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A CHARACTERIZATION OF SYMMETRIC SIEGEL DOMAINS BY CONVEXITY OF CAYLEY TRANSFORM IMAGES

CHIFUNE KAI

Abstract. We show that a homogeneous Siegel domain is symmetric if and only if its Cayley transform image is convex. Moreover, this convexity forces the parameter of the Cayley transform to be a specific one, so that the Cayley transform coincides with the inverse of the Cayley transform introduced by Korányi and Wolf.

1. Introduction

A homogeneous Siegel domain is a higher dimensional analogue of the right (or upper) half plane in $\mathbb{C}$. It is mapped to a bounded domain by the Cayley transforms introduced by [17]. Among homogeneous Siegel domains, we have a significant subclass consisting of symmetric ones. In [9], we gave a symmetry characterization for tube domains by convexity of the Cayley transform images, and in [6], for quasisymmetric Siegel domains. In this talk we finalize these works and establishes the same type of symmetry characterization theorem for general homogeneous Siegel domains.

There are some other conditions which characterize symmetric Siegel domains: a certain norm equality related to the Cayley transform image [15], the commutativity of the Berezin transform and the Laplace-Beltrami operator [16], and the harmonicity of the Poisson-Hua kernel [18]. In the latter two papers, the geometric backgrounds of the symmetry characterizations are clarified through norm equalities involving the Cayley transforms. In [3], we can find several characterizations of symmetric Siegel domains concerning the isotropy representation and the action of the automorphism group of the domain. Differential geometric characterizations by means of the Bergman metric are given in [4] and [2], and an algebraic one in terms of the defining data of Siegel domains in [23, Theorem V.3.5].

Let us present the convexity of Cayley transform image of a symmetric Siegel domain. In the case of one complex variable, the Cayley transform

$$w \mapsto \frac{w-1}{w+1} \qquad (w \in \mathbb{C})$$

maps the right half plane to the open unit disc, which is a convex set. We have a similar situation for symmetric Siegel domains. Since a symmetric Siegel domain is a Hermitian symmetric space of non-compact type, it has a canonical bounded realization, namely, the Harish-Chandra realization. In [11], Korányi and Wolf defined in a Lie-theoretic way (the inverse of) the Cayley transform which maps a

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symmetric Siegel domain to its Harish-Chandra realization. It is known that the Harish-Chandra realization coincides with the open unit ball for the spectral norm defined for the Jordan triple system canonically associated with the domain (we refer the reader to [12, §10], [11] and [6] for details). Thus the Cayley transform image of a symmetric Siegel domain is a convex set. We shall show that this convexity characterizes symmetric Siegel domains among homogeneous ones. Before proceeding, we would like to mention that it is shown in [13] that the Harish-Chandra realization of a symmetric Siegel domain is characterized essentially among bounded realizations by its convexity. In other words, the Cayley transform is essentially the only bounded convex realizaion of a symmetric Siegel domain.

In this talk we deal with the family of Cayley transforms defined by Nomura [17]. We parametrize this family by relative invariants on the homogeneous convex cone associated with the Siegel domain. If the domain is quasisymmetric and the parameter is a specific one, the corresponding Cayley transform is the same as Dorfmeister's one given in [5] which we used in [6], and in particular if the domain is symmetric, our Cayley transform with the specific parameter coincides with Korányi-Wolf's one. Moreover, our family includes Penney's Cayley transform defined in [19] which is associated with Vinberg's $*$-map of the underlying cone of the domain, and Nomura's one associated with the Bergman kernel (resp. the Szegö kernel) of the domain appearing in [14], [15] and [16] (resp. [18]).

2. Homogeneous Siegel domains

Let V be a finite-dimensional vector space over $\mathbb{R}$. Let $\Omega \subset V$ be a *homogeneous convex cone*, that is, an open convex cone containing no entire straight line such that the group

$$G(\Omega) := \{g \in GL(V) \mid g(\Omega) = \Omega\}$$

acts transitively on Ω. We put $W := V_{\mathbb{C}}$, the complexification of V and denote by $w \mapsto w^*$ the complex conjugation of W relative to the real form V. Let U be another complex vector space. We suppose that a sesquilinear map $Q : U \times U \to W$ is Hermitian and Ω-positive:

$$Q(u, u') = Q(u', u)^* \qquad (u, u' \in U),$$
$$Q(u, u) \in \overline{\Omega} \setminus \{0\} \text{ for all } u \in U \setminus \{0\}.$$

The *Siegel domain* corresponding to these data is defined by

$$D := \{(u, w) \in U \times W \mid \operatorname{Re} w - \tfrac{1}{2}Q(u, u) \in \Omega\}.$$

In this talk we suppose that D is irreducible and homogeneous.

3. Relative invariants on the cone

We know that there exists a split solvable subgroup H of $G(\Omega)$ acting simply transitively on Ω. A function $\Delta : \Omega \to \mathbb{R}$ is called a *relative invariant* if there exists a one-dimensional representation χ of H such that $\Delta(hx) = \chi(h)\Delta(x)$ ($h \in H, x \in \Omega$). We fix a base point $E \in \Omega$. We say that a relative invariant Δ on Ω is *admissible* (only in this talk) if the bilinear form

$$\langle x|y\rangle_\Delta := D_x D_y \log \Delta(E) \qquad (x, y \in V)$$

defines a positive definite inner product on V, where for a C^∞-function f on Ω, $v \in V$ and $x \in \Omega$, we define $D_v f(x) := \frac{d}{dt} f(x+tv)\big|_{t=0}$.

Since D is holomorphically equivalent to a bounded domain, the Bergman space has the reproducing kernel called *Bergman kernel*, which we denote by κ. It is known that there exists a relative invariant Δ_{B} on Ω such that it is analytically continued to a function on $\Omega + iV$ and we have for some $c > 0$,

$$\kappa(z_1, z_2) = c\,\Delta_{\mathrm{B}}(w_1 + w_2^* - Q(u_1, u_2)) \qquad (z_j = (u_j, w_j) \in D, j = 1, 2).$$

We see that Δ_{B} is admissible.

Let Δ be an admissible relative invariant on Ω. We extend the inner product $\langle\cdot|\cdot\rangle_\Delta$ on V to a comlex bilinear form on W. Then the bilinear form

$$(u|u')_\Delta := \langle Q(u, u')|E\rangle_\Delta \qquad (u, u' \in U)$$

defines a positive definite Hermitian inner product on U. For every $w \in W$, we define a linear operator $\varphi(w)$ on U by

$$(\varphi(w)u|u')_\Delta = \langle Q(u, u')|w\rangle_\Delta \qquad (u, u' \in U).$$

The assignment $w \mapsto \varphi(w)$ is also complex linear and $\varphi(E) = I$. We denote by Ω^Δ the dual cone of Ω realized in V by means of the inner product $\langle\cdot|\cdot\rangle_\Delta$:

$$\Omega^\Delta := \{x \in V \mid \forall y \in \overline{\Omega} \setminus \{0\}, \langle x|y\rangle_\Delta > 0\}.$$

4. Cayley transforms of quasisymmetric Siegel domains

Before proceeding to homogeneous domains, let us see what the Cayley transforms look like in the case of quasisymmetric Siegel domains. Let D be the homogeneous Siegel domain defined in §2. If Ω is self-dual with respect to the inner product $\langle\cdot|\cdot\rangle_{\Delta_{\mathrm{B}}}$, that is, $\Omega = \Omega^{\Delta_{\mathrm{B}}}$, then the Siegel domain is said to be *quasisymmetric*. It is known that symmetric Siegel domains are quasisymmetric. We introduce a commutative and non-associative product $\circ$ on V by

$$\langle x \circ y|z\rangle_{\Delta_{\mathrm{B}}} = -\tfrac{1}{2} D_x D_y D_z \log \Delta_{\mathrm{B}}(E) \qquad (x, y, z \in V).$$

We see that E is the unit element. It holds that D is quasisymmetric if and only if V with the product $\circ$ is a Jordan algebra. This means that in addition to the commutativity, we have

$$x \circ (x^2 \circ y) = x^2 \circ (x \circ y) \qquad (x, y \in V).$$

Let D be a quasisymmetric Siegel domain. Then V with the above product $\circ$ is a (Euclidean) Jordan algebra and W is a semisimple complex Jordan algebra in a natural way. Moreover we have

Proposition 4.1 ([5, Theorem 2.1 (6)]). *The linear map $\varphi : w \mapsto \varphi(w)$ is a $*$-representation of the Jordan algebra W:*

$$\varphi(w^*) = \varphi(w)^* \qquad (w \in W),$$
$$\varphi(w_1 \circ w_2) = \tfrac{1}{2}(\varphi(w_1)\varphi(w_2) + \varphi(w_2)\varphi(w_1)) \qquad (w_1, w_2 \in W),$$

where, if A is a complex linear operator on U, then A^* stands for the adjoint operator of A with respect to $(\cdot|\cdot)_{\Delta_B}$.

The *Cayley transform of D* is given by

$$\mathcal{C}(u,w) := \big(2\varphi\big((w+E)^{-1}\big)u, (w-E)\circ(w+E)^{-1}\big) \qquad ((u,w)\in U\times W).$$

It is proved by Dorfmeister that $\mathcal{C}(D)$ is bounded. Moreover, if D is symmetric, then $\mathcal{C}$ is the inverse of the Cayley transform introduced by Korányi and Wolf.

Remark 4.2. *For an invertible $v \in V$, the Jordan algebra inverse of v is characterized by*

$$\langle v^{-1}|x\rangle_{\Delta_B} = -D_x \log \Delta_B(v) \qquad (x\in V).$$

Also we note that $(w-E)\circ(w+E)^{-1} = E - 2(w+E)^{-1}$.

Our theorem for quasisymmetric Siegel domains is as follows:

Theorem 4.3 ([6, Theorem 2.6])**.** *Let D be an irreducible quasisymmetric Siegel domain. Then $\mathcal{C}(D)$ is convex if and only if D is symmetric.*

5. Cayley transforms of homogeneous Siegel domains

Let D be the homogeneous Siegel domain defined in §2. We suppose that a relative invariant Δ on Ω is admissible. For $x \in \Omega$, we define the *pseudoinverse* $\mathcal{I}_\Delta(x) \in V$ of x by

$$\langle \mathcal{I}_\Delta(x)|y\rangle_\Delta = -D_y \log \Delta(x) \qquad (y\in V).$$

We call $\mathcal{I}_\Delta : \Omega \to V$ the *pseudoinverse map.* Let us present the key properties of $\mathcal{I}_\Delta$:

1. (1) $\mathcal{I}_\Delta$ gives a bijection from Ω onto Ω^Δ.
2. (2) One has $\mathcal{I}_\Delta(E) = E$.
3. (3) $\mathcal{I}_\Delta$ is analytically continued to a birational map $W \to W$ which is holomorphic on $\Omega + iV$.
4. (4) Let $H_{\mathbb{C}}$ be the complexification of H. We see that $\mathcal{I}_\Delta$ is $H_{\mathbb{C}}$-equivariant: $\mathcal{I}_\Delta(hx) = {}^\Delta h^{-1}\mathcal{I}_\Delta(x)$ ($h \in H_{\mathbb{C}}$), where ${}^\Delta h$ is the transpose of h with respect to $\langle\cdot|\cdot\rangle_\Delta$. In particular, $\mathcal{I}_\Delta(\lambda x) = \lambda^{-1}\mathcal{I}_\Delta(x)$ for $\lambda > 0$.
5. (5) If D is quasisymmetric and $\Delta = \Delta_{\mathrm{B}}^p$ for some $p > 0$, then $\mathcal{I}_\Delta$ coincides with the Jordan algebra inverse map of W.

We define the *Cayley transform of D* by

$$\mathcal{C}_\Delta(u,w) := \big(2\varphi(\mathcal{I}_\Delta(w+E))u, E - 2\mathcal{I}_\Delta(w+E)\big) \qquad ((u,w)\in U\times W).$$

We know that $\mathcal{C}_\Delta(D)$ is bounded. Our main theorem is stated as follows:

Theorem 5.1. *Let D be an irreducible homogeneous Siegel domain. We suppose that a relative invariant Δ on Ω is admissible. Then $\mathcal{C}_\Delta(D)$ is convex if and only if D is symmetric and $\Delta = \Delta_B^p$ for some $p > 0$.*

6. Appendix

We present here the Cayley transform of the symmetric Siegel domain isomorphic to the bounded symmetric domain of type $I_{p,q}$ $(1 \leq p < q)$. We put $Z := \mathrm{Mat}(p, q; \mathbb{C})$, the vector space of (p, q) complex matrices. The bounded symmetric domain of type $I_{p,q}$ is given by

$$B := \{z \in Z \mid I_p - zz^* \gg 0\},$$

where I_p is the unit matrix of order p. We set

$$V := \mathrm{Herm}(p; \mathbb{C}), \qquad \Omega := \{X \in V \mid X \gg 0\}, \qquad U := \mathrm{Mat}(p, q - p; \mathbb{C}).$$

The cone Ω is a homogeneous convex cone. The complexification of V is $W := \mathrm{Mat}(p, p; \mathbb{C})$. We identify $U \oplus W$ with Z by $U \oplus W \ni (u, w) \mapsto (u\, w) \in Z$. We define a sesquilinear map $Q : U \times U \to W$ by $Q(u_1, u_2) := u_1 u_2^*$. Then it is Hermitian and Ω-positive. The corresponding symmetric Siegel domain is

$$D = \{(u, w) \in U \oplus W \mid w + w^* - uu^* \gg 0\}.$$

The relative invariant Δ_{B} is given by $\Delta_{\mathrm{B}}(w) = \det(w)^{-(p+q)/2}$ $(w \in \Omega)$ and the corresponding inner product is $\langle x|y\rangle_{\Delta_{\mathrm{B}}} = \frac{1}{2}(p + q)\mathrm{trace}(xy)$ $(x, y \in V)$. The vector space W with the product $A \circ B := \frac{1}{2}(AB + BA)$ $(A, B \in W)$ is a complex Jordan algebra. For an invertible $w \in W$, the Jordan algebra inverse w^{-1} coincides with the inverse matrix of w. The $*$-representation φ of W is given by $\varphi(w)u = wu$ $(w \in W, u \in U)$. In this case the Cayley transform of D is

$$\mathcal{C}(u, w) = (2(w + E)^{-1}u, (w - E)(w + E)^{-1}) \qquad ((u, w) \in U \oplus W),$$

where we note that the matrices $w - E$ and $(w + E)^{-1}$ are commutative. We define a linear operator T on $U \oplus W$ by $T(u, w) := (\sqrt{2}u, w)$ $(u \in U, w \in W)$. We can check easily that

$$\mathcal{C}(D) = T(B).$$

For $z \in Z$, we regard z as a linear operator from $\mathbb{C}^q$ to $\mathbb{C}^p$ and denote by $\|z\|$ the operator norm of z with respect to the standard norms of $\mathbb{C}^p$ and $\mathbb{C}^q$. Then we have $B = \{\|z\| < 1\}$ so that B is a convex set. Thus the Cayley transform image $\mathcal{C}(D)$ is convex.

References

[1] H. Asano, *On the irreducibility of homogeneous convex cones*, J. Fac. Sci. Univ. Tokyo **15** (1968), 201–208.

[2] K. Azukawa, *Curvature operator of the Bergman metric on a homogeneous bounded domain*, Tohoku Math. J. **37** (1985), 197–223.

[3] J. E. D'Atri, J. Dorfmeister and Zhao Yan Da, *The isotropy representation for homogeneous Siegel domains*, Pacific J. Math. **120** (1985), 295–326.

[4] J. E. D'Atri and I. Dotti Miatello, *A characterization of bounded symmetric domains by curvature*, Trans. Amer. Math. Soc. **276** (1983), 531–540.

[5] J. Dorfmeister, *Quasisymmetric Siegel domains and the automorphisms of homogeneous Siegel domains*, Amer. J. Math. **102** (1980), 537–563.

[6] C. Kai, *A symmetry characterization of quasisymmetric Siegel domains by convexity of Cayley transform images*, J. Lie Theory **16** (2006), 47–56.

[7] C. Kai, *A characterization of symmetric Siegel domains by convexity of Cayley transform images*, preprint.
[8] C. Kai and T. Nomura, *A Characterization of symmetric cones through pseudoinverse maps*, J. Math. Soc. Japan **57** (2005), 195–215.
[9] C. Kai and T. Nomura, *A characterization of symmetric tube domains by convexity of Cayley transform images*, Diff. Geom. Appl. **23** (2005), 38–54.
[10] S. Kaneyuki, *On the automorphism groups of homogeneous bounded domains*, J. Fac. Sci. Univ. Tokyo **14** (1967), 89–130.
[11] A. Korányi and J. A. Wolf, *Realization of Hermitian symmetric spaces as generalized half-planes*, Ann. of Math. **81** (1965), 265–288.
[12] O. Loos, *Bounded symmetric domains and Jordan pairs*, Lecture Notes, Univ. California at Irvine, 1977.
[13] N. Mok and I-H. Tsai, *Rigidity of convex realizations of irreducible bounded symmetric domains of rank* ≥ 2, J. Reine Angew. Math. **431** (1992), 91–122.
[14] T. Nomura, *On Penney's Cayley tranform of a homogeneous Siegel domain*, J. Lie Theory **11** (2001), 185–206.
[15] T. Nomura, *A characterization of symmetric Siegel domains through a Cayley transform*, Transform. Groups **6** (2001), 227–260.
[16] T. Nomura, *Berezin transforms and Laplace-Beltrami operators on homogeneous Siegel domains*, Diff. Geom. Appl. **15** (2001), 91–106.
[17] T. Nomura, *Family of Cayley transforms of a homogeneous Siegel domain parametrized by admissible linear forms*, Diff. Geom. Appl. **18** (2003), 55–78.
[18] T. Nomura, *Geometric norm equality related to the harmonicity of the Poisson kernel for homogeneous Siegel domains*, J. Funct. Anal. **198** (2003), 229–267.
[19] R. Penney, *The Harish-Chandra realization for non-symmetric domains in* $\mathbb{C}^n$, in Topics in Geometry in Memory of Joseph D'Atri, Ed. by S. Gindikin, Birkhauser, Boston, 1996, 295–313.
[20] I. I. Pyatetskii-Shapiro, *Automorphic Functions and the Geometry of Classical Domains*, Gordon and Breach, New York, 1969.
[21] H. Rossi, *Lectures on Representations of Groups of Holomorphic Transformations of Siegel Domains*, Lecture Notes, Brandeis University, 1972.
[22] H. Rossi and M. Vergne, *Representations of certain solvable Lie groups on Hilbert spaces of holomorphic functions and the application to the holomorphic discrete series of a semisimple Lie group*, J. Funct. Anal. **13** (1973), 324–389.
[23] I. Satake, Albebraic structures of symmetric domains, Iwanami Shoten and Princeton Univ. Press, Tokyo-Princeton, 1980.
[24] E. B. Vinberg, *The structure of the group of automorphisms of a homogeneous convex cone*, Tr. Mosk. Mat. Ob. **13** (1965), 56–83; Trans. Mosc. Math. Soc. **13** (1967), 63–93.

Department of Mathematics, Faculty of Science, Kyoto University, 606-8502 Sakyo-ku, Japan

E-mail address: `kai@math.kyoto-u.ac.jp`