H,G}^{1/2}}$ is given as follows : any $\phi \in \mathcal{H}_\tau^\infty$ being a $C^\infty$-function on $G$ ([44]), put $\delta_\tau(\phi) = \langle \delta_\tau, \phi \rangle = \overline{\phi(e)}$ with the unit element $e$ of $G$.

When $\mathfrak{h} \in I(f, \mathfrak{g})$, $\tau$ turns out to be irreducible and the theorem 4.1 says that the space $(\mathcal{H}_\pi^{-\infty})^{H,\chi_f \Delta_{H,G}^{1/2}}, \pi \in \hat{G}$, is equal to $\mathbb{C}\delta_\pi$ if $\pi \simeq \tau$ and trivial if $\pi \not\simeq \tau$. When $\mathfrak{h} \notin I(f, \mathfrak{g})$, we decompose $(\tau, \delta_\tau)$ in irreducibles :

$$\tau \simeq \int_{\hat{G}}^{\oplus} m(\pi)\pi d\mu(\pi), \delta_\tau \simeq \int_{\hat{G}}^{\oplus} a_\pi d\mu(\pi).$$

The $a_\pi$ being $\mu$-almost everywhere in $(m(\pi)\mathcal{H}_\pi^{-\infty})^{H,\chi_f \Delta_{H,G}^{1/2}}$, we have by the uniqueness of disintegration [43] that $a_\pi = (a_\pi^k)_{1 \le k \le m(\pi)}$ with $a_\pi^k \in (\mathcal{H}_\pi^{-\infty})^{H,\chi_f \Delta_{H,G}^{1/2}}$ and, for $\phi \in \mathcal{D}(G)$,

$$\langle \tau(\phi)\delta_\tau, \delta_\tau \rangle = \int_{\hat{G}} \sum\nolimits_{k=1}^{m(\pi)} \langle \pi(\phi) a_\pi^k, a_\pi^k \rangle d\mu(\pi).$$

For $\phi \in \mathcal{D}(G)$, we produce, making the choice of a left Haar measure $dh$ on $H$, an element $\phi_H^f$ of $\mathcal{H}_\tau^\infty$ by

$$\phi_H^f(g) = \int_H \phi(gh)\chi_f(h)\Delta_{H,G}^{-1/2}(h)dh \ (g \in G).$$

We compute : for $\psi \in \mathcal{H}_\tau^\infty$,

$$\begin{aligned}
\langle \tau(\phi)\delta_\tau, \psi \rangle &= \langle \int_G \phi(g)\tau(g)\delta_\tau dg, \psi \rangle = \langle \delta_\tau, \int_G \overline{\phi(g)}\tau(g^{-1})\psi dg \rangle \\
&= \int_G \phi(g)\overline{\psi(g)}dg = \oint_{G/H} d\mu_{G,H}(g) \int_H \phi(gh)\Delta_{H,G}^{-1/2}(h)\overline{\psi(gh)}dh \\
&= \oint_{G/H} \overline{\psi(g)}d\mu_{G,H}(g) \int_H \phi(gh)\chi_f(h)\Delta_{H,G}^{-1/2}(h)dh = \langle \phi_H^f, \psi \rangle.
\end{aligned}$$

We have in this manner $\tau(\phi)\delta_\tau = \phi_H^f \in \mathcal{H}_\tau^\infty$ and therefore

$$\langle \tau(\phi)\delta_\tau, \delta_\tau \rangle = \int_H \phi(h)\chi_f(h)\Delta_{H,G}^{-1/2}(h)dh = \phi_H^f(e)$$

for any $\phi \in \mathcal{D}(G)$.

We rewrite in this way the abstract Plancherel formula for the monomial cyclic representation $(\tau, \delta_\tau)$.

Theorem 5.1. ([21], [43]) The canonical central decomposition of $\tau = \hat{\rho}(f, \mathfrak{h}, G)$ being

$$\tau \simeq \int_{\hat{G}}^{\oplus} m(\pi)\pi d\mu(\pi),$$

there exists in $(\mathcal{H}_\pi^{-\infty})^{H,\chi_f\Delta_{G,H}^{1/2}}$, for $\mu$-almost all $\pi \in \hat{G}$, elements $a_\pi^k,\ 1 \leq k \leq m(\pi)$, with which the formula

$$\phi_H^f(e) = \int_{\hat{G}} \sum_{k=1}^{m(\pi)} \langle \pi(\phi) a_\pi^k, a_\pi^k \rangle d\mu(\pi) \tag{5.1}$$

holds for every $\phi \in \mathcal{D}(G)$.

As in the symmetric case of Benoist [6], we'll carry out in certains cases explicit calculations of $(a_\pi^k)_{\pi\in\hat{G},1\leq k\leq m(\pi)}$ to obtain a concrete Plancherel formula.

Theorem 5.2. ([21]) When $G = \exp \mathfrak{g}$ is nilpotent and $\tau$ is of finite multiplicity, the generalized vectors constructed by the formula (4.1) satisfy the Plancherel formula (5.1) under suitable normalisation of the measures $d\dot{h}$ used in their constructions.

Beyond the nilpotent case we know little. We mention here a special case. Let $G = \exp \mathfrak{g}$ an exponential group, $f \in \mathfrak{g}^*$ et $\mathfrak{h} \in M(f, \mathfrak{g})$. The disintegration of $\tau = \mathrm{ind}_H^G \chi_f$ was obtained by M. Vergne [47], which offered us a starting point toward the theorem 2.1.7. Note $U(f, \mathfrak{h})$ the set of orbits $\Omega \in \mathfrak{g}^*/G$ which encounter $\Gamma_\tau$ in a non empty open set of $\Gamma_\tau$.

Theorem 5.3. Let $\mathfrak{h} \in M(f, \mathfrak{g})$, then :
1) $U(f, \mathfrak{h})$ is a finite set ;
2) if $\Omega \in U(f, \mathfrak{h})$, the number $c(\Omega)$ of connected components of $\Omega \cap \Gamma_\tau$ is finite ;
3) $\tau \simeq \sum_{\Omega\in U(f,\mathfrak{h})} c(\Omega)\hat{\rho}(\Omega)$.

To study the concrete Plancherel formula for $\tau$, we begin by showing the :

Lemma 5.4. Let always $\mathfrak{h} \in M(f, \mathfrak{g})$. There exists $\mathfrak{b} \in I(f, \mathfrak{g})$ having the following properties. Let's put $B = \exp \mathfrak{b}, \pi = \mathrm{ind}_B^G \chi_f$ and note $(\mathcal{H}_\pi^\infty)_0$ the subspace of $\mathcal{H}_\pi^\infty$ constituted by the functions with compact support modulo $B$. Then we have :
1) $\Delta_{H\cap B,H}(h)\Delta_{H\cap B,B}(h) = 1$ for any $h \in H \cap B$ ;
2) $HB$ is closed in $G$ ;
3) Because of 1) and 2), we are able to produce an anti-linear form

$$a : (\mathcal{H}_\pi^\infty)_0 \ni \phi \mapsto \oint_{H/(H\cap B)} \overline{\phi(h)\chi_f(h)} \Delta_{H,G}^{-1/2}(h) d\mu_{H,H\cap B}(h) \in \mathbb{C}$$

which extends to an element of $(\mathcal{H}_\pi^{-\infty})^{H,\chi_f\Delta_{H,G}^{1/2}}$.

Considering always the same situation as before, we verify the :

Theorem 5.5. When $\Omega$ runs $U(f, \mathfrak{h})$, we take $\ell_\Omega^k$ $(1 \leq k \leq c(\Omega))$ arbitrarily in each connected component $C_\Omega^k$ of $\Omega \cap \Gamma_\tau$. At these points $\ell_\Omega^k \in \mathfrak{g}^*$ $(\Omega \in U(f, \mathfrak{h}),\ 1 \leq k \leq c(\Omega))$ we can choose $\mathfrak{b}_\Omega^k \in I(\ell_\Omega^k, \mathfrak{g})$ of the lemma 5.4, which permits us to produce $a_\Omega^k \in \left(\mathcal{H}_{\hat{\rho}(\Omega)}^{-\infty}\right)^{H,\chi_f\Delta_{H,G}^{1/2}}$, so that a concrete Plancherel formula for $\tau$ is written in terms of the matrix coefficients for these $a_\Omega^k$ suitably normalized : for any $\phi \in \mathcal{D}(G)$,

$$\phi_H^f(e) = \sum_{\Omega \in U(f,\mathfrak{h})} \sum_{k=1}^{c(\Omega)} \langle \hat{\rho}(\Omega) a_\Omega^k, a_\Omega^k \rangle.$$

As far as here we have studied the abstract Plancherel formula due to Penney for $\tau = \mathrm{ind}_H^G \chi_f$ having finite multiplicity. In nilpotent case, we can study in this manner the abstract Plancherel formula due to Bonnet [9]. Please see [24] for the detail.

## §6. Commutativity conjecture : induction case

Let $G$ be a connected and simply connected nilpotent Lie group with Lie algebra $\mathfrak{g}$ and $H = \exp\ \mathfrak{h}$ an analytic subgroup of $G$ with Lie algebra $\mathfrak{h}$. Being given a unitary character $\chi$ of $H$, we construct the monomial representation $\tau = \mathrm{ind}_H^G \chi$ and propose to study the algebra $D_\tau(G/H)$ of $G$-invariant differential operators on the line bundle $G \times_H \mathbb{C}$ associated to $\chi$. Our aim is the commutativity conjecture due to Duflo [17] and Corwin-Greenleaf [14]. The latter proved one implication : if $\tau$ is of finite multiplicity, then $D_\tau(G/H)$ is commutative. We are hence interested in the inverse direction.

Let's take $f \in \mathfrak{g}^*$ verifying $d\chi = if|_{\mathfrak{h}}, i = \sqrt{-1}$, and put as before $\Gamma_\tau = f + \mathfrak{h}^\perp$.

Theorem 6.1. ([29]) Suppose that $\tau$ is of infinite multiplicity. Let $\mathfrak{g}_0$ be a subalgebra of codimension 1 containing $\mathfrak{h}$ such that $\tau_0 = \mathrm{ind}_H^{G_0} \chi$, where $G_0 = \exp\ \mathfrak{g}_0$, is of finite multiplicity. Suppose that there exists $W \in \mathcal{U}(\mathfrak{g}, \tau)$ such that $W \notin \mathcal{U}(\mathfrak{g}_0) + \mathcal{U}(\mathfrak{g})\mathfrak{a}_\tau$. Then there exists $T \in \mathcal{U}(\mathfrak{g}_0, \tau_0)$ such that $[W, T] \notin \mathcal{U}(\mathfrak{g})\mathfrak{a}_\tau$.

Proof. We proceed by induction on $\dim \mathfrak{g} + \dim(\mathfrak{g}/\mathfrak{h})$. Remark first that we can assume $\mathfrak{h}$ contains the center $\mathfrak{z}$ of $\mathfrak{g}$. In fact, if $\mathfrak{z} \not\subset \mathfrak{h}$, take $0 \neq Z \in \mathfrak{z}$ in the outside of $\mathfrak{h}$. We write the representative elements in $\mathcal{U}(\mathfrak{g}_0, \tau_0)$ using an basis $\{Z, X_1, X_2, \ldots, X_p\}$ adapted to $\mathfrak{h}$ in $\mathfrak{g}_0$, where $\{X_j\}_{j=1}^p$ is that to $\mathfrak{h}' = \mathfrak{h} + \mathbb{R}Z$ in $\mathfrak{g}_0$. For generic $\ell \in \Gamma_\tau$ we note $\alpha = \ell(Z)$. Then $\tau^\alpha = \mathrm{ind}_H^{G_0} \chi_\ell$, where $\chi_\ell$ denotes the usual character of $H' = \exp\ \mathfrak{h}'$ defined by $d\chi_\ell = i\ell|_{\mathfrak{h}'}$, is of finite multiplicity. By means of $\{X_j\}_{j=1}^p$, we see that there exists for generic $\alpha$ a system of rational generators whose every element in $\mathcal{U}(\mathfrak{g}_0, \tau^\alpha) \setminus \mathcal{U}(\mathfrak{g}_0)\mathfrak{a}_{\tau^\alpha}$ is represented by

$$T(\alpha) = \sum_{k,J} c_{k,J} (-i\alpha)^k X_1^{j_1} X_2^{j_2} \cdots X_p^{j_p}$$

with a certain element

$$T = \sum_{k,J} c_{k,J} Z^k X_1^{j_1} X_2^{j_2} \cdots X_p^{j_p}$$

of $\mathcal{U}(\mathfrak{g}_0, \tau_0)$. Here, we used the notation $J = (j_1, j_2, \ldots, j_p)$ for $p$-set of non-negative integers.

Let $\mathfrak{g} = \mathfrak{g}_0 + \mathbb{R}X$ and we write with the help of the basis $\{Z, X_1, \ldots, X_p, X\}$ a representative element of $W$ and construct $W(\alpha) \in \mathcal{U}(\mathfrak{g}, \tau_\alpha), \tau_\alpha = \mathrm{ind}_{G_0}^G \tau^\alpha$. From what precedes, if $[W, T] \in \mathcal{U}(\mathfrak{g})\mathfrak{a}_\tau$ for any $T \in \mathcal{U}(\mathfrak{g}_0, \tau_0)$, then

$$[W, T(\alpha)] = [W(\alpha) + \tilde{W}(Z + i\alpha), T(\alpha)] \in \mathcal{U}(\mathfrak{g})\mathfrak{a}_{\tau_\alpha}$$

with a certain $\tilde{W} \in \mathcal{U}(\mathfrak{g})$. $\alpha$ being chosen in such a fashion as we have $W(\alpha) \notin \mathcal{U}(\mathfrak{g}_0) + \mathcal{U}(\mathfrak{g})\mathfrak{a}_{\tau_\alpha}$, this contradicts the induction hypothesis applied to the pair $(\mathfrak{h}', \chi_\ell)$.

Suppose hereafter that $\mathfrak{z} \subset \mathfrak{h}$. If $\mathfrak{z} \cap \ker f \neq \{0\}$, everything can pass to the quotient to give us the desired result. It only remains for us to examine the case where $\dim \mathfrak{z} = 1$ and where $f|_\mathfrak{z} \neq 0$. Take as usual a Heisenberg triple $(\tilde{X}, Y, Z)$ such that $\mathfrak{z} = \mathbb{R}Z, [\tilde{X}, Y] = Z, Y \in \mathfrak{g}_0$ and that $\mathfrak{g} = \mathbb{R}\tilde{X} + \mathfrak{k}$, where $\mathfrak{k}$ denotes the centralizer of $Y$ in $\mathfrak{g}$.

Let $\mathfrak{h} \subset \mathfrak{k}$. If $Y \in \mathfrak{h}$, then $\tau' = \mathrm{ind}_H^K \chi$ ($K = \exp\ \mathfrak{k}$) must be of infinite multiplicity. $\mathfrak{g}_0$ being given, $\tau_0' = \mathrm{ind}_H^{G_0 \cap K} \chi$ is of finite multiplicity and it suffices for us to apply to $\tau'$ the induction hypothesis. If $Y \notin \mathfrak{h}$, then $T = Y \in \mathcal{U}(\mathfrak{g}_0, \tau_0)$ has the required property when $\mathfrak{g}_0 = \mathfrak{k}$. Suppose therefore that $\mathfrak{g}_0 \neq \mathfrak{k}$. If $W \notin \mathcal{U}(\mathfrak{k}) + \mathcal{U}(\mathfrak{g})\mathfrak{a}_\tau$, we can always choose $T = Y$. Suppose that $W \in \mathcal{U}(\mathfrak{k}) + \mathcal{U}(\mathfrak{g})\mathfrak{a}_\tau$. Since $\tau_0 = \mathrm{ind}_{G_0 \cap K}^{G_0} \left(\mathrm{ind}_H^{G_0 \cap K} \chi\right)$ is of finite multiplicity, $\tau'$ must be of infinite multiplicity. Otherwise, $W \in \mathcal{U}(\mathfrak{k} \cap \mathfrak{g}_0) + \mathcal{U}(\mathfrak{g})\mathfrak{a}_\tau$, what is excluded by hypothesis. It suffices for us to apply to $\mathfrak{k}$ our induction hypothesis.

Let finally $\mathfrak{h} \not\subset \mathfrak{k}$. Let's consider $\mathfrak{h}_0 = \mathfrak{h} \cap \mathfrak{k}, H_0 = \exp\ \mathfrak{h}_0$ and $f_0 = f|_\mathfrak{k} \in \mathfrak{k}^*$. It follows that $\tau_1 = \mathrm{ind}_{H_0}^K \chi_{f_0}$ is of infinite multiplicity. Put $\mathfrak{m} = \mathfrak{g}_0 \cap \mathfrak{k}$ and $M = \exp\ \mathfrak{m}$. Remark that $\tau_2 = \mathrm{ind}_{H_0}^M \chi_{f_0}$ is of finite multiplicity. $W$ being represented by an element of $\mathcal{U}(\mathfrak{k})$, the induction hypothesis applied to $\mathfrak{k}$ implies that there exists $T \in \mathcal{U}(\mathfrak{m}, \tau_2)$ such that $[W, T] \notin \mathcal{U}(\mathfrak{k})\mathfrak{a}_{\tau_1}$. Here we can assume that $T$ is one of rational generators of $\mathcal{U}(\mathfrak{m}, \tau_2)$ introduced in Corwin-Greenleaf [14]. But the help of a weak Malcev basis adapted to $\mathfrak{h}$ in $\mathfrak{g}_0$, we notice that $\{Y, \gamma_2, \ldots, \gamma_q\}$ form such a system of rational generators for $\mathcal{U}(\mathfrak{m}, \tau_2)$, $\{\gamma_j\}_{j=2}^q$ being that for $\mathcal{U}(\mathfrak{g}_0, \tau_0)$. Since $[W, Y] = 0$, we can admit the existence of such a $T$ in $\mathcal{U}(\mathfrak{g}_0, \tau_0)$. c.q.f.d.

We keep certains notations introduced in the beginning of the chapter 3. In what follows, if $\mathfrak{g} \neq \mathfrak{h}$, $\mathfrak{g}'$ denotes always an ideal of codimension 1 in $\mathfrak{g}$ which contains $\mathfrak{h}$. Furthermore, we'll choose the flag (3.1) in such a way that $\mathfrak{g}_{n-1} = \mathfrak{g}'$. Similarly, if $\dim \mathfrak{h} \geq 1$, $\mathfrak{h}'$ denotes always a subalgebra of codimension 1 in $\mathfrak{h}$ and the flag (3.3) will be such that $\mathfrak{h}_{d-1} = \mathfrak{h}'$. Then, when $\mathfrak{g}'$, $\mathfrak{h}'$ both exist, $\dim \mathfrak{g} \geq 2$ and $\mathfrak{g} \supsetneq \mathfrak{g}' \supset \mathfrak{h} \supsetneq \mathfrak{h}'$. Suppose that $\mathfrak{h} \neq \{0\}$ and put $\tau' = \mathrm{ind}_{H'}^G \chi_f$ with $H' = \exp\ \mathfrak{h}'$.

We would like to assure the existence of the element $W$ in the theorem 6.1. Our first significant result in this direction is :

Theorem 6.2. ([3]) We assume that

$$\mathcal{U}(\mathfrak{g}, \mathfrak{h}') \not\subset \mathcal{U}(\mathfrak{g}') + \mathcal{U}(\mathfrak{g})\mathfrak{a}_{\tau'} \text{ and } \mathcal{U}(\mathfrak{g}, \tau') \cap \mathcal{U}(\mathfrak{g}') \not\subset \mathcal{U}(\mathfrak{g}, \tau) \cap \mathcal{U}(\mathfrak{g}').$$

Then we have $\mathcal{U}(\mathfrak{g}, \tau) \not\subset \mathcal{U}(\mathfrak{g}') + \mathcal{U}(\mathfrak{g})\mathfrak{a}_\tau$.

The theorems 6.1 and 6.2 offer us good tools to attack the commutativity conjecture in many situations, but it remains some cases which slip out their range. It's a matter of treating the case where $\mathcal{U}(\mathfrak{g}, \tau') \cap \mathcal{U}(\mathfrak{g}') \subset \mathcal{U}(\mathfrak{g}, \tau) \cap \mathcal{U}(\mathfrak{g}')$ in terms of the theorem 6.2. To settle these rather exceptional cases, the lemma 3.4 plays a key to open the final door to the following main theorem.

Theorem 6.3. ([31]) Let $G$ be a connected and simply connected nilpotent Lie group with Lie algebra $\mathfrak{g}$ such that $\dim \mathfrak{g} \geq 1$, $H$ a closed connected proper subgroup of $G$ with Lie algebra $\mathfrak{h}$ and $f$ an element of the linear dual $\mathfrak{g}^*$ of $\mathfrak{g}$ verifying $f([\mathfrak{h}, \mathfrak{h}]) = \{0\}$. Let $\mathfrak{g}'$ be an ideal of codimension 1 of $\mathfrak{g}$ containing $\mathfrak{h}$. Then the following properties are equivalent :
(i) $\mathcal{U}(\mathfrak{g}, \tau) \subset \mathcal{U}(\mathfrak{g}') + \mathcal{U}(\mathfrak{g})\mathfrak{a}_\tau$.
(ii) The $H$-orbit $H \cdot \ell$ is saturated in the direction ${\mathfrak{g}'}^\perp$ for generic $\ell \in \Gamma_\tau$.

We are now ready to prove the commutativity conjecture.

Corollary 6.4. We keep the notations. The algebra $D_\tau(G/H)$ is commutative if and only if $\tau = \mathrm{ind}_H^G \chi_f$ is of finite multiplicity.

Proof. Corwin-Greenleaf [14] already showed one direction : if $\tau$ is of finite multiplicity, the algebra $D_\tau(G/H)$ is commutative. Therefore, we only need to show the inverse direction. Suppose that $\tau$ is of infinite multiplicity, and let's see by induction on $\dim \mathfrak{g}$ that $D_\tau(G/H)$ is non commutative. Recall first [11] : generically on $\Gamma_\tau$,

$$\begin{aligned} \tau \text{ is of finite multiplicity} &\iff \dim H \cdot \ell = \frac{1}{2} \dim G \cdot \ell \\ &\iff 2(\dim \mathfrak{h} - \dim \mathfrak{h}(\ell)) = \dim \mathfrak{g} - \dim \mathfrak{g}(\ell) \end{aligned}$$

Consequently, it suffices to prove that $2(\dim \mathfrak{h} - \dim \mathfrak{h}(\ell)) < \dim \mathfrak{g} - \dim \mathfrak{g}(\ell)$ for generic $\ell \in \Gamma_\tau$ means that $D_\tau(G/H)$ is non commutative. In this case, evidently $\mathfrak{h} \neq \mathfrak{g}$. Let $\mathfrak{g}'$ be an ideal of codimension 1 in $\mathfrak{g}$ containing $\mathfrak{h}$, $G' = \exp\ \mathfrak{g}'$ and $\tau^* = \mathrm{ind}_H^{G'} \chi_f$. If $D_{\tau^*}(G'/H) \subset D_\tau(G/H)$ is already non commutative, nothing to do. Suppose that $D_{\tau^*}(G'/H)$ is commutative, what means that $2(\dim \mathfrak{h} - \dim \mathfrak{h}(\ell')) = \dim \mathfrak{g}' - \dim \mathfrak{g}'(\ell')$ for generic $\ell \in \Gamma_\tau$. From what,

$$2(\dim \mathfrak{h}(\ell') - \dim \mathfrak{h}(\ell)) < 1 + \dim \mathfrak{g}'(\ell') - \dim \mathfrak{g}(\ell) \leq 2$$

and hence $\mathfrak{h}(\ell') = \mathfrak{h}(\ell)$ for generic $\ell \in \Gamma_\tau$. Then, the theorem 6.3 claims that there exists an element $W \in \mathcal{U}(\mathfrak{g}, \tau)$ such that $W \notin \mathcal{U}(\mathfrak{g}') + \mathcal{U}(\mathfrak{g})\mathfrak{a}_\tau$. Finally, making use of the theorem 6.1, we obtain an element $T \in \mathcal{U}(\mathfrak{g}', \tau^*)$ such that $[W, T] \notin \mathcal{U}(\mathfrak{g})\mathfrak{a}_\tau$. It's in this manner that the algebra $D_\tau(G/H)$ is non commutative. c.q.f.d.

Remark 6.5. These studies concerning induced representations could be translated in nilpotent case for the restriction of unitary representations to subgroups (cf. [4], [5]). However, a plenty of questions remains open for exponential groups. It may be instructive to treat a completely solvable Lie group whose Lie algebra is a normal $j$-algebra in the sense of Pjatetskii-Shapiro because it possesses a remarkable algebraic structure.

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# Errata for the proceedings of Symposium on Representation Theory 2005

- “可解リー群のユニタリ表現に関する幾つかの話題（Some topics for unitary representations of solvable Lie groups）” ⋯ 藤原 英徳 (Hidenori Fujiwara)：

  **Pages 143–175:**
  At first, the abbreviation c.q.f.d. at the end of each proof must be replaced by q.e.d..

  **Page 143, line 8 from the bottom:**
  **§1. Orbite method** $\longrightarrow$ **§1. Orbit method**

  **Page 146, line 18:**
  Let $\mathfrak{g}_{\mathbb{C}}$ the complexification $\ldots$ $\longrightarrow$ Let $\mathfrak{g}_{\mathbb{C}}$ be the complexification $\ldots$

  **Page 146, line 19:**
  Let $\mathfrak{p}$ a complex $\ldots$ $\longrightarrow$ Let $\mathfrak{p}$ be a complex $\ldots$

  **Page 148, line 9:**
  $\ldots$ $D$-orbit $Df$ dans $\mathfrak{g}^*$. $\longrightarrow$ $\ldots$ $D$-orbit $Df$ in $\mathfrak{g}^*$.

  **Page 153, line 18:**
  $\ldots$ $\{X\}$ ou $\{X_1, X_2\}$, $\ldots$ $\longrightarrow$ $\ldots$ $\{X\}$ or $\{X_1, X_2\}$, $\ldots$

  **Page 154, line 2 from the bottom:**
  that there exists $\ldots$ $\longrightarrow$ that there exist $\ldots$

  **Page 155, line 17 from the bottom:**
  Contrary to the nilpotent cas, $\ldots$ $\longrightarrow$ Contrary to the nilpotent case, $\ldots$

  **Page 155, line 12 from the bottom:**
  $f + \mathfrak{h}^{\perp} \subset G{\cdot}f$ et $\ldots$ $\longrightarrow$ $f + \mathfrak{h}^{\perp} \subset G{\cdot}f$ and $\ldots$

  **Page 156, line 14:**
  Then, for $\phi \in \mathcal{H}(f, \mathfrak{h}_1, G)$ et $g \in G$, $\ldots$ $\longrightarrow$ Then, for $\phi \in \mathcal{H}(f, \mathfrak{h}_1, G)$ and $g \in G$, $\ldots$

  **Page 158, the last line:**
  $g^{\nu}(X) \longrightarrow \ell^{\nu}(X)$

  **Page 160, line 13:**
  de $\pi$ à $G_0$. $\longrightarrow$ of $\pi$ to $G_0$.

  **Page 160, line 17:**
  où $\ell_s = \exp(sX){\cdot}\ell_0$. $\longrightarrow$ where $\ell_s = \exp(sX){\cdot}\ell_0$.

**Page 161, line 16:**

$\ldots$ the restriction $\pi|_K$ of $\pi$ à $K$. $\longrightarrow$ $\ldots$ the restriction $\pi|_K$ of $\pi$ to $K$.

**Page 162, line 9:**

$\ldots$ if $\pi_\ell(A)$, où $\pi_\ell = \hat{\rho}_G(\ell)$, $\ldots$ $\longrightarrow$ $\ldots$ if $\pi_\ell(A)$, where $\pi_\ell = \hat{\rho}_G(\ell)$, $\ldots$

**Page 162, line 19:**

$\pi_\ell(A_j) = \varphi_j(\ell)Id$, où $\ldots$ $\longrightarrow$ $\pi_\ell(A_j) = \varphi_j(\ell)Id$, where $\ldots$

**Page 165, line 15:**

$\ldots$ let $C_k$ $(1 \leq k \leq m(\pi))$ the connected $\longrightarrow$ $\ldots$ let $C_k$ $(1 \leq k \leq m(\pi))$ be the connected

**Page 166, line 8:**

set $g_j \cdot (\ell + \mathfrak{b}^\perp) \cap \Gamma_\tau$ de $\Omega(\pi) \cap \Gamma_\tau$ $\ldots$ $\longrightarrow$ set $g_j \cdot (\ell + \mathfrak{b}^\perp) \cap \Gamma_\tau$ of $\Omega(\pi) \cap \Gamma_\tau$ $\ldots$

**Page 167, line 10 from the bottom:**

where $\nu = \mu_{H, H \cap g_k B g_k^{-1}}$. $\longrightarrow$ where $\nu = \mu_{H, H \cap g_k B g_k^{-1}}$?

**Page 167, the last line:**

$\ldots$ in this chapiter $\ldots$ $\longrightarrow$ $\ldots$ in this chapter $\ldots$

**Page 169, line 1:**

there exists in $\ldots$ $\longrightarrow$ there exist in $\ldots$

**Page 169, line 10:**

$\ldots$ Let $G = \exp \mathfrak{g}$ $\longrightarrow$ $\ldots$ Let $G = \exp \mathfrak{g}$ be

**Page 173, line 1:**

Références $\longrightarrow$ References