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Factor representations and their characters for the wreath products of compact groups with the infinite symmetric group

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ABSTRACT. Let T be a compact group, and denote by $G = \mathfrak{S}_\infty(T)$ the wreath product of T with the infinite symmetric group $\mathfrak{S}_\infty$. (1) We study first characters of facter representations of finite type of G and give a general character formula. (2) Then, for each character f, we give a realization of a factor representation π corresponding to it with a trace-element v such that $f(g) = \langle \pi(g)v, v\rangle$. (3) Further we study limits of irreducible characters of $G_n = \mathfrak{S}_n(T)$ as $n \to \infty$, and show that all the characters are obtained as such limits.

The following text contains the part (1) and summaris of the parts (2) and (3).

1 Characters of factor representations of finite type

We begin with two theorems in the general theory of representations of topological groups, which give us important backgrounds for our study.

Let G be a Hausdorff topological group, $K(G)$ the set of continuous positive definite class functions on G, and $K_1(G)$ the set of $f \in K(G)$ normalized as $f(e) = 1$ at the identity element $e \in G$, and $E(G) = \mathrm{Extr}\big(K_1(G)\big)$ the set of extremal points in the convex set $K_1(G)$.

On the other hand, let π be a continuous unitary representation (= UR) of G, and $\mathfrak{U} = \pi(G)''$ the von Neumann algebra generated by $\{\pi(g); g \in G\}$. Then π is called *factorial* if $\mathfrak{U}$ is a factor. If the factor is of finite type (type $\mathrm{I}_n, n < \infty$, or II_1), there exists a unique faithful finite normal trace t on the set $\mathfrak{U}^+$ of non-negative elements in $\mathfrak{U}$, normalized as $t(I) = 1$ at the identity operator I. The unique extention of t to a linear form on $\mathfrak{U}$ is denoted by ϕ, and the function

$$f(g) = \phi(\pi(g)) \qquad (g \in G) \tag{1}$$

is called a *character* of π. It naturally belongs to $K_1(G)$, and determines the quasi-equivalence class $[\pi]$ of π.

Theorem 1.1 ([HH3, Theorem 1.6.1]). *For a (Hausdorff) topological group G, let* $\mathrm{URfi}(G)$ *be the set of all quasi-equivalence classes of continuous unitary representations of G, factorial of finite type. Then there exists a canonical bijective correspondence between* $\mathrm{URfi}(G)$ *and $E(G)$ through (1) above. The converse map is given by $f \to [\pi_f]$, where π_f denotes the Gelfand-Raikov representation associated to f* [GR].

In [Dix, 17.3], the above canonical bijection is asserted under the condition that G is locally compact and unimodular. In the work [Voic] on factorial representations of $U(\infty)$, this point is not mentioned.

Let G be a topological group and N its normal subgroup with the relative topology. Denote by $K_1(N, G)$ the set of all $f \in K_1(N)$ which are G-invariant, that is, $f(g\xi g^{-1}) = f(\xi)$ $(\xi \in N, g \in G)$, and by $E(N, G)$ the set of extremal points $\mathrm{Extr}\big(K_1(N, G)\big)$. The following result is necessary to deduce Theorem 6.1 from Theorem 5.1, where $E(N, G) = E(N)$ in particular.

Theorem 1.2 ([HH6, Theorem 14.1]). *Let N be a normal subgroup of G. For any $F \in E(G)$, its restriction $f = F|_N$ onto N belongs to $E(N, G)$.*

2 Wreath products of compact groups with the infinite symmetric group

For a set I, we denote by $\mathfrak{S}_I$ the group of all finite permutations on A. A permutation σ on I is called *finite* if its support $\mathrm{supp}(\sigma) := \{ i \in I \,;\, \sigma(i) \neq i \}$ is finite. We call the *infinite symmetric group* the permutation group $\mathfrak{S}_{\boldsymbol{N}}$ on the set of natural numbers $\boldsymbol{N}$. The index $\boldsymbol{N}$ is frequently replaced by ∞. The symmetric group $\mathfrak{S}_n$ is naturally imbedded in $\mathfrak{S}_\infty$ as the permutation group of the set $I_n := \{1, 2, \ldots, n\} \subset \boldsymbol{N}$.

Let T be a compact group. We consider a wreath product group $\mathfrak{S}_I(T)$ of T with a permutation group $\mathfrak{S}_I$ as follows:

$$\mathfrak{S}_I(T) = D_I(T) \rtimes \mathfrak{S}_I \,, \quad D_I(T) = \textstyle\prod'_{i\in I} T_i \,, \quad T_i = T \ (i \in I), \tag{2}$$

where the symbol $\prod'$ means the restricted direct product, and $\sigma \in \mathfrak{S}_I$ acts on $D_I(T)$ as

$$D_I(T) \ni d = (t_i)_{i\in I} \stackrel{\sigma}{\longmapsto} \sigma(d) = (t'_i)_{i\in I} \in D_I(T), \quad t'_i = t_{\sigma^{-1}(i)} \ (i \in I). \tag{3}$$

Identifying groups $D_I(T)$ and $\mathfrak{S}_I$ with their images in the semidirect product $\mathfrak{S}_I(T)$, we have $\sigma\, d\, \sigma^{-1} = \sigma(d)$. The groups $D_{I_n}(T)$ and $\mathfrak{S}_{I_n}(T)$ are denoted by $D_n(T)$ and $\mathfrak{S}_n(T)$ respectively, then $G := \mathfrak{S}_\infty(T)$ is an inductive limit of $G_n := \mathfrak{S}_n(T) = D_n(T) \rtimes \mathfrak{S}_n$. Since T is compact, G_n is also compact, and the inductive system $(G_n)_{n\geq 1}$ is an example of countable LCG inductive systems in [TSH]. We introduce in G its inductive limit topology τ_{ind}. Then G with τ_{ind} becomes a topological groups (cf. **2.7** in [TSH]). By definition, a subset $B \subset G$ is τ_{ind}-open if and only if $B \cap G_n$ is open in G_n for any $n \geq 1$. A general theory of unitary representations of the inductive limit group G of a countable LCG inductive system is carried out in [TSH, §5] using continuous positive definite functions on the group, and we know that G with τ_{ind} has sufficiently many URs.

Lemma 2.1. (i) *The topology τ_{ind} on $G = \mathfrak{S}_\infty(T) = D_\infty(T) \rtimes \mathfrak{S}_\infty$ is discrete if T is finite, and is not locally compact if T is continuous.*

(ii) *In the topology τ_{ind} on $G = \mathfrak{S}_\infty(T)$, the subgroup $D_\infty(T)$ is open. Denote by τ^D_{ind} the inductive limit topology on $D_\infty(T)$ of the topologies on $D_n(T)$, then τ_{ind} on G is the product of τ^D_{ind} and the discrete topology $\tau^{\mathfrak{S}}_{disc}$ on $\mathfrak{S}_\infty$.*

Put $\Pi_I(T) = \prod_{i\in I} T_i$ be the direct product of $T_i = T$ over $i \in I$, and let τ_{prod} denote the product topology on $\Pi_I(T)$.

When T is a non-trivial finite group, the topology τ_{prod} on $\Pi_{\boldsymbol{N}}(T)$ is not discrete but totally disconnected, whereas the topology τ^D_{ind} on $D_\infty(T)$ is discrete. Thus τ_{ind} in $G = \mathfrak{S}_\infty(T)$ is discrete, and this case is worked out in [HH2]–[HH3].

When T is infinite, τ_{ind} is not discrete, and a subset $\{(d, \mathbf{1}) \,;\, d \in D_\infty(T)\} \cong D_\infty(T)$ is an open neighbourhood of the identity element e of G, where $\mathbf{1}$ denotes the trivial permutation on $\boldsymbol{N}$.

A natural subgroup of $G = \mathfrak{S}_\infty(T)$ is given as a wreath product of T with the alternating group $\mathfrak{A}_\infty$ as $G' := \mathfrak{A}_\infty(T) = D_\infty(T) \rtimes \mathfrak{A}_\infty$. Furthermore, in the case where T is abelian, we put

$$P_I(d) = \prod_{i\in I} t_i \quad \text{for} \quad d = (t_i)_{i\in I} \in D_I(T), \tag{4}$$

and, for a subgroup S of T, we define a subgroup of $\mathfrak{S}_I(T)$ as

$$\mathfrak{S}_I^S(T) = D_I^S(T) \rtimes \mathfrak{S}_I \quad \text{with} \quad D_I^S(T) := \{\, d = (t_i)_{i\in I} \;;\; P_I(d) \in S \,\}. \tag{5}$$

The subgroup $G^S := \mathfrak{S}_I^S(T)$ is normal in G and $[G : G^S] = [T : S]$. If S is trivial or $S = \{e_T\}$ (and T is finite), we denote $G^S = \mathfrak{S}_I^S(T)$ simply by $G^e = \mathfrak{S}_I^e(T)$. This kind of groups $\mathfrak{S}_\infty(T)$ and $\mathfrak{S}_\infty^S(T)$ with T abelian contain, as their special cases, the infinite Weyl groups of classical types, $W_{\mathbf{A}_\infty} = \mathfrak{S}_\infty$ of type $\mathbf{A}_\infty$, $W_{\mathbf{B}_\infty} = \mathfrak{S}_\infty(\boldsymbol{Z}_2)$ of type $\mathbf{B}_\infty/\mathbf{C}_\infty$, and $W_{\mathbf{D}_\infty} = \mathfrak{S}_\infty^{\{e_T\}}(\boldsymbol{Z}_2)$ of type $\mathbf{D}_\infty$, and moreover the inductive limits $\mathfrak{S}_\infty(\boldsymbol{Z}_r) = \lim_{n\to\infty} G(r,1,n)$ of complex reflexion groups $G(r,1,n) = \mathfrak{S}_n(\boldsymbol{Z}_r)$ (cf. [AK], [Kaw], [Sho]).

3 Structure of wreath product groups $\mathfrak{S}_\infty(T) = D_\infty(T) \rtimes \mathfrak{S}_\infty$

Fix a compact group T, and take the wreath product group $\mathfrak{S}_\infty(T)$ of T with the symmetric group $\mathfrak{S}_\infty$:

$$\mathfrak{S}_\infty(T) = D_\infty(T) \rtimes \mathfrak{S}_\infty, \quad D_\infty(T) := \textstyle\prod'_{i\in \boldsymbol{N}} T_i\,, \quad T_i = T \quad (i \in \boldsymbol{N}). \tag{6}$$

An element $g = (d, \sigma) \in G = \mathfrak{S}_\infty(T)$ is called *basic* in the following two cases:

Case 1: σ is cyclic and $\mathrm{supp}(d) \subset \mathrm{supp}(\sigma)$;
Case 2: $\sigma = \mathbf{1}$ and for $d = (t_i)_{i\in\boldsymbol{N}}$, $t_q \neq e_T$ only for one $q \in \boldsymbol{N}$.

The element $(d, \mathbf{1})$ in Case 2 is denoted by ξ_q, and put $\mathrm{supp}(\xi_q) := \mathrm{supp}(d) = \{\, q \,\}$.

For a cyclic permutation $\sigma = (i_1\ \ i_2\ \ \cdots\ \ i_\ell)$ of ℓ integers, we define its *length* as $\ell(\sigma) = \ell$, and for the identity permutation $\mathbf{1}$, put $\ell(\mathbf{1}) = 1$ for convenience. In this connection, ξ_q is also denoted by $(t_q, (q))$ with a trivial cyclic permutation (q) of length 1. In Cases 1 and 2, put $\ell(g) = \ell(\sigma)$ for $g = (d, \sigma)$, and $\ell(\xi_q) = 1$. It is very helpful for us to illustrate these basic elements by permutation matrices with entries from T or more correctly from the group algebra of T. For $g = (d, \sigma)$ with $d = (t_1, t_2, \ldots, t_\ell)$, $\sigma = (1\ \ 2\ \ 3\ \ \cdots\ \ \ell)$, and $\xi_q = (t_q, (q))$, their expressions in matrix form are respectively

$$\begin{pmatrix} 0 & 0 & \cdots & 0 & 0 & t_1 \\ t_2 & 0 & \cdots & 0 & 0 & 0 \\ 0 & t_3 & \ddots & 0 & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots & \vdots \\ 0 & \cdots & 0 & t_{\ell-1} & 0 & 0 \\ 0 & \cdots & 0 & 0 & t_\ell & 0 \end{pmatrix}, \qquad \begin{array}{cc} \begin{pmatrix} e_T & \cdots & 0 & 0 & 0 & \cdots \\ \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & e_T & 0 & 0 & \cdots \\ 0 & \cdots & 0 & t_q & 0 & \cdots \\ 0 & \cdots & 0 & 0 & e_T & \cdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots \end{pmatrix} & q\text{-th.} \end{array}$$

An arbitrary element $g = (d, \sigma) \in G$, is expressed as a product of basic elements as

$$g = \xi_{q_1}\xi_{q_2}\cdots\xi_{q_r}g_1g_2\cdots g_m \tag{7}$$

with $g_j = (d_j, \sigma_j)$ in Case 1, in such a way that the supports of these components, $q_1, q_2, \ldots,$ q_r, and $\mathrm{supp}(g_j) = \mathrm{supp}(\sigma_j)$ $(1 \le j \le m)$, are mutually disjoint. This expression of g is unique up to the orders of ξ_{q_k}'s and g_j's, and is called *standard decomposition* of g. Note that, for $\mathfrak{S}_\infty$-components, $\sigma = \sigma_1\sigma_2\cdots\sigma_m$ gives the cycle decomposition of σ.

To write down conjugacy class of $g = (d, \sigma)$, there appear products of components t_i of $d = (t_i)$, where the orders of taking products are crucial when T is not abelian. So we should fix notations well. We denotes by $[t]$ the conjugacy class of $t \in T$, and by $T/\!\sim$ the set of all conjugacy classes of T, and $t \sim t'$ denotes that $t, t' \in T$ are mutually conjugate in T. For a basic component $g_j = (d_j, \sigma_j)$ of g, let $\sigma_j = (i_{j,1}\ \ i_{j,2}\ \ \ldots\ \ i_{j,\ell_j})$ and put $K_j := \mathrm{supp}(\sigma_j) = \{\, i_{j,1},\, i_{j,2},\, \ldots,\, i_{j,\ell_j} \,\}$ with $\ell_j = \ell(\sigma_j)$. For $d_j = (t_i)_{i \in K_j}$, we put

$$P_{\sigma_j}(d_j) := \left[t'_{\ell_j}t'_{\ell_j-1}\cdots t'_2t'_1\right] \in T/\!\sim \qquad \text{with} \quad t'_k = t_{i_{j,k}} \ \ (1 \le k \le \ell_j). \tag{8}$$

Note that the product $P_{\sigma_j}(d_j)$ is well-defined, because, for $t_1, t_2, \ldots, t_\ell \in T$, we have $t_1t_2\cdots t_\ell \sim t_kt_{k+1}\cdots t_\ell t_1\cdots t_{k-1}$ for any k, that is, the conjugacy class does not depend on any cyclic permutation of $(t_1, t_2, \ldots, t_\ell)$.

Lemma 3.1. (i) *Let $\sigma \in \mathfrak{S}_\infty$ be a cycle, and put $K = \mathrm{supp}(\sigma)$. Then, an element $g = (d, \sigma) \in \mathfrak{S}_K(T)(=: G_K$ (put)) is conjugate in it to $g' = (d', \sigma) \in G_K$ with $d' = (t'_i)_{i\in K}, t'_i = e_T$ $(i \ne i_0), [t'_{i_0}] = P_\sigma(d)$ for some $i_0 \in K$.*

(ii) *Identify $\tau \in \mathfrak{S}_\infty$ with its image in $G = \mathfrak{S}_\infty(T)$. Then we have, for $g = (d, \sigma)$, $\tau\, g\tau^{-1} = (\tau(d), \tau\sigma\tau^{-1}) =: (d', \sigma')$ (put), and $P_{\sigma'}(d') = P_\sigma(d)$.*

Applying this lemma to each basic components $g_j = (d_j, \sigma_j)$ of $g \in G$ in (7), we get

Theorem 3.2. *Let T be a compact group. Take an element $g \in G = \mathfrak{S}_\infty(T)$ and let its standard decomposition into basic elements be $g = \xi_{q_1}\xi_{q_2}\cdots\xi_{q_r}g_1g_2\cdots g_m$ in (7), with $\xi_{q_k} = (t_{q_k}, (q_k)\,)$, and $g_j = (d_j, \sigma_j)$, σ_j cyclic, $\mathrm{supp}(d_j) \subset \mathrm{supp}(\sigma_j)$. Then the conjugacy class of g is determined by*

$$[t_{q_k}] \in T/\!\sim \quad (1 \le k \le r) \quad \textit{and} \quad (P_{\sigma_j}(d_j), \ell(\sigma_j)\,) \quad (1 \le j \le m), \tag{9}$$

where $P_{\sigma_j}(d_j) \in T/\!\sim$ and $\ell(\sigma_j) \ge 2$.

Note that we put $\ell(\xi_{q_k}) = 1$, $\ell(g_j) = \ell(\sigma_j) \ge 2$.

Lemma 3.3. *A finite-dimensional continuous irreducible unitary representation (= IUR) π of $\mathfrak{S}_\infty(T)$ is a one-dimensional character, and is given in the form $\pi = \pi_{\zeta,\varepsilon}$ with*

$$\pi_{\zeta,\varepsilon}(g) = \zeta(P(d))\ (\mathrm{sgn}_{\mathfrak{S}})^\varepsilon(\sigma) \quad \text{ for } \ g = (d, \sigma) \in \mathfrak{S}_\infty(T) = D_\infty(T) \rtimes \mathfrak{S}_\infty,$$

where ζ is a one-dimensional character of T, $P(d)$ is a product of components t_i of $d = (t_i)$, and $\mathrm{sgn}_{\mathfrak{S}}(\sigma)$ denotes the usual sign of σ and $\varepsilon = 0, 1$. (Since $\zeta(P(d)) = \prod_{i\in\boldsymbol{N}}\zeta(t_i)$, the order of taking product for $P(d)$ has no meaning even if T is not abelian.)

In the case where T is abelian and S a closed subgroup of T, we have a closed subgroup $G^S = \mathfrak{S}^S_\infty(T)$ of $G = \mathfrak{S}_\infty(T)$. Then a similar assertion as Lemma 3.3 hold for G^S too.

4 Factorizable positive definite class functions

Let T be a compact group, and f a continuous positive definite class function on $G = \mathfrak{S}_\infty(T)$ or $f \in K(G)$. An $f \in K(G)$ is called *factorizable* if it has the following properties which are mutually equivelent:

(FTP) *For any* $g = (d, \sigma) \in G$, *let* $g = \xi_{q_1}\xi_{q_2}\cdots\xi_{q_r}g_1g_2\cdots g_m, \xi_q = \big(t_q, (q)\big),\ g_j = (d_j, \sigma_j)$, *be its standard decomposition. Then,*

$$f(g) = \textstyle\prod_{1\le k\le r} f(\xi_{q_k}) \ \times\ \prod_{1\le j\le m} f(g_j). \tag{10}$$

(FTP$'$) *For any two elements* g, g' *with disjoint supports in* $\mathfrak{S}_\infty$, $f(gg') = f(g)f(g')$.

Let $F(G)$ be the sets of all factorizable f in $K_1(G)$.

Theorem 4.1 ([HH6, Theorem 4.2])**.** *Let* $G = \mathfrak{S}_\infty(T)$ *with a compact group* T*. An* $f \in K_1(G)$ *is extremal if and only if it is factorizable, that is,* $E(G) = F(G)$.

5 Characters of $\mathfrak{S}_\infty(T)$ with T any compact group

Let $\widehat{T}$ be the dual of T consisting of all equivalence classes of IURs. We identify every equivalence class with one of its representative. Thus $\zeta \in \widehat{T}$ is an IUR and denote by χ_ζ its character: $\chi_\zeta(t) = \mathrm{tr}(\zeta(t))$ $(t \in T)$, then $\dim\zeta = \chi_\zeta(e_T)$. Put $G = \mathfrak{S}_\infty(T)$. For a $g \in G$, let its standard decomposition into basic components be $g = \xi_{q_1}\xi_{q_2}\cdots\xi_{q_r}g_1g_2\cdots g_m$, as in (7), where $\xi_{q_k} = (t_{q_k}, (q_k)), t_{q_k} \ne e_T$, with $\ell(\xi_{q_k}) = 1$ for $1 \le k \le r$, and σ_j is a cycle of length $\ell(\sigma_j) \ge 2$ and $\mathrm{supp}(d_j) \subset K_j = \mathrm{supp}(\sigma_j)$. For $d_j = (t_i)_{i\in K_j} \in D_{K_j}(T) \hookrightarrow D_\infty(T)$, put $P_{\sigma_j}(d_j)$ as in (8).

For one-dimensional characters of $\mathfrak{S}_\infty$, we introduce simple notation as

$$\chi_\varepsilon(\sigma) := \mathrm{sgn}_{\mathfrak{S}}(\sigma)^\varepsilon \quad (\sigma \in \mathfrak{S}_\infty\,;\ \varepsilon = 0, 1). \tag{11}$$

As a parameter for characters of $G = \mathfrak{S}_\infty(T)$, we prepare a set

$$\alpha_{\zeta,\varepsilon}\ (\zeta \in \widehat{T}, \varepsilon \in \{\,0,1\,\}) \quad \text{and} \quad \mu = (\mu_\zeta)_{\zeta\in\widehat{T}}\,, \tag{12}$$

of decreasing sequences of non-negative real numbers

$$\alpha_{\zeta,\varepsilon} = (\alpha_{\zeta,\varepsilon,i})_{i\in\boldsymbol{N}}\,,\ \ \alpha_{\zeta,\varepsilon,1} \ge \alpha_{\zeta,\varepsilon,2} \ge \alpha_{\zeta,\varepsilon,3} \ge\ \cdots\ \ge 0\,;$$

and a set of non-negative $\mu_\zeta \ge 0$ $(\zeta \in \widehat{T})$, which altogether satisfy the condition

$$\sum_{\zeta\in\widehat{T}}\ \sum_{\varepsilon\in\{\,0,1\,\}} \|\alpha_{\zeta,\varepsilon}\| \ +\ \|\mu\| \ =\ 1\,, \tag{13}$$

$$\text{with} \qquad \|\alpha_{\zeta,\varepsilon}\| = \textstyle\sum_{i\in\boldsymbol{N}} \alpha_{\zeta,\varepsilon,i}\,, \quad \|\mu\| = \sum_{\zeta\in\widehat{T}} \mu_\zeta\,.$$

Note that, under the condition (14), there exists a countable subset $\widehat{T}_0 \subset \widehat{T}$ such that $\alpha_{\zeta,\varepsilon} = \boldsymbol{0}$ and $\mu_\zeta = 0$ for $\zeta \notin \widehat{T}_0$.

Theorem 5.1 ([HH6, Theorem 5.1]). *Let $G = \mathfrak{S}_\infty(T)$ be a wreath product group of a compact group T with $\mathfrak{S}_\infty$. For a parameter $A := \Big((\alpha_{\zeta,\varepsilon})_{(\zeta,\varepsilon)\in\widehat{T}\times\{0,1\}}\,;\,\mu\Big)$ in (12)–(13), the following formula gives an element in $E(G) = F(G)$: for a $g \in G$, let (7) be its standard decomposition, then put*

$$\begin{aligned} f_A(g) \;&=\; \prod_{1\le k\le r}\left\{\sum_{\zeta\in\widehat{T}}\left(\sum_{\varepsilon\in\{0,1\}}\sum_{i\in\boldsymbol{N}}\frac{\alpha_{\zeta,\varepsilon,i}}{\dim\zeta} + \frac{\mu_\zeta}{\dim\zeta}\right)\chi_\zeta(t_{q_k})\right\} \\ &\times \prod_{1\le j\le m}\left\{\sum_{\zeta\in\widehat{T}}\left(\sum_{\varepsilon\in\{0,1\}}\sum_{i\in\boldsymbol{N}}\left(\frac{\alpha_{\zeta,\varepsilon,i}}{\dim\zeta}\right)^{\ell(\sigma_j)}\chi_\varepsilon(\sigma_j)\right)\chi_\zeta\big(P_{\sigma_j}(d_j)\big)\right\}, \end{aligned} \tag{14}$$

where $\chi_\varepsilon(\sigma_j) = \mathrm{sgn}_{\mathfrak{S}}(\sigma_j)^\varepsilon = (-1)^{\varepsilon(\ell(\sigma_j)-1)}$.

Conversely any element in $E(G)$ or any character of G is given in the form of f_A.

The case of the infinite symmetric group $\mathfrak{S}_\infty$ itself is considered as an extreme case of the wreath product groups $\mathfrak{S}_\infty(T)$ with a trivial group $T = \{e_T\}$. For $\mathfrak{S}_\infty$, we have only the so-called Thoma parameters $\alpha = (\alpha_p)_{p\in\boldsymbol{N}}, \beta = (\beta_p)_{p\in\boldsymbol{N}}$ in [Tho] satisfying the inequality condition $\|\alpha\| + \|\beta\| \le 1$.

Then, for the parameter $A = \Big((\alpha_{\zeta,\varepsilon})_{(\zeta,\varepsilon)\in\widehat{T}\times\{0,1\}}\,;\,\mu\Big)$ of the character f_A, we put

$$\alpha_{0,\mathbf{1}_T} = \alpha, \quad \alpha_{1,\mathbf{1}_T} = \beta, \tag{15}$$

with the trivial representation $\mathbf{1}_T$ of T, and introduce a fake parameter $\mu = (\mu_{\mathbf{1}_T})$ for the trivial representation $\mathbf{1}_T$ of $T = \{e_T\}$ by putting $\mu_{\mathbf{1}_T} := 1 - (\|\alpha_{0,\mathbf{1}_T}\| + \|\alpha_{1,\mathbf{1}_T}\|)$. Then the equality condition (13) is established.

6 Characters of the subgroup $\mathfrak{S}_\infty^S(T) \subset \mathfrak{S}_\infty(T)$, $S \subset T$ abelian compact

Let T be abelian and S its subgroup. Then, the natural subgroup $G^S = \mathfrak{S}_\infty^S(T) = D_\infty^S(T) \rtimes \mathfrak{S}_\infty$ is defined by (4)–(5). We can deduce a general character formula for this normal subgroup $N = G^S$ from the one for $G = \mathfrak{S}_\infty(T)$, especially when S is open in T.

Take a $g \in N$ and let its standard decomposition in $G \supset N$ be $g = \xi_{q_1}\xi_{q_2}\cdots\xi_{q_r}\,g_1g_2\cdots g_m$ with $\xi_{q_k} = (t_{q_k}, (q_k))$ and $g_j = (d_j, \sigma_j), d_j = (t_i)_{i\in K_j}, K_j = \mathrm{supp}(\sigma_j)$. Note that each component ξ_{q_k} does not necessarily belong to N, and that the component $g_j = (d_j, \sigma_j)$ belongs to N if and only if $P(d_j) = \prod_{i\in K_j} t_i \in S$. However, we know that $E(N, G) = E(G)$ in this case, and discussing relations between N and G, and using Theorem 1.2, we obtain the following result for $N = G^S$ from the result for G.

Theorem 6.1 ([HH6, Theorem 7.1]). *Let T be abelian and S a subgroup of T, and let $N = G^S = \mathfrak{S}_\infty^S(T)$ be the subgroup of $G = \mathfrak{S}_\infty(T)$ given by (4)–(5).*

(i) *For any character $f \in E(G)$ of a factor representaion of finite type of G, the restriction $f^S = f|_N$ on N is again such a character of N or $f^S \in E(N)$.*

(ii) *For* $f_A \in E(G)$ *with a parameter* $A = \Big((\alpha_{\zeta,\varepsilon})_{(\zeta,\varepsilon)\in\widehat{T}\times\{0,1\}}\,;\,\mu\Big)$ *in* (12)–(13)*, its restriction* $f_A^S := f_A|_N$ *is in* $E(N)$*, and is given as follows: for a* $g \in N$*, let its standard decomposition in* G *be as above, then*

$$\begin{aligned} f_A^S(g) \;&=\; \prod_{1\le k\le r}\left\{\sum_{\zeta\in\widehat{T}}\left(\sum_{\varepsilon\in\{0,1\}}\sum_{i\in\boldsymbol{N}}\alpha_{\zeta,\varepsilon,i} \;+\; \mu_\zeta\right)\zeta(t_{q_k})\right\} \\ &\times\;\prod_{1\le j\le m}\left\{\sum_{\zeta\in\widehat{T}}\left(\sum_{\varepsilon\in\{0,1\}}\sum_{i\in\boldsymbol{N}}(\alpha_{\zeta,\varepsilon,i})^{\ell(\sigma_j)}\cdot\chi_\varepsilon(\sigma_j)\right)\zeta(d_j)\right\}, \end{aligned} \tag{16}$$

where $\chi_\varepsilon(\sigma_j) = \mathrm{sgn}_{\mathfrak{S}}(\sigma_j)^\varepsilon = (-1)^{\varepsilon(\ell(\sigma_j)-1)}$*, and* $\zeta(d_j) = \zeta(P(d_j))$*.*

(iii) *Assume* S *is open in* T*. Then any character of* $N = G^S$ *is given in the form of* f_A^S*, that is,* $E(N) = \{f_A^S;\ A \text{ in } (12)\text{–}(13)\}$*.*

Similar results are obtained for other subgroups as $G' = \mathfrak{A}_\infty(T)$ and ${G'}^S = G' \cap G^S$.

To describe the correspondence of parameters, we introduce a translation $R(\zeta_0)$ on A by an element $\zeta_0 \in \widehat{T}$ as follows: $R(\zeta_0)A := \Big((\alpha'_{\zeta,\varepsilon})_{(\zeta,\varepsilon)\in\widehat{T}\times\{0,1\}}\,;\,R(\zeta_0)\mu\Big)$ with $\alpha'_{\zeta,\varepsilon} = \alpha_{\zeta{\zeta_0}^{-1},\varepsilon}\ \big((\zeta,\varepsilon)\in\widehat{T}\times\{0,1\}\big)$, $R(\zeta_0)\mu = (\mu'_\zeta)_{\zeta\in\widehat{T}}$, $\mu'_\zeta = \mu_{\zeta{\zeta_0}^{-1}}$.

Proposition 6.2. *Assume that two parameters of characters of* G

$$A = \Big((\alpha_{\zeta,\varepsilon})_{(\zeta,\varepsilon)\in\widehat{T}\times\{0,1\}}\,;\,\mu\Big) \quad \textit{and} \quad A' = \Big((\alpha'_{\zeta,\varepsilon})_{(\zeta,\varepsilon)\in\widehat{T}\times\{0,1\}}\,;\,\mu'\Big)$$

satisfy the normalization condition (13) *respectively. Then, they determine the same function on* $N = G^S$*, or* $f_A^S = f_{A'}^S$*, if and only if* $A' = R(\zeta_S)A$ *for some* $\zeta_S \in \widehat{T}$ *which is trivial on* S*. In this case, as elements in* $E(G)$ *for the bigger group* G*, we have*

$$f_{A'}(g) = \pi_{\zeta_S,0}(g)\cdot f_A(g) \quad (g\in G).$$

7 Method of Proof of Theorem 5.1 in [HH6]

Let $G = \mathfrak{S}_\infty(T)$. The first part of our proof is to prepare seemingly sufficiently big family $E'(G)$ of factorizable continuous positive definite class functions f_A on G. The second part is to prove that $E'(G)$ is actually equal to $F(G) = E(G)$.

7.1. The first part of the proof. The first part has two important ingredients. The one is a method of *taking limits of centralizations* of positive definite functions. The other is *inducing up positive definite functions* from subgroups.

7.1.1. For a continuous positive definite function F on a topological group G and a compact subgroup $G' \subset G$, we define a *centralization* of F with respect to G' as

$$F^{G'}(g) := \int_{g'\in G'} F(g'g{g'}^{-1})\,d\mu_{G'}(g'), \tag{17}$$

where $\mu_{G'}$ denotes the normalized Haar measure on G'. Then $F^{G'}$ is automatically invariant under G'.

Assume that we have an increasing sequence of compact subgroups $G_N \nearrow G$. Then we can examine if the series of continuous positive definite functions F^{G_N} converges pointwise to a continuous function $F_\infty = \lim_{N\to\infty} F^{G_N}$. If it does, then F_∞ is necessarily a positive definite *class* function.

Choosing starting functions F as simple as possible, we check what we get as the limit functions F_∞ which depend heavily on the choices of the series $G_N \nearrow G$. This is a kind of 'trial and error' method. In the case of $G = \mathfrak{S}_\infty(T)$, this method woprks very well.

However, in the case of $G = GL(\infty, \boldsymbol{F})$, where $\boldsymbol{F}$ is a finite field, it doesn't, and we obtain non until now except the delta function δ_e on G supported by the identity element $e \in G$.

7.1.2. We choose appropriate subgroups H and their URs π and use their diagonal matrix elements f_π as positive definite functions on H to be induced up to G as $F = \mathrm{Ind}_H^G f_\pi$ (see Lemma 7.1 below). Then we centralize F along with some increasing sequences $G_N \nearrow G$ as F^{G_N} and check their limits $F_\infty = \lim_{N\to\infty} F^{G_N}$.

We have constructed in [Hir1]-[Hir2] a huge family of IURs of a wreath product group $G = \mathfrak{S}_\infty(T)$ with any finite group T, by taking the so-called wreath product type subgroups H in a 'saturated fashion', and their IURs π of a certain form to get IURs of G as induced representations $\rho = \mathrm{Ind}_H^G \pi$. For our present purpose of getting a big set $E'(G)(\subset F(G))$ of f_A's on G, actually it is sufficient to choose simpler subgroups H of degenerate wreath product type and their IURs π. Then the induced representations $\rho = \mathrm{Ind}_H^G \pi$ are very far from to be irreducible, but sufficient for our purpose to obtain positive definite class functions on G.

In a general setting, we have the following fact.

Lemma 7.1. *Let G be a group and H its subgroup. Take a positive definite function f on H, and extend it trivially onto G by putting zero outside of H, which is denoted by $F = \mathrm{Ind}_H^G f$. Then F is positive definite on G.*

As an example of positive definite functions f on H, we can take a matrix element of a UR π of H on a Hilbert space $V(\pi)$ as

$$f_\pi(h) = \langle \pi(h)v, v\rangle \ \ (h \in H) \quad \text{with} \quad v \in V(\pi), \|v\| = 1.$$

In the case where H is open in G, or in particular G is discrete, the trivial inducing up $F = \mathrm{Ind}_H^G f_\pi$ is a matrix element for the induced representation $\rho = \mathrm{Ind}_H^G \pi$. Let G' be a compact subgroup of G and take a centralization $F^{G'}$ of F. Since F is zero outside of H, the value of centralization $F^{G'}(g)$ is $\neq 0$ only for elements g which are conjugate under G' to some $h \in H$, and moreover, for $h \in H$,

$$F^{G'}(h) = \int_{G'} f(g'hg'^{\,-1})\, d\mu_{G'}(g'), \tag{18}$$

where we put $f(g'hg'^{\,-1}) = 0$ if $g'hg'^{\,-1} \notin H$, by definition, whence the integrand $\neq 0$ only if $g'hg'^{-1} \in H$.

A pointwise limit F_∞ of F^{G_N} as $N \to \infty$ for an increasing sequence $G_N \nearrow G$ of compact subgroups of G is certainly positive definite and invariant, and may be *continuous* or may be *not* with respect to τ_{ind}. The condition $g'hg'^{-1} \in H$ for $g' \in G_N$, is translated into certain combinatorial conditions, and to get the limit F_∞ of F^{G_N}, we have to calculate asymtotic behavior of several ratios of combinatorial numbers. In the discrete case or the case of a finite group T, the above integral turns out to be a sum which is calculated by some combinatorics [HH3].

7.2. The second part of the proof. The second part is to guarantee that actually all characters have been already obtained in the first part. For that, we prove first the equality $F(G) = E(G)$, and then $E'(G) = F(G)$, or the *completeness* of $E'(G)$.

7.2.1. To prove $E(G) = F(G)$, in the case of finite groups T, the point is that $K_{\leq 1}(G)$ is weakly compact. Then, we can use the theorem of Choquet-Bishop-K. de Leeuw (cf. [BK]) of integral representation theorem for a compact convex set.

In the case where T is continuous, the assertion that $K_{\leq 1}(G)$ is compact in a certain natural topology has not yet been established, and so we take, for a fixed $f \in K_{\leq 1}(G)$, a smaller part $K_{\leq 1}(G; f)$ of $K_{\leq 1}(G)$ consisting of f' majorized by f. It is proved that the convex set $K_{\leq 1}(G; f)$ is compact and then we can appeal to the above mentiond theorem.

7.2.2. To prove $E'(G) = F(G)$, we proceed as follows. Take an $f \in F(G)$. We can take a kind of *partial Fourier transform* of f on $G = D_\infty(T) \rtimes \mathfrak{S}_\infty$ with respect to subgroups $D_n(T) \subset D_\infty(T)$, and get a series of positive definite class functions $\mathcal{F}_{\zeta,\varepsilon,n}(f)$ on $\mathfrak{S}_n, n \geq 1$, for any $\zeta \in \widehat{T}, \varepsilon = 0, 1$. For a fixed (ζ, ε), we appeal to Korollar 1 to Satz 2 in [Tho] for the series of positive definite class functions $\mathcal{F}_{\zeta,\varepsilon,n}(f)$ on $\mathfrak{S}_n, n \geq 1$.

8 Realizations of factor representations of finite type with emphasis on their characters for wreath products of compact groups with the infinite symmetric group [HHH1]

(Rewrite of Introduction of [HHH1])

Let A be a datum which determines a character f_A of the wreath product group $\mathfrak{S}_\infty(T)$ of compact group T with the infinite symmetric group $\mathfrak{S}_\infty$. We mean by a character an extremal continuous positive definite class function on the group. The aim of this paper is to construct a nice realization of a factor representation of finite type of $\mathfrak{S}_\infty(T)$ for any A which yields f_A as its matrix element.

The character formula for $\mathfrak{S}_\infty$ was established by Thoma in [Tho]. Later in [VK2], Vershik-Kerov characterized the Thoma parameters as asymptotic frequencies of growing Young diagrams and showed that the characters of $\mathfrak{S}_\infty$ are expressed as pointwise limits of the normalized irreducible characters of $\mathfrak{S}_n$, the symmetric group of degree n. In [Hir4] the Thoma characters are captured anew by using a different kind of approximation procedure. This method has an advantage that it is applicable to general wreath product groups including the infinite Weyl groups of other types. In a series of works [HH1]–[HH6], we obtained a complete and unified character formula for the wreath product $\mathfrak{S}_\infty(T)$ of any compact group T with the infinite symmetric group $\mathfrak{S}_\infty$.

On the other hand, Vershik-Kerov constructed in [VK1] a factor representation of finite type of $\mathfrak{S}_\infty$ which realizes the Thoma character as its matrix element. It is useful to give such a nice realization of the factor representation. Among its applications, let us mention here two cases. In [BG], Bożejko-Guţă obtained a class of generalized Brownian motions associated with the Thoma characters. A positive definite function on $\mathcal{P}_2(\infty)$, the set of the pair partitions, is needed to introduce a Gaussian state of the algebra generated by the field operators on a certain Fock space. They used the realization in [VK1] to extend the Thoma character on $\mathfrak{S}_\infty$ to $\mathcal{P}_2(\infty)$. Another example is due to Biane in [Bia] concerning asymptotic concentration which is observed in irreducible decomposition of some representations of $\mathfrak{S}_n$ as $n \to \infty$. For example, in irreducible decomposition of the regular representation of $\mathfrak{S}_n$, we see that a typical irreducible component occupies a dominant size (the so-called limit shape of Young diagrams) under appropriate scaling limit. Biane showed in [Bia] that such a concentration phenomenon is observed in a sequence of the Vershik-Kerov factor representations and that the typical irreducible component is characterized by using free probability theory. See also [Hor] for a survey on this concentration phenomenon and free probability.

Motivated by the above facts, we are led to construct those realizations analogous as that in [VK1] for the explicitly given characters of $\mathfrak{S}_\infty(T)$.

Apart from expected similar applications to the case of $\mathfrak{S}_\infty$, we note that our realization gives an alternative simpler proof of the positive-definiteness for f_A in [HH6], which is given at first by a formula in the right hand side of (14) as a class function on the group. We should prove to be positive definite and extremal, and then to cover all characters of factor representations of finite type of the group.

9 Limits of irreducible characters of wreath products $\mathfrak{S}_n(T)$ of compact group T with the symmetric groups, I, – from the point of view of group representation theory – [HHH2]

Let T be a compact subgroup and $G = \mathfrak{S}_\infty(T)$ and $G_n = \mathfrak{S}_n(T)$ be as before. Then $G_n \nearrow G$. In this paper we give the following results.

(I-1) For any IUR ρ_n of G_n, we give a realization of it as an induced representation of an IUR π_n of a certain subgroup H_n, or $\rho_n \cong \mathrm{Ind}_{H_n}^{G_n} \pi_n$.

(I-2) Using this realization, we calculate explicitly the trace character

$$\chi_{\rho_n}(g_n) = \mathrm{tr}\big(\rho_n(g_n)\big) \quad (g_n \in G_n).$$

(I-3) For an "increasing" series of IURs ρ_n of G_n $(n \to \infty)$, we find a necessary and sufficient codition for that there exists the limit of normalized characters

$$\widetilde{\chi_{\rho_n}} := \chi_{\rho_n} / \dim \rho_n \,.$$

(I-4) We calculate explicitly the limit $\lim_{n\to\infty} \widetilde{\chi_{\rho_n}}$, and obtain the following.

Theorem A. *For any character* $f_A \in E(G)$ *of* $G = \lim_{n\to\infty} G_n$, *there exists an increasing sequence of IURs* ρ_n *of* G_n *such that* $f_A = \lim_{n\to\infty} \widetilde{\chi_{\rho_n}}$.

Conversely any limit $\lim_{n\to\infty} \widetilde{\chi_{\rho_n}}$, *if it exists, is equal to some of* f_A.

10 Limits of irreducible characters of wreath products $\mathfrak{S}_n(T)$ of compact group T with the symmetric groups, II, – remark on a generalized Young gragh – [HHH2]

Under this provisional subtitle, we give at least the following results.

(II-1) In (I-1), the datum (π_n, H_n) is determined by a set of Young diagrams

$$\lambda_{\zeta,\varepsilon;n} \qquad \text{for} \quad (\zeta,\varepsilon) \in \widehat{T} \times \{0,1\},$$

where the total sum of the sizes of $\lambda_{\zeta,\varepsilon;n}$ over $(\zeta,\varepsilon) \in \widehat{T} \times \{0,1\}$ is equal to n. Then, using the so-called ergodic method as in [VK2], where the inverse martingale convergence theorem plays a dicisive role, we obtain the following.

Theorem B. *For any character* f *of* G, *there exists an increasing sequence of* $(\lambda_{\zeta,\varepsilon;n})_{n\geq 1}$ *for each* $(\zeta,\varepsilon) \in \widehat{T} \times \{0,1\}$, *such that* $\widetilde{\chi_{\rho_n}}$ *converges to* f *on* G *uniformly on every compact subsets. Here* $\rho_n = \mathrm{Ind}_{H_n}^{G_n} \pi_n$, *and* (π_n, H_n) *is determinde by the datum* $(\lambda_{\zeta,\varepsilon;n})_{(\zeta,\varepsilon)\in\widehat{T}\times\{0,1\}}$.

(II-2) We know Theorem 4.1 which asserts that a continuous positive definite class function in $K_1(G)$ is a character (or *extremal*) if and only if it is *factorizable*: $E(G) = F(G)$. Then we can calculate the limit $\lim_{n\to\infty} \widetilde{\chi_{\rho_n}}$ more easily than in (I-4), applying the above ergodic method. Thus we get the explicit form of any character $f \in E(G)$ in a different way.

Thus, in this second part II of [HHH2], we can give alternative proofs of the resuts in the part I of [HHH2] which are explained in the preceding section.

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- “Factor representations and their characters for the wreath products of compact groups with the infinite symmetric group” ⋯ 平井 武（Takeshi Hirai）:

  Page 122, line 5:

  $\ldots$ permutations on A. $\ldots$ $\longrightarrow$ $\ldots$ permutations on I. $\ldots$