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Poincaré series of the Weyl groups of the elliptic root systems $A_1^{(1,1)}, A_1^{(1,1)*}$ and $A_2^{(1,1)}$

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Adstract. We calculate the Poincaré series of the elliptic Weyl group $W(A_2^{(1,1)})$, which is the Weyl group of the elliptic root system $A_2^{(1,1)}$. The generators and relations of $W(A_2^{(1,1)})$ have been already given by the author and K. Saito.

1 Introduction

We calculate the Poincaré series $W(t)$ of the elliptic Weyl groups W of types $A_1^{(1,1)}, A_1^{(1,1)^*}$ and $A_2^{(1,1)}$. In the cases of types $A_1^{(1,1)}$ and $A_1^{(1,1)*}$, they have been already examined by Wakimoto ([10]). Elliptic Weyl groups are the Weyl groups associated to the elliptic root systems introduced by K. Saito ([6], [7]), which are defined by a semi-positive definite inner product with 2-dimensional radical. The generators and their relations of elliptic Weyl groups were described from the view point of a generalization of Coxeter groups by the author and K.Saito ([8], [9]). The Poincaré series $W(t)$ is defined by

$$W(t) = \sum_{w \in W} t^{l(w)},$$

where t is an indeterminate and $l(w)$ is the length of a minimal expression of an element $w \in W$. If W is one of the finite or affine Weyl groups, it is known that

$$\sum_{w \in W} t^{l(w)} = \begin{cases} \displaystyle\prod_{i=1}^{n} \frac{1-t^{m_i+1}}{1-t} & (\text{ W : finite }), \\ \frac{1}{(1-t)^n} \displaystyle\prod_{i=1}^{n} \frac{1-t^{m_i+1}}{1-t^{m_i}} & (\text{ W : affine }), \end{cases}$$

where n is the rank and $m_1, \cdots, m_n$ are the exponents of W ([1], [2], [3], [4], [5]). In the cases of $A_1^{(1,1)}$ and $A_1^{(1,1)*}$, we give a different proof from [10], and in the similar way we calculate the case of $A_2^{(1,1)}$.

2 Poincaré series of the Weyl groups of types $A_1^{(1,1)}$ and $A_1^{(1,1)*}$

The generators and their relations of the elliptic Weyl group W of type $A_1^{(1,1)}$ are given as follows ([8], [9]):
Generators : $w_i,\ w_i^*\quad (i=0,1)$.

Relations : $w_i^2 = w_i^{*2} = 1\ (i=0,1),\quad w_0w_0^*w_1w_1^* = 1$.

The relation $w_0w_0^*w_1w_1^* = 1$ is rewritten as follows :
(2.1.1) $\qquad w_0^*w_1 = w_0w_1^*\ (\Leftrightarrow w_1^*w_0 = w_1w_0^*)$.

(It is understood that $*$ moves between w_0 and w_1 .) We set $T := w_1w_0,\ R := w_1^*w_1 = w_0w_0^*$, then we easily see the following.

Lemma 2.1 *T, R and w_1 are the generators of the Weyl group W of type $A_1^{(1,1)}$ and their relations are given by ;*

$$TR = RT,\quad w_1T = T^{-1}w_1,\quad w_1R = R^{-1}w_1,\quad w_1^2 = 1.$$

From this, we have $W = \{R^mT^nw_1,\ R^mT^n,\ m,n \in \mathbf{Z}\}$.
The elements T and w_1 are the generators of the affine Weyl group of type A_1 and all elements of the Weyl group are classified to the following:.

$$\{\text{(I) } T^n\ (n\geq 0),\quad \text{(II) } T^{-n}\ (n\geq 1),\quad \text{(III) } T^nw_1\ (n\geq 0),\quad \text{(IV) } T^{-n}w_1\ (n\geq 1)\}.$$

We multiply the elements $R^m\ (m \in \mathbf{Z})$ to the above elements from the left, and examine their minimal length in each case by using the following.

Lemma 2.2 *Let w be a minimal expression by w_0 and w_1. Then even if we attach $*$ to any components of w, the length of w does not decrease.*

(I) $T^n = (w_1w_0)^n\quad (n\geq 0)$: From the expression $Rw_1w_0 = w_1^*w_0$ and (2.1.1), we see that

$$R^kT^n = R^k(w_1w_0)^n = (w_{11}w_{10})(w_{21}w_{20})\cdots(w_{n1}w_{n0}),\ \text{for } 0\leq k\leq 2n,$$

where each $w_{i1} = w_1,\ w_1^*$ and $w_{i0} = w_0,\ w_0^*$, for all i. So, we find that

$$\sharp\{R^kT^n,\ (n\geq 0,\ k\in\mathbf{Z}) \mid l(R^kT^n) = 2n\} = 2n+1,$$

and for $m\geq 1, R^{2n+m}T^n = R^m(R^{2n}T^n) = (w_1^*w_1)^m(w_1^*w_0^*)^n$ and $R^{-m}T^n = (w_1w_1^*)^m(w_1w_0)^n$ and each length is $2n+2m$, so we get $\sharp\{R^kT^n,\ (n\geq 0,\ k\in\mathbf{Z}) \mid l(R^kT^n) = 2n+2m\} = 2$.

In the case of (II), it is similar to (I).

(III) $T^n w_1 = (w_1 w_0)^n w_1 \quad (n \geq 0)$: From $Rw_1 = w_1^*$ and (2.1.1),

$$R^k T^n w_1 = R^k (w_1 w_0)^n w_1 = (w_{11} w_{10}) \cdots (w_{n1} w_{n0}) w_{n+1,1} \quad \text{for } 0 \leq k \leq 2n+1$$

where each $w_{i1} = w_1,\ w_1^*$ and $w_{i0} = w_0,\ w_0^*$, so

$$\sharp\{R^k T^n w_1,\ (n \geq 0,\ k \in \mathbf{Z}) \mid l(R^k T^n w_1) = 2n+1\} = 2n+2, \text{ and}$$

$$\sharp\{R^k T^n w_1, (n \geq 0,\ k \in \mathbf{Z}) \mid l(R^k T^n w_1) = 2n+1+2m\} = \sharp\{R^{2n+1+m} T^n w_1,\ R^{-m} T^n w_1\} = 2.$$

(IV) $T^{-n} w_1 = (w_0 w_1)^{n-1} w_0 \quad (n \geq 1)$: From $R^{-1} w_0 = w_0^*$ and (2.1.1), we see that

$$\sharp\{R^k T^{-n} w_1,\ (n \geq 1,\ k \in \mathbf{Z}) \mid l(R^k T^{-n} w_1) = 2n-1\} = 2n,$$

and for $m \geq 1$, $R^{-(2n-1)-m} T^{-n} w_1 = R^{-m} (w_0^* w_1^*)^{n-1} w_0^* = (w_0^* w_0)^m (w_0^* w_1^*)^{n-1} w_0^*$, $R^m T^{-n} w_1 = (w_0 w_0^*)^m (w_0 w_1)^{n-1} w_0$, so $\sharp\{R^k T^{-n} w_1,\ (n \geq 1,\ k \in \mathbf{Z}) \mid l(R^k T^{-n} w_1) = 2n+2m\} = 2$.

In the case of $A_1^{(1,1)*}$, the generators and their relations are given as follows :
Generators : $w_0,\ w_1,\ w_1^*$.
Relations : $w_0^2 = w_1^2 = w_1^{*2} = (w_0 w_1 w_1^*)^2 = 1$.
This Weyl group is obtained from the Weyl group of type $A_1^{(1,1)}$ by removing one generator w_0^*, so we examine similarly to the case of $A_1^{(1,1)}$.

(I) $T^n = (w_1 w_0)^n \quad (n \geq 0)$: Form $Rw_1 = w_1^*$ and $R^n T^n = (w_1^* w_0)^n$,

$$\sharp\{R^k T^n, (n \geq 0,\ k \in \mathbf{Z}) \mid l(R^k T^n) = 2n\} = n+1,$$

and for $m \geq 1$, $R^{n+m} T^n = R^m (w_1^* w_0)^n = (w_1^* w_1)^m (w_1^* w_0)^n$ and $R^{-m} T^n = (w_1 w_1^*)^m (w_1 w_0)^n$, so $\sharp\{R^k T^n,\ (n \geq 0,\ k \in \mathbf{Z}) \mid l(R^k T^n) = 2n+2m\} = 2$.

The case of (II) is similar to (I).

(III) $T^n w_1 = (w_1 w_0)^n w_1 \quad (n \geq 0)$: From $Rw_1 = w_1^*$, and $R^{n+1} (w_1 w_0)^n w_1 = (w_1^* w_0)^n w_1^*$ we see that

$$\sharp\{R^k T^n w_1,\ (n \geq 0,\ k \in \mathbf{Z}) \mid l(R^k T^n w_1) = 2n+1\} = n+2, \text{ and}$$

$$\sharp\{R^k T^n w_1,\ k \in \mathbf{Z} \mid l(R^k T^n w_1) = 2n+1+2m\} = \sharp\{R^{n+1+m} T^n w_1,\ R^{-m} T^n w_1\} = 2.$$

(IV) $T^{-n} w_1 = (w_0 w_1)^{n-1} w_0 \quad (n \geq 1)$: From $R^{-1}(w_0 w_1) = w_0 w_1^*$ and $R^{-(n-1)} (w_0 w_1)^{n-1} w_0 = (w_0 w_1^*)^{n-1} w_0$, we see that

$$\sharp\{R^k T^{-n} w_1,\ (n \geq 1,\ k \in \mathbf{Z}) \mid l(R^k T^{-n} w_1) = 2n-1\} = n, \text{ and}$$

$\sharp\{R^kT^{-n}w_1,\ k\in\mathbf{Z} \mid l(R^kT^{-n}w_1)=2n-1+2m\}=\sharp\{R^{-n+1-m}T^{-n}w_1,\ R^mT^{-n}w_1\}=2.$
From the above argument, we obtain the following.

$A_1^{(1,1)}$	$l(w)$ $(n\geq 1, m\geq 1)$	$\sharp$
I	0 $2n$ $2m, 2(n+m)$	1 $2n+1$ 2
II	$2n$ $2(n+m)$	$2n+1$ 2
III	$2n-1$ $2(n+m)-1$	$2n$ 2
IV	$2n-1$ $2(n+m)-1$	$2n$ 2

$A_1^{(1,1)*}$	$l(w)$ $(n\geq 1, m\geq 1)$	$\sharp$
I	0 $2n$ $2m, 2(n+m)$	1 $n+1$ 2
II	$2n$ $2(n+m)$	$n+1$ 2
III	$2n-1$ $2(n+m)-1$	$n+1$ 2
IV	$2n-1$ $2(n+m)-1$	n 2

Further from this we obtain the following.

Proposition 2.3 *([10])* (i) *The length and its number of the elements of $W(A_1^{(1,1)})$ and $W(A_1^{(1,1)*})$ are given by ;*

$$W(A_1^{(1,1)}):\quad \sharp\{w\in W \mid l(w)=0\}=1,\quad \sharp\{w\in W \mid l(w)=n,\ (n\geq 1)\}=4n,$$

$$W(A_1^{(1,1)*}):\quad \sharp\{w\in W \mid l(w)=0\}=1,\quad \sharp\{w\in W \mid l(w)=n,\ (n\geq 1)\}=3n.$$

(ii) *The Poincaré series of $W(A_1^{(1,1)})$ and $W(A_1^{(1,1)*})$ are given by ;*

$$\sum_{w\in W(A_1^{(1,1)})} t^{l(w)}=\frac{(1+t)^2}{(1-t)^2},\qquad \sum_{w\in W(A_1^{(1.1)*})} t^{l(w)}=\frac{1-t^3}{(1-t)^3}.$$

3 Poincaré series of the Weyl group of type $A_2^{(1,1)}$

The elliptic Weyl group W of $A_2^{(1,1)}$ is described as follows ([8], [9]).
Generators : $w_i,\ w_i^*\quad (i=0,1,2)$.
Relations : $w_i^2=w_i^{*2}=1\quad (i=0,1,2)$,
for $i\neq j$

$$w_iw_jw_i=w_jw_iw_j,\quad w_i^*w_j^*w_i^*=w_j^*w_i^*w_j^*,$$

$$w_i^*w_jw_i^*=w_jw_i^*w_j=w_iw_j^*w_i=w_j^*w_iw_j^*,$$

and $w_0w_0^*w_1w_1^*w_2w_2^*=1$.

We set $T_1:=w_0w_2w_0w_1$, $T_2:=w_0w_1w_0w_2$, $R_1:=w_1w_1^*$, and $R_2:=w_2w_2^*$, then we have the following.

Lemma 3.1 (i) *W is generated by $w_1,\ w_2,\ T_1,\ T_2,\ R_1,\ R_2$, and $T_1,\ T_2,\ R_1,\ R_2$ generate an abelian normal subgroup of W.*

(ii) $w_1,\ w_2,\ T_1,\ T_2,\ R_1$ *and* R_2 *satisfy the following relations :*

$$\begin{cases} w_iT_i = T_i^{-1}w_i \\ w_iR_i = R_i^{-1}w_i \\ w_iT_j = T_iT_jw_i \quad (i \neq j) \\ w_iR_j = R_iR_jw_i \quad (i \neq j). \end{cases}$$

(iii) $W = \{R_1^n R_2^m T_1^k T_2^l\ w, \quad (n,m,k,l \in \mathbf{Z}) \ \mid\ w = \mathrm{id},\ w_1,\ w_2,\ w_1w_2,\ w_2w_1,\ w_1w_2w_1\}.$

We first consider minimal expressions of the elements $T_1^nT_2^m$ generated by $T_1 = w_0w_2w_0w_1$, and $T_2 = w_0w_1w_0w_2$, then by noting the following minimal expressions;

$$T_1T_2 = w_0w_1w_2w_1, \quad T_1T_2^{-1} = (w_2w_0w_1)^2, \quad T_1T_2^2 = (w_0w_1w_2)^2,$$

we have $T_1^nT_2^{n+i} = (0121)^n(012)^i = (012)^2(0121)^{n-1}(012)^{i-1}$, and from this we obtain
$T_1^nT_2^{n+i}\ (n \geq 1, i \geq 1) = \begin{cases} T_1^nT_2^{n+i} = (012)^{2i}(0121)^{n-i} & (1 \leq i < n,\ n \geq 2) \\ T_1^nT_2^{2n+i} = (0102)^i(012)^{2n} & (i \geq 0,\ n \geq 1) \end{cases}$
where for brevity , we use $0, 1, 2, 0^*, 1^*, 2^*$ for $w_0, w_1, w_2, w_0^*, w_1^*, w_2^*$, respectively. Further by considering minimal expressions of $T_1^nT_2^mw\ (w = w_1, w_2, w_1w_2, w_2w_1, w_1w_2w_1)$, we classify $T_1^nT_2^m\ (n, m \in \mathbf{Z})$ as follows.

$$(3.1.1) \quad T_1^nT_2^m\,(n,m \in \mathbf{Z}) = \begin{cases} T_1^nT_2^{n+i} = (012)^{2i}(0121)^{n-i} \quad (1 \leq i < n,\ n \geq 2) \quad (1 \leftrightarrow 2) \\ T_1^{-n}T_2^{-n-i} = (210)^{2i}(1210)^{n-i} \quad (1 \leq i \leq n,\ n \geq 1) \quad (1 \leftrightarrow 2) \\ T_1^nT_2^{2n+i} = (0102)^i(012)^{2n} \quad (i \geq 0,\ n \geq 1) \quad (1 \leftrightarrow 2) \\ T_1^{-n}T_2^{-2n-i} = (210)^{2n}(2010)^i \quad (i \geq 1,\ n \geq 0) \quad (1 \leftrightarrow 2) \\ T_1^{-n-i}T_2^{n} = (1020)^i(102)^{2n} \quad (i \geq 0,\ n \geq 1) \quad (1 \leftrightarrow 2) \\ T_1^{n+i}T_2^{-n} = (201)^{2n}(0201)^i \quad (i \geq 1,\ n \geq 0) \quad (1 \leftrightarrow 2) \\ T_1^nT_2^n = (0121)^n \quad (n \geq 1) \\ T_1^{-n}T_2^{-n} = (1210)^n \quad (n \geq 0), \end{cases}$$

where $(1 \leftrightarrow 2)$ means that we consider the element obtained by exchanging T_1 and T_2.

Similar to the case of $A_1^{(1,1)}$, we use the following.

Lemma 3.2 *Let w be a minimal expression by w_0, w_1 and w_2. Then even if we attach $*$ to any components of w, the length of that does not decrease.*

In each case we multiply $R_1^kR_2^l$ from the left, and examine their minimal length.
For $1 \leq i < n,\ T_1^nT_2^{n+i} = (012)^{2i}(0121)^{n-i}$, by noting the expressions ;

$$\begin{cases} 0^*12012 = (R_1R_2)\ 012012 \\ 01^*2012 = R_2\ 012012 \\ 012^*012 = (R_1R_2)\ 012012 \\ 0120^*12 = R_2\ 012012 \\ 01201^*2 = (R_1R_2)\ 012012 \\ 012012^* = R_2\ 012012 \end{cases} \qquad \begin{cases} 0^*121 = (R_1R_2)\ 0121 \\ 01^*21 = R_2\ 0121 \\ 012^*1 = (R_1R_2)\ 0121 \\ 0121^* = R_1\ 0121, \end{cases}$$

we consider how many R_1, R_2 and R_1R_2 can be contained in $(012)^{2i}(0121)^{n-i}$ by attaching $*$ to arbitrary components. From the above, $(012)^2$ can contain $3 \times R_1R_2$ and $3 \times R_2$, and 0121 can contain $2\times R_1R_2,\ 1\times R_1,\ 1\times R_2$, so by the relation , $(012)^2R_j = R_j(012)^2\ (j = 1, 2)$, we see that $(012)^{2i}(0121)^{n-i}$ can contain $(n-i) \times R_1,\ (n+2i) \times R_2$ and $(2n+i) \times R_1R_2$.

Lemma 3.3 *For* $1 \le i < n$

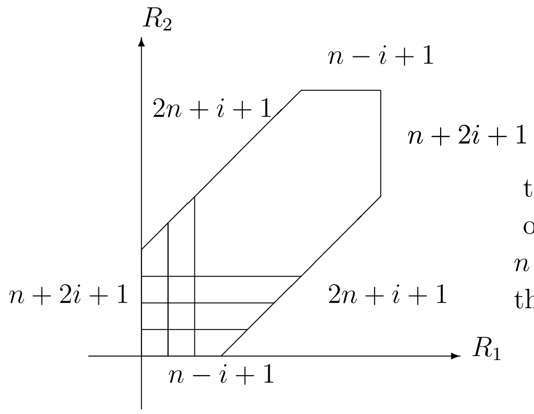
$$\begin{aligned} & R_1^k R_2^l (R_1R_2)^m T_1^n T_2^{n+i} \\ &= R_1^k R_2^l (R_1R_2)^m (012)^{2i}(0121)^{n-i} \\ &= (w_{10}w_{11}w_{12})\cdots(w_{2i,0}w_{2i,1}w_{2i,2})(w'_{10}w'_{11}w'_{12}w''_{11})\cdots(w'_{n-i,0}w'_{n-i,1}w'_{n-i,2}w''_{n-i,1}) \end{aligned}$$

where w_{ij}, *and* $w'_{ij} = w_j,\ w_j^*$ $(j = 0, 1, 2)$ *and* $w''_{i1} = w_1,\ w_1^*$,
for any $0 \le k \le n-i, \quad 0 \le l \le n+2i, \quad 0 \le m \le 2n+i.$

We count the number

$$\sharp\{R_1^k R_2^l T_1^n T_2^{n+i},\ (1 \le i < n,\ n \ge 2,\ k, l \in \mathbf{Z}) \mid l(R_1^k R_2^l T_1^n T_2^{n+i}) = l(T_1^n T_2^{n+i}) = 4n + 2i\ \}.$$

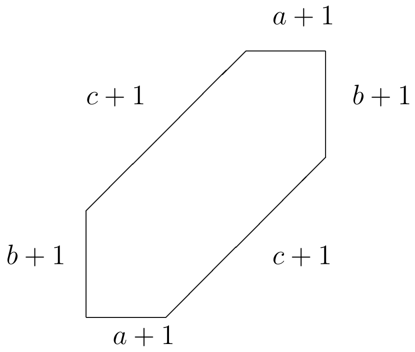
For the purpose we use the following figure ;



then the number is equal to the number of the vertices of the lattices, where $n-i+1,\ n+2i+1,$ and $2n+i+1$ are the number of vertices on each edge.

Then we use the following.

Lemma 3.4



In the left figure, the number of the vertices of the lattices is $ab + bc + ca + a + b + c + 1$.

By multiplying $R_1^{\pm 1}$, $R_2^{\pm 1}$, and $(R_1R_2)^{\pm 1}$ $(R_1 = w_1w_1^*,\ R_2 = w_2w_2^*,\ R_1R_2 = w_0^*w_0)$, we obtain the elements whose length are $4n + 2i + 2$, and actually we have only to multiply to the boundary in the figure, and iterating this procedure we get the following.

Lemma 3.5

$$\sharp\{R_1^m R_2^l T_1^n T_2^{n+i},\ (1 \le i < n,\ n \ge 2,\ m, l \in \mathbf{Z})\ \mid\ l(R_1^k R_2^l T_1^n T_2^{n+i}) = 4n{+}2i{+}2k,\ (k \ge 1)\} = 8n{+}4i{+}6k.$$

Next we consider the elements $T_1^n T_2^{n+i} w$, for $w = w_1, w_2, w_1w_2, w_2w_1$, and $w_1w_2w_1$, then we have the following:

$$\begin{cases} T_1^n T_2^{n+i} = (012)^{2i}(0121)^{n-i} \\ T_1^n T_2^{n+i}1 = (012)^{2i}(0121)^{n-i-1}012 \\ T_1^n T_2^{n+i}2 = (012)^{2i}(0121)^{n-i-1}021 \\ T_1^n T_2^{n+i}12 = (012)^{2i}(0121)^{n-i-1}01 \\ T_1^n T_2^{n+i}21 = (012)^{2i}(0121)^{n-i-1}02 \\ T_1^n T_2^{n+i}121 = (012)^{2i}(0121)^{n-i-1}0. \end{cases}$$

In the similar method to the case of $T_1^n T_2^{n+i}$, in this case and for other cases we count how many $R_1^{\pm 1}, R_2^{\pm 1}$ and $(R_1R_2)^{\pm 1}$ can be contained in a minimal expression. By the figure of the number of $R_1^{\pm 1}, R_2^{\pm 1}$ and $(R_1R_2)^{\pm 1}$, we count the number of a minimal expression of the elements of the Weyl group and that of increasing length by 2 , which is equal to $\sharp$(boundary) + 6. In the sequal, we examine the number of the vertices on each ede of the figure in a minimal expression, first we have;

$$\begin{cases} 2^*10210 = R_2^{-1}\ 210210 \\ 21^*0210 = (R_1R_2)^{-1}\ 210210 \\ 210^*210 = R_2^{-1}\ 210210 \\ 2102^*10 = (R_1R_2)^{-1}\ 210210 \\ 21021^*0 = R_2^{-1}\ 210210 \\ 210210^* = (R_1R_2)^{-1}\ 210210 \end{cases} \quad \begin{cases} 1^*210 = R_1^{-1}\ 1210 \\ 12^*10 = (R_1R_2)^{-1}\ 1210 \\ 121^*0 = R_2^{-1}\ 1210 \\ 1210^* = (R_1R_2)^{-1}\ 1210 \end{cases} \quad \begin{cases} 0^*102 = (R_1R_2)\ 0102 \\ 01^*02 = R_2\ 0102 \\ 010^*2 = R_1^{-1}\ 0102 \\ 0102^* = R_2\ 0102 \end{cases}$$

$$\begin{cases} 2^*010 = R_2^{-1}\ 2010 \\ 20^*10 = R_1\ 2010 \\ 201^*0 = R_2^{-1}\ 2010 \\ 2010^* = (R_1R_2)^{-1}\ 2010 \end{cases} \quad \begin{cases} 1^*020 = R_1^{-1}\ 1020 \\ 10^*20 = R_2\ 1020 \\ 102^*0 = R_1^{-1}\ 1020 \\ 1020^* = (R_1R_2)^{-1}\ 1020 \end{cases} \quad \begin{cases} 1^*02102 = R_1^{-1}\ 102102 \\ 10^*2102 = R_2\ 102102 \\ 102^*102 = R_1^{-1}\ 102102 \\ 1021^*02 = R_2\ 102102 \\ 10210^*2 = R_1^{-1}\ 102102 \\ 102102^* = R_2\ 102102 \end{cases}$$

From these and (3.1.1), we obtain the following 8 cases.

(I) $\mathbf{T_1^nT_2^{n+i} = (012)^{2i}(0121)^{n-i}}$ $(\mathbf{1 \le i < n,\ n \ge 2})$			
$(012)^{2i}(0121)^{n-i}w$	$\sharp R_1^{\pm1}$	$\sharp R_2^{\pm1}$	$\sharp(R_1R_2)^{\pm1}$
$(012)^{2i}(0121)^{n-i}$	$n-i$	$n+2i$	$2n+i$
$(012)^{2i}(0121)^{n-i-1}012$	$n-i-1$	$n+2i$	$2n+i$
$(012)^{2i}(0121)^{n-i-1}021$	$n-i$	$n+2i-1$	$2n+i$
$(012)^{2i}(0121)^{n-i-1}01$	$n-i-1$	$n+2i$	$2n+i-1$
$(012)^{2i}(0121)^{n-i-1}02$	$n-i$	$n+2i-1$	$2n+i-1$
$(012)^{2i}(0121)^{n-i-1}0$	$n-i-1$	$n+2i-1$	$2n+i-1$

(VII) $\mathbf{T_1^nT_2^n = (0121)^n}$ $(\mathbf{n \ge 1})$			
$(0121)^n$	n	n	$2n$
$(0121)^{n-1}012$	$n-1$	n	$2n$
$(0121)^{n-1}021$	n	$n-1$	$2n$
$(0121)^{n-1}01$	$n-1$	n	$2n-1$
$(0121)^{n-1}02$	n	$n-1$	$2n-1$
$(0121)^{n-1}0$	$n-1$	$n-1$	$2n-1$

(II) $\mathbf{T_1^{-n}T_2^{-n-i} = (210)^{2i}(1210)^{n-i}}$ $(\mathbf{1 \le i \le n,\ n \ge 1})$			
$(210)^{2i}(1210)^{n-i}$	$n-i$	$n+2i$	$2n+i$
$(210)^{2i}(1210)^{n-i}1$	$n-i+1$	$n+2i$	$2n+i$
$(210)^{2i}(1210)^{n-i}2$	$n-i$	$n+2i+1$	$2n+i$
$(210)^{2i}(1210)^{n-i}12$	$n-i+1$	$n+2i$	$2n+i+1$
$(210)^{2i}(1210)^{n-i}21$	$n-i$	$n+2i+1$	$2n+i+1$
$(210)^{2i}(1210)^{n-i}121$	$n-i+1$	$n+2i+1$	$2n+i+1$

(VIII) $\mathbf{T_1^{-n}T_2^{-n} = (1210)^n}$ $(\mathbf{n \ge 0})$			
$(1210)^n$	n	n	$2n$
$(1210)^n1$	$n+1$	n	$2n$
$(1210)^n2$	n	$n+1$	$2n$
$(1210)^n12$	$n+1$	n	$2n+1$
$(1210)^n21$	n	$n+1$	$2n+1$
$(1210)^n121$	$n+1$	$n+1$	$2n+1$

(III) $\mathbf{T_1^nT_2^{2n+i} = (0102)^i(012)^{2n}}$ $(\mathbf{i \ge 0,\ n \ge 1})$			
$(0102)^i(012)^{2n}$	i	$3n+2i$	$3n+i$
$(0102)^i(012)^{2n}1$	$i+1$	$3n+2i$	$3n+i$
$(0102)^i(012)^{2n-2}01201$	i	$3n+2i-1$	$3n+i$
$(0102)^i(012)^{2n-2}012021$	$i+1$	$3n+2i$	$3n+i-1$
$(0102)^i(012)^{2n-2}0120$	i	$3n+2i-1$	$3n+i-1$
$(0102)^i(012)^{2n-2}01202$	$i+1$	$3n+2i-1$	$3n+i-1$

(IV) $\mathbf{T_1^{-n}T_2^{-2n-i}} = \mathbf{(210)^{2n}(2010)^i}$ $(\mathbf{i \geq 1,\ n \geq 0})$			
$(210)^{2n}(2010)^i$	i	$3n+2i$	$3n+i$
$(210)^{2n}(2010)^{i-1}210$	$i-1$	$3n+2i$	$3n+i$
$(210)^{2n}(2010)^i2$	i	$3n+2i+1$	$3n+i$
$(210)^{2n}(2010)^{i-1}2102$	$i-1$	$3n+2i$	$3n+i+1$
$(210)^{2n}(2010)^i21$	i	$3n+2i+1$	$3n+i+1$
$(210)^{2n}(2010)^{i-1}21021$	$i-1$	$3n+2i+1$	$3n+i+1$

(V) $\mathbf{T_1^{-n-i}T_2^{n}} = \mathbf{(1020)^i(102)^{2n}}$ $(\mathbf{i \geq 0,\ n \geq 1})$			
$(1020)^i(102)^{2n}$	$3n+2i$	$3n+i$	i
$(1020)^i(102)^{2n}1$	$3n+2i+1$	$3n+i$	i
$(1020)^i(102)^{2n-2}10210$	$3n+2i$	$3n+i-1$	i
$(1020)^i(102)^{2n}12$	$3n+2i+1$	$3n+i$	$i+1$
$(1020)^i(102)^{2n-2}102101$	$3n+2i$	$3n+i-1$	$i+1$
$(1020)^i(102)^{2n-2}1021012$	$3n+2i+1$	$3n+i-1$	$i+1$

(VI) $\mathbf{T_1^{n+i}T_2^{-n}} = \mathbf{(201)^{2n}(0201)^i}$ $(\mathbf{i \geq 1,\ n \geq 0})$			
$(201)^{2n}(0201)^i$	$3n+2i$	$3n+i$	i
$(201)^{2n}(0201)^{i-1}202$	$3n+2i-1$	$3n+i$	i
$(201)^{2n}(0201)^i2$	$3n+2i$	$3n+i+1$	i
$(201)^{2n}(0201)^{i-1}20$	$3n+2i-1$	$3n+i$	$i-1$
$(201)^{2n}(0201)^{i-1}2012$	$3n+2i$	$3n+i+1$	$i-1$
$(201)^{2n}(0201)^{i-1}201$	$3n+2i-1$	$3n+i+1$	$i-1$

From the above tables, we find the following.

Lemma 3.6 (i) *By the suitable rearrangements of lows and columns, all tables are rewritten as ;*

$\sharp R_1^{\pm 1}$, $\sharp R_2^{\pm 1}$, $\sharp (R_1R_2)^{\pm 1}$		
a	b	c
a	$b+1$	c
a	$b+1$	$c+1$
$a+1$	b	c
$a+1$	b	$c+1$
$a+1$	$b+1$	$c+1$

and

	a	b	c
I	$n-i-1$	$n+2i-1$	$2n+i-1$
II	$n-i$	$n+2i$	$2n+i$
III	$3n+i-1$	i	$3n+2i-1$
IV	$3n+i$	$i-1$	$3n+2i$
V	i	$3n+i-1$	$3n+2i$
VI	$i-1$	$3n+i$	$3n+2i-1$
VII	$n-1$	$n-1$	$2n-1$
VIII	n	n	$2n$

(ii) *In all 8 cases, the minimal length of each element is equal to* $\sharp R_1^{\pm 1} + \sharp R_2^{\pm 1} + \sharp (R_1R_2)^{\pm 1}$.

From this Lemma we obtain the main result.

Theorem 3.7 *The Poincaré series of the Weyl group of type $A_2^{(1,1)}$ is given by*

$$\sum_{w \in W} t^{l(w)} = \frac{1+4t+17t^2+19t^3+17t^4+4t^5+t^6}{(1-t)^4(1+t)^2} = \frac{(1+t+t^2)(1+3t+13t^2+3t^3+t^4)}{(1-t)^4(1+t)^2}.$$

Cororally 3.8 $\sharp\{w \in W \mid l(w) = n \ (n \geq 1)\} = \frac{3}{16}n(21 + 3(-1)^n + 14n^2)$.

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