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Connection relations for a generalization of spherical functions

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1 Introduction

The purpose of this talk is to explain the study on the connection relations for solutions to a system of differential equations related with the symmetric space. The system in question has a solution which is a zonal spherical function or its generalization studied by Heckman and Opdam.

2 Connection relations for Gaussian hypergenometric functions

Let $F(a,b,c;x) = \sum_{n=0}^{\infty} \frac{(a,n)(b,n)}{n!(c,n)} x^n$ be the Gaussian hypergeometric series. Its convergent radius is 1. We first recall the connection formula for hypergeometric functions:

$$\begin{aligned} & F(a,b,c;x) \\ = & \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} F(a,b,a+b-c+1;1-x) \\ & + \frac{\Gamma(c)\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)} (1-x)^{c-a-b} F(c-a,c-b,c-a-b+1;1-x) \end{aligned} \tag{1}$$

$$\begin{aligned} & F(a,b,c;x) \\ = & \frac{\Gamma(c)\Gamma(b-a)}{\Gamma(b)\Gamma(c-a)} |x|^{-a} F(a,a-c+1,a-b+1;1/x) \\ & + \frac{\Gamma(c)\Gamma(a-b)}{\Gamma(a)\Gamma(c-b)} |x|^{-b} F(b,b-c+1,b-a+1;1/x) \end{aligned} \tag{2}$$

Since the convergent area of the power series $F(a,b,c;x)$ is $|x| < 1$ and that of the two power series $F(a,b,a+b-c+1;1-x)$, $F(c-a,c-b,c-a-b+1;1-x)$ is $|1-x| < 1$, the formula (1) holds in the usual sense in the interval $0 < x < 1$. On the contrary, since the convergent area of the two power series $F(a,a-c+1,a-b+1;1/x)$, $F(b,b-c+1,b-a+1;1/x)$ is $1 < |x|$, the formula (2) does not in the usual sense. But it has a meaning if we consider $F(a,b,c;x)$, $F(a,a-c+1,a-b+1;1/x)$, $F(b,b-c+1,b-a+1;1/x)$ as real analytic functions on the real half line $x < 0$ by analytic continuation.

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3 Zonal spherical functions on Riemannian symmetric spaces

(The contents of this section are due to Harish-Chandra [1].)

Let G be a connected linear semisimple Lie group of the non-compact type and let G/K be the corresponding Riemannian symmetric space, where K is the maximal compact subgroup of G. A zonal spherical function $\varphi(xK)$ is a real analytic function on G/K with the conditions: (a) $D\varphi = \chi(D)\varphi$ ($\forall D \in \mathbf{D}(G/K)$), (b) $\varphi(kxK) = \varphi(xK)$ ($\forall k \in K$), (c) $\varphi(eK) = 1$. Here $\mathbf{D}(G/K)$ is the algebra of invariant differential operators on G/K and χ is a character of $\mathbf{D}(G/K)$. Due the the Cartan decomposition $G = KAK$, the zonal spherical function $\varphi(xK)$ is determined by its restriction to A, a maximal split abelian subspace of G. If $\mathfrak{a}$ is the Lie algebra of A, zonal spherical functions are parametrized by $\mathfrak{a}_c$, the complexifiaction of $\mathfrak{a}$. Indeed if $\varphi(aK)$ is a zonal spherical function, there is $\lambda \in \mathfrak{a}_c$ such that $\varphi(aK) = \int_K e^{(\lambda-\rho)(H(ak))}dk (= \varphi_\lambda(aK))$. If W is the Weyl group of (G, A), $\varphi_{w\lambda} = \varphi_\lambda$ ($\forall w \in W$). Let Σ be the root system of (G, A) and let $\Psi = \{\alpha_1, \ldots, \alpha_r\}$ be the system of simple roots. Let $\mathfrak{a}_+$ be the positive Weyl chamber. Namely $\mathfrak{a}_+ = \{Y \in \mathfrak{a};\ \alpha_j(Y) > 0\ (j = 1, 2, \ldots, r)\}$. Put $y_j(a) = e^{-2\alpha_j(\log a)}$ ($j = 1, \ldots, r$). Then clearly $0 < y_j(a) < 1$ if $a \in A_+$, where $A_+ = \exp \mathfrak{a}_+$. It is possible to define a linear function $s_j(\lambda)$ ($j = 1, \ldots, r$) on $\mathfrak{a}_c$ by the condition $a^{\lambda-\rho} = y_1(a)^{s_1(\lambda)} \cdots y_r(a)^{s_r(\lambda)} (= y(a)^{s(\lambda)})$. It is Harish-Chandra who defined a power series $f_\lambda(y)$ convergent in the demain $\{y = (y_1, \ldots, y_r) \in \mathbf{R}^r;\ |y_j| < 1\ (j = 1, \ldots, r)\}$ of the y-space such that $f_\lambda(0) = 1$ and that

$$\varphi_\lambda(aK) = \sum_{w\in W} c(w\lambda) y(a)^{s(\lambda)} f_{w\lambda}(y(a)) \quad (a \in A_+), \tag{3}$$

where $c(\lambda)$ is the c-function. The formula (3) is regarded as an analogue of (1). Actually, not only $f_\lambda(y)$ but also $\varphi_\lambda(aK)$ are regarded as real analytic functions on the y-space and solutions of a system of differential equations $Du = \chi_\lambda(D)u$ ($\forall D \in \mathbf{D}(G/K)$). Moreover the former are local solutions near the origin, whereas $\varphi_\lambda(aK)$ is a real analytic solution near the point $y = (1, 1, \ldots, 1)$.

4 Left K-invariant eigenfunctions on symmetric spaces of the form G/K_ε

(The contents of this section are due to Oshima-Sekiguchi [2].)

We first explain the definition of the symmetric space G/K_ε briefly. Let ε be a signature of roots in Σ, that is, ε is a map of Σ to $\{\pm 1\}$ with the conditions (s.a) $\varepsilon(\alpha) = \pm 1$ ($\forall \alpha \in \Sigma$), (s.b) $\varepsilon(\alpha + \beta) = \varepsilon(\alpha)\varepsilon(\beta)$ if $\alpha, \beta, \alpha + \beta \in \Sigma$, (s.c) $\varepsilon(-\alpha) = \varepsilon(\alpha)$ ($\forall \alpha \in \Sigma$). Let θ be the Cartan involution of G with respect to K. Take $Y_\varepsilon \in \mathfrak{a}$ such that $\alpha_j(Y_\varepsilon) = \frac{1}{2}(1 - \varepsilon(\alpha))$ ($j = 1, \ldots, r$) and put $a_\varepsilon = \exp(\pi\sqrt{-1}Y_\varepsilon)$. Then $\mathrm{Ad}(a_\varepsilon^2) = \mathrm{Id}_{\mathfrak{g}_c}$ and $\mathrm{Ad}(a_\varepsilon)$ leaves $\mathfrak{g}$ invariant in $\mathfrak{g}_c$, the complexification of $\mathfrak{g}$. Moreover $\mathrm{Ad}(a_\varepsilon)$ commutes with θ and so $\theta_\varepsilon = \mathrm{Ad}(a_\varepsilon) \circ \theta$ is also an involution of $\mathfrak{g}$. Let $\mathfrak{g} = \mathfrak{k}_\varepsilon + \mathfrak{p}_\varepsilon$ be the corresponding direct sum decomposition of $\mathfrak{g}$. Now assume that θ_ε is lifted to an involution of G (e.g. if G

is the adjoint group of $\mathfrak{g}$, this is true). Let K_ε be the closed Lie subgroup of G generated by $\exp \mathfrak{k}_\varepsilon$ and $M(= Z_K(\mathfrak{a}))$. The coset space G/K_ε is a symmetric space. The rank of G/K_ε is equal to r which is also the rank of G/K and in particular, the algebra of invariant differential operators which is denoted by $\mathbf{D}(G/K_\varepsilon)$ is isomorphic to $\mathbf{D}(G/K)$. The functions which play a role of zonal spherical functions on G/K_ε are left K-invariant joint eigenfunctions of all invariant differential operators on G/K_ε. For $\lambda \in \mathfrak{a}_c$, let $S(G/K_\varepsilon;\chi_\lambda)^K$ be the space of functions $\varphi(xK_\varepsilon)$ on G/K_ε such that (a) $Du = \chi_\lambda u$ $(\forall D \in \mathbf{D}(G/K_\varepsilon))$ and (b) $\varphi(kxK_\varepsilon) = \varphi(xK_\varepsilon)$ $(\forall k \in K)$. Then any $\varphi \in S(G/K_\varepsilon;\chi_\lambda)^K$ is real analytic and, if λ is generic, $\dim S(G/K_\varepsilon;\chi_\lambda)^K = [W : W_\varepsilon]$, where $W_\varepsilon = (N_K(\mathfrak{a}) \cap K_\varepsilon)/M$. The number $[W : W_\varepsilon]$ coincides with that of open K_ε-orbits of the flag manifold G/P ($P = MAN$ is the minimal parabolic of G). Indeed, for each open K_ε-orbit $\mathcal{O}$ of G/P, there is a distribution $h_{\mathcal{O},\lambda}(g)$ on G such that (a) $\mathrm{Supp}(h_{\mathcal{O},\lambda}) = \overline{\mathcal{O}}$ and (b) $h_{\mathcal{O},\lambda}(gman) = h_{\mathcal{O},\lambda}(kg)a^{\lambda-\rho}$ $(\forall k \in K_\varepsilon, \forall m \in M, \forall a \in A, \forall n \in N)$. If $\mathcal{O}_j$ $(j = 1,2,\ldots,[W : W_\varepsilon])$ are all the open K_ε-orbits of G/P, then $\varphi^j_\lambda(xK_\varepsilon) = \int_K h_{\mathcal{O}_j,-\lambda}(g^{-1}k)dk$ $(j = 1,2,\ldots,[W : W_\varepsilon])$ constitute a basis of $S(G/K_\varepsilon;\chi_\lambda)^K$ (if λ is generic). We studied in [2] the formulas for $\varphi^j_\lambda(xK_\varepsilon)$ $(j = 1,2,\ldots,[W : W_\varepsilon])$ analogues to (3).

5 Generalized hypergeometric functions associated with root systems (due to Heckman and Opdam)

Suppose we have given complex numbers k_α for $\alpha \in \Sigma$ such that $k_{w\alpha} = k_\alpha$ for all $\alpha \in \Sigma$ and $w \in W$. If m denotes the number of W-orbits of roots in Σ, then $k = (k_\alpha)_{\alpha\in\Sigma}$ lies in a parameter space $K \simeq \mathbf{C}^m$. Fix a basis $Y_1,\ldots,Y_r$ for $\mathfrak{a}$ and consider the differential operator

$$L(= L(k)) = \sum_{j=1}^{r} \partial(Y_j)^2 + \sum_{\alpha\in\Sigma^+} k_\alpha \frac{1+a^{-2\alpha}}{1-a^{-2\alpha}}\partial(Y_\alpha) \tag{4}$$

Here $Y_\alpha \in \mathfrak{a}$ is defined by $\beta(Y_\alpha) = (\beta,\alpha)$ for all $\alpha,\beta \in \mathfrak{a}^*$. Introduce also

$$\rho = \rho(k) = \frac{1}{2}\sum_{\alpha\in\Sigma^+} k_\alpha \alpha$$

Remark. *If G/K is a Riemannian symmetric space of the non-compact type and A is as before, the radial part of the Laplace-Beltrami operator on G/K with respect to the action of K on G/K is a W-invariant differential operator on A of the form (4). In this case, the parameter k_α is the multiplicity of root α, namely, $k_\alpha = m_\alpha$.*

Let $\mathbf{D}(k)$ be the algebra of W-invariant differential operators $D(k)$ on A' (the totality of regular elements of A) such that $D(k)$ commute with L and that the principal symbols of $D(k)$ are contained in $S(\mathfrak{a}_c)^W$. Then $\mathbf{D}(k)$ is commutative. Noting this, for any character $\chi_\lambda : \mathbf{D}(k) \to \mathbf{C}$ depending on $\lambda \in \mathfrak{a}_c$, we define the system of differential equations

$$\mathcal{M}^{(k)}_\lambda : Du = \chi_\lambda(D)u \quad (\forall D \in \mathbf{D}(k))$$

on A'_c. Then by the same way to the case of Riemannian symmetric spaces, it is possible to construct a power series solution $y^{s(\lambda)}f_{\lambda,(k)}(y)$ to $\mathcal{M}_\lambda^{(k)}$ convergent $|y_j| < 1 (j = 1, \ldots, r)$ and $f_{\lambda,(k)}(0) = 1$. Now define

$$c_\alpha(\lambda, k) = \frac{B(\frac{(\lambda,\alpha)}{(\alpha,\alpha)}, \frac{k_\alpha}{2})B(\frac{(\lambda,\alpha)}{2(\alpha,\alpha)} + \frac{k_\alpha}{4}, \frac{k_{2\alpha}}{2})}{B(\frac{(\rho(k),\alpha)}{(\alpha,\alpha)}, \frac{k_\alpha}{2})B(\frac{(\rho(k),\alpha)}{2(\alpha,\alpha)} + \frac{k_\alpha}{4}, \frac{k_{2\alpha}}{2})} \quad \text{for } \alpha \in \Sigma^+,\ \frac{1}{2}\alpha \notin \Sigma$$

and put $c(\lambda, k) = \prod_{\alpha\in\Sigma^+,\ \frac{1}{2}\alpha\notin\Sigma} c_\alpha(\lambda, k)$. Then any solution to $\mathcal{M}_\lambda^{(k)}$ in $U \cap A'_c$ which can be extended to $a = e$ coincides with

$$F_{\lambda,(k)}(y) = \sum_{w\in W} c(w\lambda, k)y^{s(w\lambda)}f_{w\lambda,(k)}(y) \tag{5}$$

up to a constant factor, where U is a small neighbourhood of $a = e$ in A_c. In particular, $F_{\lambda,(k)}(1, \ldots, 1) = 1$. For this reason, (5) is regarded as a generalization of (3).

In the sequel, we put $\tilde{f}_{\lambda,(k)}(y) = y^{s(\lambda)}f_{\lambda,(k)}(y)$ for simplicity.

6 Formulation of the problem

The motivation of this talk is what happens if we reformulate the results of [2] from the view point of Heckman-Opdam theory. For this purpose, we prepare some notation. For a signature of roots ε, we define a domain

$$\mathbf{R}^r_\varepsilon = \{y = (y_1, \ldots, y_r)|\ \varepsilon(\alpha_j)y_j > 0\ (j = 1, \ldots, r)\}$$

in $\mathbf{R}^r$. Then $\mathbf{R}^r_\varepsilon$ is identified with A and the function $f_{\lambda,(k)}(y)$ introduced in the previous section is regarded as a real analytic function on $\mathbf{R}^r_\varepsilon$. Moreover, the system $\mathcal{M}_\lambda^{(k)}$ is defined on a complement of the union of hypersurfaces (related roots) in $\mathbf{R}^r_\varepsilon$.

For $\lambda \in \mathfrak{a}_c$, put

$$S(\mathbf{R}^r_\varepsilon, \mathcal{M}_\lambda^{(k)}) = \{\varphi \in \mathcal{A}(\mathbf{R}^r_\varepsilon);\ \varphi \text{ is a solution of } \mathcal{M}_\lambda^{(k)}\},$$

where $\mathcal{A}(D)$ is the space of real analytic functions on a domain D of $\mathbf{R}^r$.

Problem 1.
(i) Determine $\dim S(\mathbf{R}^r_\varepsilon, \mathcal{M}_\lambda^{(k)})$.
(ii) Construct the basis of $S(\mathbf{R}^r_\varepsilon, \mathcal{M}_\lambda^{(k)})$.

It is clear from the definition that $\dim S(\mathbf{R}^r_\varepsilon, \mathcal{M}_\lambda^{(k)}) \le |W|$.

We give here some examples which are known.

(I) As a trivial example, we consider the case $k_\alpha = 0\,(\forall\alpha \in \Sigma)$. Then $\tilde{f}_{\lambda,(0)}(y) = |y|^{s(\lambda)}$ is a real analytic solution of $\mathcal{M}_\lambda^{(0)}$ on $\mathbf{R}^r_\varepsilon$. In particular, if $\lambda \in \mathfrak{a}_c$ is generic, $\tilde{f}_{\lambda,(0)}(y)$ $(w \in W)$ form a basis of $S(\mathbf{R}^r_\varepsilon, \mathcal{M}_\lambda^{(0)})$.

(II) Zonal spherical functions on Riemannian symmetric spaces and the generalized hypergeometric functions due to Heckman and Opdam

If ε is trivial, then $\dim S(\mathbf{R}^r_\varepsilon, \mathcal{M}^{(k)}_\lambda) = 1$ and $F_{\lambda,(k)}$ is a generator of the space $S(\mathbf{R}^r_\varepsilon, \mathcal{M}^{(k)}_\lambda)$. In particular, if k_α coincides with the multiplicity of α for any $\alpha \in \Sigma$, then $S(\mathbf{R}^r_\varepsilon, \mathcal{M}^{(k)}_\lambda)$ is spanned by the zonal spherical function φ_λ.

(III) Left K-invariant eigenfunctions on G/K_ε

We treat the case where k_α coincides with the multiplicity of α for any $\alpha \in \Sigma$. Then as we explained before, if $\lambda \in \mathfrak{a}_c$ is generic, then $\dim S(\mathbf{R}^r_\varepsilon, \mathcal{M}^{(k)}_\lambda) = [W : W_\varepsilon]$ and $S(\mathbf{R}^r_\varepsilon, \mathcal{M}^{(k)}_\lambda)$ is spanned by φ^j_λ ($j = 1, 2, \ldots, [W : W_\varepsilon]$).

7 Some results due to E. Opdam

The speaker went to Amsterdam in this March and August to discuss with E. Opdam on this problem. In particular, at the occasion of my first visit to Amsterdam, Eric told that Heckman [4] discussed problems deeply related with that introduced in the previous section.

The idea of E. Opdam to attack the problem is to consider complex analytic solutions to $\mathcal{M}^{(k)}_\lambda$ in a neighbourhood of the point a_ε in the space A_c or its quotient space A_c/W_ε and translate the problem to those in terms of Heck algebra which is naturally arised via the monodromy representation of multivalued solutions to $\mathcal{M}^{(k)}_\lambda$ in the complex domain. Then he showed that under the condtion (A) $2k_\alpha \notin \mathbf{Z}$ for any $\alpha \in \Sigma$, the number $[W : W_\varepsilon]$ is the dimension of local holomorphic solutions to $\mathcal{M}^{(k)}_\lambda$ near the point a_ε. As a consequence, Problem 1, (i) holds under the condition (A). His proof needs the so called Tits' deformation lemma for Hecke algebra and quite indirect.

To state the result for Problem 1, (ii), we note that for any nontrivial signature of roots ε, there is $w \in W$ and a unique j_0 ($1 \leq j_0 \leq r$) such that $\varepsilon(w\alpha_j) = 1$ ($j \neq j_0$). For this reason, we may assume from the first that the signature ε is so taken that, there is a unique j_0 ($1 \leq j_0 \leq r$) such that $\varepsilon(w\alpha_j) = 1$ ($j \neq j_0$). As a direct consequence, the group W_ε contains all the reflections with respect to simple roots α_j ($j \neq j_0$). This implies in particular that $\mathrm{rank}(W_\varepsilon) = \mathrm{rank}(W)$ or $\mathrm{rank}(W) - 1$. Now we assume that $\mathrm{rank}(W_\varepsilon) = \mathrm{rank}(W) - 1$. Then, E. Opdam proved that the function

$$\sum_{w \in W_\varepsilon} c(w\lambda, k) \tilde{f}_{w\lambda,(k)}(y)$$

is a unique holomorphic function up to a constant factor in the linear space spanned by the restrictions to a neighpurhood of a_ε of multivalued functions $\tilde{f}_{w\lambda}(y)$ ($w \in W_\varepsilon$), This contains an answer to Problem 1, (ii). On the other hand, in the case where $\mathrm{rank}(W_\varepsilon) = \mathrm{rank}(W)$, it seems not so easy to solve Problem 1, (ii).

8 Connection relations

This section is the main part of this note.

8.1 Type A_1 case

We first treat the simplest case, that is type A_1 case. Then

$$L = \vartheta_y^2 - \frac{k}{2}\frac{1+y}{1-y}\vartheta_y$$

up to a non-zero constant factor, where $\vartheta_y = y\frac{d}{dy}$. The differential equation $\mathcal{M}_\lambda^{(k)}$ turns out to be

$$Lu = \frac{s^2-k^2}{16}u \tag{6}$$

for a constant $s \in \mathbf{C}$. From the definition, the equation (6) is defined if $y \neq 1$.

From now on, *we assume that the parameters k and s are generic* and consider real analytic solutions to (6) on the real positive half line and the real negative half line, separately.

Noting that $y^{(s+k)/4}F(\frac{s+k}{2}, \frac{k}{2}, \frac{s}{2}+1; y)$ is a solution to (6), we put $u_{s,k}(y) = F(\frac{s+k}{2}, \frac{k}{2}, \frac{s}{2}+1; y)$ and $\sigma_k(s) = \frac{\Gamma(s/2)}{\Gamma((s+k)/2)}$ for simplicity.

(B.1) Real analytic functions on the half line $y > 0$ which are solutions on the two intervals $0 < y < 1$ and $1 < y$.

It is known that

$$y^{(s+k)/4}F(\frac{s+k}{2}, \frac{k}{2}, k; 1-y), \quad y^{(s+k)/4}(1-y)^{1-k}F(\frac{s+2-k}{2}, \frac{2-k}{2}, 2-k; 1-y)$$

are solutions to (6) near the point $y = 1$; the former is real analytic whereas the latter is not. Moreover,

$$F(\frac{s+k}{2}, \frac{k}{2}, k; 1-y) = \frac{1}{\sigma_k(k)}\{\sigma_k(-s)u_{s,k}(y) + \sigma_k(s)y^{-s/2}u_{-s,k}(y)\} \tag{7}$$

This is a special case of Formula (1). As a consequence, we have the following: Put

$$F_{s,k}(y) = \frac{1}{\sigma_k(k)}\{\sigma_k(-s)y^{(s+k)/4}u_{s,k}(y) + y^{(-s+k)/4}\sigma_k(s)u_{-s,k}(y)\}.$$

(This is a prototype of (5).) Then $F_{s,k}(y) = y^{(s+k)/4}F(\frac{s+k}{2}, \frac{k}{2}, k; 1-y)$ and if $u(y)$ is a real analytic function on $y > 0$ and $u(y)$ is a solution to (6) outside $y = 1$, $u(y)$ coincides with $F_{s,k}(y)$ up to a constant factor.

(B.2) Real analytic functions on the half line $y < 0$ which are solutions to (6).

Since (6) has no singular point on the negative half line, any solution to (6) on $y < 0$ is automatically real analytic. As a consequence, $|y|^{(s+k)/4}u_{s,k}(y)$ and $|y|^{(-s+k)/4}u_{-s,k}(y)$ form a basis of the space of real analytic functions on $y < 0$ that are solutions to (6). On the other hand, as a special case of Formula (2), we have on $y < 0$

$$u_{s,k}(y) = -\frac{\sin\frac{\pi k}{2}}{\sin\frac{\pi s}{2}}|y|^{-\frac{s+k}{2}}u_{s,k}\left(\frac{1}{y}\right) + \frac{\sigma_k(s)}{\sigma_k(-s)}\frac{\sin\frac{\pi(s-k)}{2}}{\sin\frac{\pi s}{2}}|y|^{-\frac{k}{2}}u_{-s,k}\left(\frac{1}{y}\right) \tag{8}$$

Put $\tilde{u}_{s,k}(y) = \sigma(-s)|y|^{(s+k)/4}u_{s,k}(y)$. Then, as a direct consequence of (8), we have

$$(\tilde{u}_{s,k}(y), \tilde{u}_{-s,k}(y)) = (\tilde{u}_{s,k}\left(\frac{1}{y}\right), \tilde{u}_{-s,k}\left(\frac{1}{y}\right)) \begin{pmatrix} -\frac{\sin\frac{\pi k}{2}}{\sin\frac{\pi s}{2}} & \frac{\sin\frac{\pi(s+k)}{2}}{\sin\frac{\pi s}{2}} \\ \frac{\sin\frac{\pi(s-k)}{2}}{\sin\frac{\pi s}{2}} & \frac{\sin\frac{\pi k}{2}}{\sin\frac{\pi s}{2}} \end{pmatrix} \tag{9}$$

Remark. *(B.1) corresponds to $SL(2, \mathbf{R})/SO(2)$ and (B.2) does to $SL(2, \mathbf{R})/SO(1,1)$.*

8.2 Rank two case

Irreducible root systems of rank two are A_2, $B_2(= C_2)$, BC_2, G_2.

In this subsection, I focus my attention to the case $B_2(= C_2)$ and study connection relations among solutions to $\mathcal{M}_\lambda^{(k)}$ in order to give an answer to Problem 1, (ii).

I start the argument with giving some notation. Let Σ be the root system of type C_2. (I take C_2 realization instead of B_2.) The roots of Σ are $\pm e_1 \pm e_2$, $\pm 2e_1$, $\pm 2e_2$ and $\alpha_1 = e_1 - e_2$, $\alpha_2 = 2e_2$ form a fundamental system of simple roots for Σ. Then we put $y_j = e^{-2\alpha_j}$ $(j = 1, 2)$. The multiplicity functions are so taken that $p = k_{\alpha_2}$, $q = k_{\alpha_1}$.

As is known, there are eight Weyl chambers and to each Weyl chamber, there associates a fundamental system of simple roots. They are

$$\begin{array}{l}(\alpha_1, \alpha_2), (-\alpha_1, 2\alpha_1 + \alpha_2), (\alpha_1 + \alpha_2, -2\alpha_1 - \alpha_2), (-\alpha_1 - \alpha_2, \alpha_2), \\ (-\alpha_1, -\alpha_2), (\alpha_1, -2\alpha_1 - \alpha_2), (-\alpha_1 - \alpha_2, 2\alpha_1 + \alpha_2), (\alpha_1 + \alpha_2, -\alpha_2).\end{array}$$

Corresponding to these, the coordinate (y_1, y_2) are changed to

$$(y_1, y_2), (\frac{1}{y_1}, y_1^2 y_2), (y_1 y_2, \frac{1}{y_1^2 y_2}), (\frac{1}{y_1 y_2}, y_2), (\frac{1}{y_1}, \frac{1}{y_2}), (y_1, \frac{1}{y_1^2 y_2}), (\frac{1}{y_1 y_2}, y_1^2 y_2), (y_1 y_2, \frac{1}{y_2}).$$

It is easy to show that the Laplace-Beltrami operator is given as follows:

$$\begin{aligned} L &= \vartheta_1^2 - 2\vartheta_1\vartheta_2 + 2\vartheta_2^2 + p\left\{-\frac{1 + y_1^2 y_2}{1 - y_1^2 y_2}\vartheta_1 + \frac{1 + y_2}{1 - y_2}(\vartheta_1 - 2\vartheta_2)\right\} \\ &\quad + q\left\{-\frac{1 + y_1}{1 - y_1}(\vartheta_1 - \vartheta_2) - \frac{1 + y_1 y_2}{1 - y_1 y_2}\vartheta_2\right\} \end{aligned}$$

where $\vartheta_j = y_j \partial_{y_j}$ $(j = 1, 2)$. In this case, there is a differential operator D_4 of order four which commutes with L and L and D_4 are algebraically independent. If $\lambda = (\lambda_1, \lambda_2)$ is an element of $\mathfrak{a}_c$, the system $\mathcal{M}_\lambda^{(k)}$ is expressed as follows:

$$\mathcal{M}_\lambda : \left\{ \begin{array}{lcl} Lu &=& \frac{1}{2}\{\lambda_1^2 + \lambda_2^2 - (p + q)^2 - p^2\}u \\ D_4 u &=& \lambda_1^2 \lambda_2^2 u \end{array} \right. \tag{10}$$

(I don't write down D_4 here.)

There are eight linearly independent solutions

$$\begin{array}{l}
y_1^{-\lambda_1+p+q}y_2^{(-\lambda_1-\lambda_2+2p+q)/2}f_{\lambda_1,\lambda_2}(y_1,y_2),\\
y_1^{-\lambda_1+p+q}y_2^{(-\lambda_1+\lambda_2+2p+q)/2}f_{\lambda_1,-\lambda_2}(y_1,y_2),\\
y_1^{\lambda_1+p+q}y_2^{(\lambda_1-\lambda_2+2p+q)/2}f_{-\lambda_1,\lambda_2}(y_1,y_2),\\
y_1^{\lambda_1+p+q}y_2^{(\lambda_1+\lambda_2+2p+q)/2}f_{-\lambda_1,-\lambda_2}(y_1,y_2),\\
y_1^{-\lambda_2+p+q}y_2^{(-\lambda_1-\lambda_2+2p+q)/2}f_{\lambda_2,\lambda_1}(y_1,y_2),\\
y_1^{-\lambda_2+p+q}y_2^{(\lambda_1-\lambda_2+2p+q)/2}f_{\lambda_2,-\lambda_1}(y_1,y_2),\\
y_1^{\lambda_2+p+q}y_2^{(-\lambda_1+\lambda_2+2p+q)/2}f_{-\lambda_2,\lambda_1}(y_1,y_2),\\
y_1^{\lambda_2+p+q}y_2^{(\lambda_1+\lambda_2+2p+q)/2}f_{-\lambda_2,-\lambda_1}(y_1,y_2),
\end{array}$$

where $f_{\lambda_1,\lambda_2}(y_1,y_2)$ is a power series of y_1,y_2 convergent in a neighbourhood of $(y_1,y_2)=(0,0)$ and $f_{\lambda_1,\lambda_2}(0,0)=1$.

By direct computation, we have

$$\begin{aligned}
&y_1^{\lambda_1-p-q}\{L-\tfrac{1}{2}\{\lambda_1^2+\lambda_2^2-(p+q)^2-p^2\}\}\circ y_1^{-\lambda_1+p+q}|_{y_1=0}\\
=\ &-\frac{2}{1-y_2}\left\{\left(\vartheta_2+\frac{\lambda_1+\lambda_2-2p-q}{2}\right)\left(\vartheta_2+\frac{\lambda_1-\lambda_2-2p-q}{2}\right)\right.\\
&\left.\qquad -y_2\left(\vartheta_2+\frac{\lambda_1+\lambda_2-q}{2}\right)\left(\vartheta_2+\frac{\lambda_1-\lambda_2-q}{2}\right)\right\}
\end{aligned}$$

This implies that

$$f_{\lambda_1,\lambda_2}(0,y_2)=F(-\lambda_2+p,p,-\lambda_2+1;y_2)$$

Similarly, we have

$$\begin{aligned}
&y_2^{(\lambda_1+\lambda_2-2p-q)/2}\{L-\tfrac{1}{2}\{\lambda_1^2+\lambda_2^2-(p+q)^2-p^2\}\}\circ y_2^{(-\lambda_1-\lambda_2+2p+q)/2}|_{y_2=0}\\
=\ &-\frac{2}{1-y_1}\{(\vartheta_1+\lambda_1-p-q)(\vartheta_1+\lambda_2-p-q)-y_1(\vartheta_1+\lambda_1-p)(\vartheta_1+\lambda_2-p)\}
\end{aligned}$$

and

$$f_{\lambda_1,\lambda_2}(y_1,0)=F(\lambda_2-\lambda_1+q,q,\lambda_2-\lambda_1+1;y_1)$$

For simplicity, we put

$$v_{s,k}(x)=F(s+k,k,s+1;x),\quad \tau_k(s)=\frac{\Gamma(s)}{\Gamma(s+k)}$$

Then

$$F(s+k,k,2k;1-x)=\frac{1}{\tau_k(k)}\{\tau_k(-s)v_{s,k}(x)+\tau_k(s)x^{-s}v_{-s,k}(x)\}$$

We consider the differential equations for the cases (A) $\varepsilon(\alpha_1)=-1,\varepsilon(\alpha_2)=1$ and (B) $\varepsilon(\alpha_1)=1,\varepsilon(\alpha_2)=-1$.

The case (A)

In this case, we change the coordinate (y_1, y_2) by $(-y_1, y_2)$. Then L turns out to be

$$
\begin{aligned}
L &= \vartheta_1^2 - 2\vartheta_1\vartheta_2 + 2\vartheta_2^2 + p\left\{-\frac{1+y_1^2y_2}{1-y_1^2y_2}\vartheta_1 + \frac{1+y_2}{1-y_2}(\vartheta_1 - 2\vartheta_2)\right\} \\
&\quad +q\left\{-\frac{1-y_1}{1+y_1}(\vartheta_1-\vartheta_2) - \frac{1-y_1y_2}{1+y_1y_2}\vartheta_2\right\}
\end{aligned}
$$

We consider solutions of

$$
\left\{\begin{array}{lcl} Lu &=& \frac{1}{2}\{\lambda_1^2+\lambda_2^2-(p+q)^2-p^2\}u \\ D_4u &=& \lambda_1^2\lambda_2^2u \end{array}\right.
$$

which are analytic on the domain $y_1 > 0$, $y_2 > 0$.

There are eight linearly independent solutions

$$
\begin{array}{l}
y_1^{-\lambda_1+p+q}y_2^{(-\lambda_1-\lambda_2+2p+q)/2}f_{\lambda_1,\lambda_2}(-y_1,y_2), \\
y_1^{-\lambda_1+p+q}y_2^{(-\lambda_1+\lambda_2+2p+q)/2}f_{\lambda_1,-\lambda_2}(-y_1,y_2), \\
y_1^{\lambda_1+p+q}y_2^{(\lambda_1-\lambda_2+2p+q)/2}f_{-\lambda_1,\lambda_2}(-y_1,y_2), \\
y_1^{\lambda_1+p+q}y_2^{(\lambda_1+\lambda_2+2p+q)/2}f_{-\lambda_1,-\lambda_2}(-y_1,y_2), \\
y_1^{-\lambda_2+p+q}y_2^{(-\lambda_1-\lambda_2+2p+q)/2}f_{\lambda_2,\lambda_1}(-y_1,y_2), \\
y_1^{-\lambda_2+p+q}y_2^{(\lambda_1-\lambda_2+2p+q)/2}f_{\lambda_2,-\lambda_1}(-y_1,y_2), \\
y_1^{\lambda_2+p+q}y_2^{(-\lambda_1+\lambda_2+2p+q)/2}f_{-\lambda_2,\lambda_1}(-y_1,y_2), \\
y_1^{\lambda_2+p+q}y_2^{(\lambda_1+\lambda_2+2p+q)/2}f_{-\lambda_2,-\lambda_1}(-y_1,y_2),
\end{array}
$$

in a neighbourhood of $(y_1, y_2) = (0, 0)$. By direct computation, we have

$$
\begin{aligned}
& y_1^{\lambda_1-p-q}\{L - \tfrac{1}{2}\{\lambda_1^2+\lambda_2^2-(p+q)^2-p^2\}\} \circ y_1^{-\lambda_1+p+q}|_{y_1=0} \\
=& -\frac{2}{1-y_2}\left\{\left(\vartheta_2+\frac{\lambda_1+\lambda_2-2p-q}{2}\right)\left(\vartheta_2+\frac{\lambda_1-\lambda_2-2p-q}{2}\right)\right. \\
& \left. -y_2\left(\vartheta_2+\frac{\lambda_1+\lambda_2-q}{2}\right)\left(\vartheta_2+\frac{\lambda_1-\lambda_2-q}{2}\right)\right\}
\end{aligned}
$$

This implies that

$$
f_{\lambda_1,\lambda_2}(0, y_2) = v_{-\lambda_2,p}(y_2)
$$

Since

$$
\tau_p(\lambda_2)v_{-\lambda_2,p}(y_2) + \tau_p(-\lambda_2)y_2^{\lambda_2}v_{\lambda_2,p}(y_2) = \tau_p(p)F(-\lambda_2+p, p, 2p; 1-y_2),
$$

it follows that

$$
\begin{aligned}
& \tau_p(p)y_2^{(-\lambda_1-\lambda_2+2p+q)/2}F(-\lambda_2+p,p,2p;1-y_2) \\
=& \tau_p(\lambda_2)y_2^{(-\lambda_1-\lambda_2+2p+q)/2}f_{\lambda_1,\lambda_2}(0,y_2) + \tau_p(-\lambda_2)y_2^{(-\lambda_1+\lambda_2+2p+q)/2}f_{\lambda_1,-\lambda_2}(0,y_2)
\end{aligned}
$$

On the other hand, since

$$
F(a, b, c; x) = (1-x)^{c-a-b}F(c-a, c-b, c; x),
$$

it follows that

$$y_2^{(-\lambda_1-\lambda_2+2p+q)/2}F(-\lambda_2+p,p,2p;1-y_2)=y_2^{(-\lambda_1+\lambda_2+2p+q)/2}F(\lambda_2+p,p,2p;1-y_2)$$

Therefore the function of y_2 defined by

$$\tau_p(\lambda_2)y_2^{(-\lambda_1-\lambda_2+2p+q)/2}f_{\lambda_1,\lambda_2}(0,y_2)+\tau_p(-\lambda_2)y_2^{(-\lambda_1+\lambda_2+2p+q)/2}f_{\lambda_1,-\lambda_2}(0,y_2)$$

extends to a neighbourhood of $y_2=1$ analytically and invariant by $\lambda_2\leftrightarrow-\lambda_2$. Noting this, we put

$$\begin{aligned}&F_{\lambda_1,\lambda_2}(y_1,y_2)\\ =\ &y_1^{-\lambda_1+p+q}\{\tau_p(\lambda_2)y_2^{(-\lambda_1-\lambda_2+2p+q)/2}f_{\lambda_1,\lambda_2}(-y_1,y_2)+\tau_p(-\lambda_2)y_2^{(-\lambda_1+\lambda_2+2p+q)/2}f_{\lambda_1,-\lambda_2}(-y_1,y_2)\}\end{aligned}$$

Then it is easy to show that

$$F_{\lambda_1,\lambda_2}\left(y_1y_2,\frac{1}{y_2}\right)=F_{\lambda_1,\lambda_2}(y_1,y_2)$$

By direct computation, we find that

$$\begin{aligned}&y_2^{(\lambda_1+\lambda_2-2p-q)/2}\{L-\tfrac{1}{2}\{\lambda_1^2+\lambda_2^2-(p+q)^2-p^2\}\}\circ y_2^{(-\lambda_1-\lambda_2+2p+q)/2}|_{y_2=0}\\ =\ &-\frac{2}{1+y_1}\{(\vartheta_1+\lambda_1-p-q)(\vartheta_1+\lambda_2-p-q)+y_1(\vartheta_1+\lambda_1-p)(\vartheta_1+\lambda_2-p)\}\end{aligned}$$

and

$$f_{\lambda_1,\lambda_2}(y_1,0)=F(\lambda_2-\lambda_1+q,q,\lambda_2-\lambda_1+1;-y_1)=v_{\lambda_2-\lambda_1,q}(-y_1)$$

Now recall the connection formula

$$\begin{aligned}v_{s,k}(x)&=\frac{\Gamma(s+1)\Gamma(-s)}{\Gamma(k)\Gamma(1-k)}|x|^{-s-k}v_{s,k}(\frac{1}{x})+\frac{\Gamma(s+1)\Gamma(s)}{\Gamma(s+k)\Gamma(s-k+1)}|x|^{-k}v_{-s,k}(\frac{1}{x})\\ &=-\frac{\sin\pi k}{\sin\pi s}|x|^{-s-k}v_{s,k}(\frac{1}{x})+\frac{\tau_k(s)}{\tau_k(-s)}\frac{\sin\pi(s-k)}{\sin\pi s}|x|^{-k}v_{-s,k}(\frac{1}{x}).\end{aligned}$$

This holds when $x<0$. Putting $\tilde{v}_{s,k}(x)=\tau_k(-s)|x|^{(s+k)/2}v_{s,k}(x)$, we have

$$\tilde{v}_{s,k}(x)=-\frac{\sin\pi k}{\sin\pi s}\tilde{v}_{s,k}\left(\frac{1}{x}\right)+\frac{\sin\pi(s-k)}{\sin\pi s}\tilde{v}_{-s,k}\left(\frac{1}{x}\right)$$

and this implies that

$$\begin{aligned}&y_1^{-\lambda_1+p+q}f_{\lambda_1,\lambda_2}(y_1,0)\\ =\ &y_1^{-\lambda_1+p+q}v_{\lambda_2-\lambda_1,q}(-y_1)\\ =\ &\frac{\sin\pi q}{\sin\pi(\lambda_1-\lambda_2)}y_1^{-\lambda_2+p}v_{\lambda_2-\lambda_1,q}\left(-\frac{1}{y_1}\right)\\ &-\frac{\sin\pi(\lambda_2-\lambda_1-q)}{\sin\pi(\lambda_1-\lambda_2)}\frac{\tau_q(\lambda_2-\lambda_1)}{\tau_q(\lambda_1-\lambda_2)}y_1^{-\lambda_1+p}v_{\lambda_1-\lambda_2}\left(-\frac{1}{y_1}\right)\\ =\ &\frac{\sin\pi q}{\sin\pi(\lambda_1-\lambda_2)}y_1^{-\lambda_2+p}f_{\lambda_1,\lambda_2}\left(-\frac{1}{y_1},0\right)\\ &-\frac{\sin\pi(\lambda_2-\lambda_1-q)}{\sin\pi(\lambda_1-\lambda_2)}\frac{\tau_q(\lambda_2-\lambda_1)}{\tau_q(\lambda_1-\lambda_2)}y_1^{-\lambda_1+p}f_{\lambda_2,\lambda_1}\left(-\frac{1}{y_1},0\right)\end{aligned}$$

On the other hand,

$$\begin{aligned}
& \left(\frac{1}{y_1}\right)^{-\lambda_1+p+q}(y_1^2y_2)^{(-\lambda_1-\lambda_2+2p+q)/2}f_{\lambda_1,\lambda_2}\left(-\frac{1}{y_1},y_1^2y_2\right)\\
= \; & y_1^{-\lambda_2+p}y_2^{(-\lambda_1-\lambda_2+2p+q)/2}f_{\lambda_1,\lambda_2}\left(-\frac{1}{y_1},y_1^2y_2\right)
\end{aligned}$$

Therefore

$$\begin{aligned}
& y_1^{-\lambda_1+p+q}y_2^{(-\lambda_1-\lambda_2+2p+q)/2}f_{\lambda_1,\lambda_2}(y_1,y_2)\\
= \; & \frac{\sin\pi q}{\sin\pi(\lambda_1-\lambda_2)}\left(\frac{1}{y_1}\right)^{-\lambda_1+p+q}(y_1^2y_2)^{\frac{-\lambda_1-\lambda_2+2p+q}{2}}f_{\lambda_1,\lambda_2}\left(-\frac{1}{y_1},y_1^2y_2\right)\\
& -\frac{\sin\pi(\lambda_2-\lambda_1-q)}{\sin\pi(\lambda_1-\lambda_2)}\frac{\tau_q(\lambda_2-\lambda_1)}{\tau_q(\lambda_1-\lambda_2)}\left(\frac{1}{y_1}\right)^{-\lambda_2+p+q}(y_1^2y_2)^{\frac{-\lambda_1-\lambda_2+2p+q}{2}}f_{\lambda_2,\lambda_1}\left(-\frac{1}{y_1},y_1^2y_2\right)
\end{aligned}$$

As a consequence, we have

$$\begin{aligned}
& F_{\lambda_1,\lambda_2}(y_1,y_2)\\
= \; & \tau_p(\lambda_2)\times\\
& \left\{\frac{\sin\pi q}{\sin\pi(\lambda_1-\lambda_2)}\left(\frac{1}{y_1}\right)^{-\lambda_1+p+q}(y_1^2y_2)^{\frac{-\lambda_1-\lambda_2+2p+q}{2}}f_{\lambda_1,\lambda_2}\left(-\frac{1}{y_1},y_1^2y_2\right)\right.\\
& \left.+\frac{\sin\pi(\lambda_1-\lambda_2+q)}{\sin\pi(\lambda_1-\lambda_2)}\frac{\tau_q(\lambda_2-\lambda_1)}{\tau_q(\lambda_1-\lambda_2)}\left(\frac{1}{y_1}\right)^{-\lambda_2+p+q}(y_1^2y_2)^{\frac{-\lambda_1-\lambda_2+2p+q}{2}}f_{\lambda_2,\lambda_1}\left(-\frac{1}{y_1},y_1^2y_2\right)\right\}\\
& +\tau_p(-\lambda_2)\times\\
& \left\{\frac{\sin\pi q}{\sin\pi(\lambda_1+\lambda_2)}\left(\frac{1}{y_1}\right)^{-\lambda_1+p+q}(y_1^2y_2)^{\frac{-\lambda_1+\lambda_2+2p+q}{2}}f_{\lambda_1,-\lambda_2}\left(-\frac{1}{y_1},y_1^2y_2\right)\right.\\
& \left.+\frac{\sin\pi(\lambda_1+\lambda_2+q)}{\sin\pi(\lambda_1+\lambda_2)}\frac{\tau_q(-\lambda_1-\lambda_2)}{\tau_q(\lambda_1+\lambda_2)}\left(\frac{1}{y_1}\right)^{\lambda_2+p+q}(y_1^2y_2)^{\frac{-\lambda_1+\lambda_2+2p+q}{2}}f_{-\lambda_2,\lambda_1}\left(-\frac{1}{y_1},y_1^2y_2\right)\right\}
\end{aligned}$$

Put $y_1'=1/y_1$, $y_2'=y_1^2y_2$. Then

$$\begin{aligned}
& F_{\lambda_1,\lambda_2}(y_1,y_2)\\
= \; & \tau_p(\lambda_2)\times\\
& \left\{\frac{\sin\pi q}{\sin\pi(\lambda_1-\lambda_2)}y_1'^{-\lambda_1+p+q}y_2'^{\frac{-\lambda_1-\lambda_2+2p+q}{2}}f_{\lambda_1,\lambda_2}(-y_1',y_2')\right.\\
& \left.+\frac{\sin\pi(\lambda_1-\lambda_2+q)}{\sin\pi(\lambda_1-\lambda_2)}\frac{\tau_q(\lambda_2-\lambda_1)}{\tau_q(\lambda_1-\lambda_2)}y_1'^{-\lambda_2+p+q}y_2'^{\frac{-\lambda_1-\lambda_2+2p+q}{2}}f_{\lambda_2,\lambda_1}(-y_1',y_2')\right\}\\
& +\tau_p(-\lambda_2)\times\\
& \left\{\frac{\sin\pi q}{\sin\pi(\lambda_1+\lambda_2)}y_1'^{-\lambda_1+p+q}y_2'^{\frac{-\lambda_1+\lambda_2+2p+q}{2}}f_{\lambda_1,-\lambda_2}(-y_1',y_2')\right.\\
& \left.+\frac{\sin\pi(\lambda_1+\lambda_2+q)}{\sin\pi(\lambda_1+\lambda_2)}\frac{\tau_q(-\lambda_1-\lambda_2)}{\tau_q(\lambda_1+\lambda_2)}y_1'^{\lambda_2+p+q}y_2'^{\frac{-\lambda_1+\lambda_2+2p+q}{2}}f_{-\lambda_2,\lambda_1}(-y_1',y_2')\right\}
\end{aligned}$$

For simplicity, we put

$$\begin{aligned}
c_{p,q}(\lambda_1,\lambda_2) &= \tau_p(\lambda_1)\tau_p(\lambda_2)\tau_q(\lambda_1-\lambda_2)\tau_q(\lambda_1+\lambda_2)\\
\tilde{f}_{\lambda_1,\lambda_2}(-y_1,y_2) &= c_{p,q}(\lambda_1,\lambda_2)y_1^{-\lambda_1+p+q}y_2^{\frac{-\lambda_1-\lambda_2+2p+q}{2}}f_{\lambda_1,\lambda_2}(-y_1,y_2)\\
\tilde{F}_{\lambda_1,\lambda_2}(y_1,y_2) &= \tau_p(\lambda_2)\tau_q(\lambda_1-\lambda_2)\tau_q(\lambda_1+\lambda_2)F_{\lambda_1,\lambda_2}(y_1,y_2)
\end{aligned}$$

Then

$$\tilde F_{\lambda_1,\lambda_2}(y_1,y_2)=\tilde f_{\lambda_1,\lambda_2}(-y_1,y_2)+\tilde f_{\lambda_1,-\lambda_2}(-y_1,y_2)$$

$$\begin{aligned}&\tilde F_{\lambda_1,\lambda_2}(y_1,y_2)\\ =\ &\frac{\sin\pi q}{\sin\pi(\lambda_1-\lambda_2)}\tilde f_{\lambda_1,\lambda_2}(-y_1',y_2')+\frac{\sin\pi(\lambda_1-\lambda_2+q)}{\sin\pi(\lambda_1-\lambda_2)}\tilde f_{\lambda_2,\lambda_1}(-y_1',y_2')\\ &+\frac{\sin\pi q}{\sin\pi(\lambda_1+\lambda_2)}\tilde f_{\lambda_1,-\lambda_2}(-y_1',y_2')+\frac{\sin\pi(\lambda_1+\lambda_2+q)}{\sin\pi(\lambda_1+\lambda_2)}\tilde f_{-\lambda_2,\lambda_1}(-y_1',y_2')\end{aligned}$$

Let V_1 be the linear space spanned by

$$\tilde F_{\lambda_1,\lambda_2}(y_1,y_2),\ \tilde F_{\lambda_2,\lambda_1}(y_1,y_2),\ \tilde F_{-\lambda_1,\lambda_2}(y_1,y_2),\ \tilde F_{-\lambda_2,\lambda_1}(y_1,y_2)$$

and let V_2 be the linear space spanned by

$$\tilde F_{\lambda_1,\lambda_2}(y_1',y_2'),\ \tilde F_{\lambda_2,\lambda_1}(y_1',y_2'),\ \tilde F_{-\lambda_1,\lambda_2}(y_1',y_2'),\ \tilde F_{-\lambda_2,\lambda_1}(y_1',y_2').$$

Then it is possible to show that $\dim V_1\cap V_2=2$.

Indeed, we put

$$\begin{aligned}&G_{\lambda_1,\lambda_2}(y_1,y_2)\\ =\ &\sin\pi(\lambda_1-\lambda_2)\sin\pi(\lambda_1+\lambda_2-q)\tilde F_{\lambda_2,\lambda_1}(y_1,y_2)-2\sin\pi q\sin\pi\lambda_1\cos\lambda_2\tilde F_{-\lambda_1,\lambda_2}(y_1,y_2)\\ &+\sin\pi(\lambda_1+\lambda_2)\sin\pi(\lambda_1-\lambda_2-q)\tilde F_{-\lambda_2,\lambda_1}(y_1,y_2).\end{aligned}$$

Then it is easy to show that

$$\begin{aligned}&G_{\lambda_1,\lambda_2}(y_1,y_2)\\ =\ &-\frac{\sin\pi q}{\sin\pi(\lambda_1-\lambda_2)}G_{\lambda_1,\lambda_2}(y_1',y_2')-\frac{\sin\pi(\lambda_1-\lambda_2-q)}{\sin\pi(\lambda_1-\lambda_2)}G_{\lambda_2,\lambda_1}(y_1',y_2')\end{aligned}$$

By changing $\lambda_1\leftrightarrow\lambda_2$, we also obtain

$$\begin{aligned}&G_{\lambda_2,\lambda_1}(y_1,y_2)\\ =\ &-\frac{\sin\pi(\lambda_1-\lambda_2+q)}{\sin\pi(\lambda_1-\lambda_2)}G_{\lambda_1,\lambda_2}(y_1',y_2')+\frac{\sin\pi q}{\sin\pi(\lambda_1-\lambda_2)}G_{\lambda_2,\lambda_1}(y_1',y_2')\end{aligned}$$

In virtue of these identities, we easily conclude that $V_1\cap V_2$ is spanned by $G_{\lambda_1,\lambda_2}(y_1,y_2)$ and $G_{\lambda_2,\lambda_1}(y_1,y_2)$.

By using vector notation, we have

$$\begin{aligned}&(G_{\lambda_1,\lambda_2}(y_1,y_2)\ \ G_{\lambda_2,\lambda_1}(y_1,y_2))\\ =\ &(G_{\lambda_1,\lambda_2}(y_1',y_2')\ \ G_{\lambda_2,\lambda_1}(y_1',y_2'))\begin{pmatrix}-\frac{\sin\pi q}{\sin\pi(\lambda_1-\lambda_2)} & -\frac{\sin\pi(\lambda_1-\lambda_2+q)}{\sin\pi(\lambda_1-\lambda_2)}\\ -\frac{\sin\pi(\lambda_1-\lambda_2-q)}{\sin\pi(\lambda_1-\lambda_2)} & \frac{\sin\pi q}{\sin\pi(\lambda_1-\lambda_2)}\end{pmatrix}\end{aligned}$$

On the other hand,

$$(G_{\lambda_1,\lambda_2}(y_1,y_2)\ \ G_{\lambda_2,\lambda_1}(y_1,y_2))=(G_{\lambda_1,\lambda_2}(z_1,z_2)\ \ G_{\lambda_2,\lambda_1}(z_1,z_2))$$

where $z_1 = y_1 y_2$, $z_2 = 1/y_2$. As a consequence, we obtain

$$(G_{\lambda_1,\lambda_2}(1/y_1, y_1^2 y_2)\ G_{\lambda_2,\lambda_1}(1/y_1, y_1^2 y_2)) = (G_{\lambda_1,\lambda_2}(1/z_1, z_1^2 z_2)\ G_{\lambda_2,\lambda_1}(1/z_1, z_1^2 z_2))$$

I explain the meaning of these identity equations. At first, the functions $f_{w\lambda}(y_1, y_2)(w \in W)$ are defined by power series convergent in the domain $|y_j| < 1$ $(j = 1, 2)$. Our purpose is to define $f_{w\lambda}(y_1, y_2)$ $(w \in W)$ outside the domain $|y_j| < 1$ $(j = 1, 2)$ by analytic continuation. Some of $f_{w\lambda}(y_1, y_2)$ $(w \in W)$ may have singularities in the set $\{(y_1, y_2) :\ y_1 > 0, y_2 > 0\}$. The identity equations above guarantee that $G_{\lambda_1,\lambda_2}(y_1, y_2)$ is real analytic near the points

$$\begin{aligned}&(y_1, y_2) = (0, 0), (\tfrac{1}{y_1}, y_1^2 y_2) = (0, 0), (y_1 y_2, \tfrac{1}{y_1^2 y_2}) = (0, 0), (\tfrac{1}{y_1 y_2}, y_2) = (0, 0),\\ &(y_1 y_2, \tfrac{1}{y_1}) = (0, 0), (\tfrac{1}{y_1 y_2}, y_1^2 y_2) = (0, 0), (y_1, \tfrac{1}{y_1^2 y_2}) = (0, 0), (\tfrac{1}{y_1}, \tfrac{1}{y_2}) = (0, 0).\end{aligned}$$

This means that $G_{\lambda_1,\lambda_2}(y_1, y_2)$ is real analytic on each Weyl chanber of A' and moreover has no singularities at generic points of the singular elemtns of A. This implies that $G_{\lambda_1,\lambda_2}(y_1, y_2)$ is real analytic on A.

The case (B)
In this case, we cahnge the coordinate (y_1, y_2) by $(y_1, -y_2)$. Then L turns out to be

$$\begin{aligned} L &= \vartheta_1^2 - 2\vartheta_1\vartheta_2 + 2\vartheta_2^2 + p\left\{-\frac{1 - y_1^2 y_2}{1 + y_1^2 y_2}\vartheta_1 + \frac{1 - y_2}{1 + y_2}(\vartheta_1 - 2\vartheta_2)\right\} \\ &\quad + q\left\{-\frac{1 + y_1}{1 - y_1}(\vartheta_1 - \vartheta_2) - \frac{1 - y_1 y_2}{1 + y_1 y_2}\vartheta_2\right\}\end{aligned}$$

We consider solutions of

$$\mathcal{M}_{\lambda_1,\lambda_2} : \begin{cases} Lu &= \frac{1}{2}\{\lambda_1^2 + \lambda_2^2 - (p + q)^2 - p^2\}u \\ D_4 u &= \lambda_1^2\lambda_2^2 u \end{cases}$$

analytic on the domain $y_1 > 0$, $y_2 > 0$.

There are eight linearly independent solutions in a neighbourhood of $(y_1, y_2) = (0, 0)$; one of them is given by

$$y_1^{-\lambda_1+p+q} y_2^{(-\lambda_1-\lambda_2+2p+q)/2} f_{\lambda_1,\lambda_2}(y_1, -y_2).$$

Put

$$\tilde{G}_{\lambda_1,\lambda_2}(y_1, y_2) = \tilde{f}_{\lambda_1,\lambda_2}(y_1, -y_2) + \tilde{f}_{\lambda_2,\lambda_1}(y_1, -y_2).$$

Then it is possible to show (by the result of E. Opdam) that the four functions

$$\tilde{G}_{\lambda_1,\lambda_2}(y_1, y_2), \tilde{G}_{\lambda_1,-\lambda_2}(y_1, y_2), \tilde{G}_{-\lambda_1,\lambda_2}(y_1, y_2), \tilde{G}_{-\lambda_1,-\lambda_2}(y_1, y_2)$$

defined on the domain $\{(y_1, y_2);\ 0 < y_j < 1\ (j = 1, 2)\}$ are real analytically extended to the space $\{(y_1, y_2);\ 0 < y_j\ (j = 1, 2)\}$.

Remark. In the case (A), $W_\varepsilon \simeq S_2 \times S_2$ and its rank equals that of W. On the oher hand, in the case (B), $W_\varepsilon \simeq S_2$ and its rank does not equal that of W.

References

[1] Harish-Chandra: Spherical functions on a semi-simple Lie group I. Amer. J. Math., **80** (1958), 241-310.

[2] T. Oshima and J. Sekiguchi: Eigenspaces of invariant differential operators on an affine symmetric space. Invent. Math. **57**(1980), 1-81.

[3] G. Heckman: Root systems and hypergeometric functions II. Compositio Math. **64** (1987), 353-373.

[4] G. Heckman: Are K-invariant Eisenstein integrals for G/H hypergeometric functions? In "Harmonic Analysis and Special Functions on Symmetric Spaces", by G. Heckman and H. Schlichtkrull, 189-208.

[5] G. Heckman and E. Opdam: Root systems and hypergeometric functions I. Compositio Math. **64** (1987), 329-352.

[6] E. Opdam: Root systems and hypergeometric functions III, IV. Compositio Math. **67** (1988), 21-49, Compositio Math. **67** (1988), 191-209.