

重複度 1 の表現と複素多様体上の 可視的な作用

Multiplicity-free representations and visible actions on complex manifolds

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アブストラクト :

正則ペクトル束に群作用が与えられているときに、その正則な大域切断の空間に実現された表現を考える。この表現は、1) 底空間が非コンパクトならば一般に無限次元表現であり、2) 底空間における群軌道が無限個ならば一般に既約とは程遠い表現になる。

しかし、ある種の幾何的な条件 (『可視的な作用』) が満たされていれば、重複度が 1 という性質がファイバーへの表現⇒大域切断における表現に伝播することが証明できる。

この概説講演では、有限次元および無限次元の表現における重複度 1 の表現のたくさん例を紹介し、それが『可視的な作用』という幾何的な条件からどのように理解できるかを説明する予定です。

- [1] Multiplicity-free theorem in branching problems of unitary highest weight modules, Proc. Symposium on Representation Theory held at Saga, Kyushu 1997 (ed. K. Mimachi), (1997), 9–17.
- [2] Multiplicity one theorem in the orbit method (with Nasrin), in memory of Karpelevic “Lie Groups and Symmetric Spaces”, (eds. S. Gindikin), 161–169 Amer. Math. Soc. 2003
- [3] Geometry of Multiplicity-free representations of $GL(n)$, visible actions on flag varieties, and triunity, Acta Appl. Math. 81 (2004), 129–146.
- [4] Multiplicity-free representations and visible actions on complex manifolds, Publ. RIMS 41 (2005), 497–549.

Eg. 1 (Eigenspace decomposition)

$\mathcal{H} : \text{Vector sp.}/\mathbb{C}, \dim < \infty$

$A \in \text{End}_{\mathbb{C}}(\mathcal{H})$

①

s.t. $\begin{cases} A \text{ is diagonalizable,} \\ \text{all eigenvalues are distinct.} \end{cases}$

$$\Rightarrow \mathcal{H} = \mathbb{C}e_1 \oplus \cdots \oplus \mathbb{C}e_n$$

$\simeq \mathbb{C}^n$ (canonical)

① & $\det A \neq 0$

$\Downarrow$

$\pi_A : \mathbb{Z} \longrightarrow GL_{\mathbb{C}}(\mathcal{H})$ is MF

$\cup$

$\cup$

$n \longmapsto A^n$

Eg. 2 (Fourier series expansion)

$$L^2(S^1) \simeq \sum_{n \in \mathbb{Z}}^{\oplus} \mathbb{C} e^{inx}$$

$$f(x) = \sum_{n \in \mathbb{Z}} a_n e^{inx}$$

Translation ($\Rightarrow$ rep. of the group S^1)

$$f(\cdot) \mapsto f(\cdot - c) \quad (c \in S^1 \simeq \mathbb{R}/2\pi\mathbb{Z})$$

$S^1 \curvearrowright L^2(S^1)$ is MF

$\pi : G \rightarrow GL_{\mathbb{C}}(\mathcal{H})$
group

Def. (naive)

$(\pi, \mathcal{H})$ is MF

multiplicity-free

if $\dim \operatorname{Hom}_G(\tau, \pi) \leq 1$

$(\forall \tau : \text{irred. rep. of } G).$

$$L^2(S^1) \overset{\exists 1}{\hookrightarrow} e^{inx} \\ \Leftrightarrow \dim \operatorname{Hom}_{S^1}(\tau, L^2(S^1)) = 1 \\ (\forall \tau : \text{irred. rep. of } S^1)$$

$\Rightarrow S^1 \curvearrowright L^2(S^1)$ is MF

Eg. 3 (Taylor expansion,

Laurent expansion)

$$f(z_1, \dots, z_n) = \sum_{\alpha=(\alpha_1, \dots, \alpha_n)} a_\alpha z_1^{\alpha_1} \dots z_n^{\alpha_n}$$

Point (too obvious)

$\exists 1 \ a_\alpha \in \mathbb{C} \quad \text{for each } \alpha$

$\uparrow$

$\dim \text{Hom}_{(S^1)^n}(\tau, \mathcal{O}(\{0\})) \leq 1$

$(\forall \tau : \text{irred. rep. of } (S^1)^n)$

i.e. MF

Eg. 4 (Peter-Weyl)

G : compact (Lie) group

$$L^2(G) \simeq \sum_{\tau \in \widehat{G}}^{\oplus} \tau \boxtimes \tau^*$$

irred. rep. of $G \times G$

Translation ($\Rightarrow$ rep. of $G \times G$)

$$f(\cdot) \mapsto f(g_1^{-1} \cdot g_2)$$

$\Rightarrow \quad G \times G \curvearrowright L^2(G)$ **is MF**

Eg. 5 (Spherical harmonics)

$$\mathcal{H}_l := \left\{ f \in C^\infty(S^{n-1}) : \right.$$

$$\Delta_{S^{n-1}} f = -l(l+n-2)f \left. \right\}$$

$$L^2(S^{n-1}) \simeq \sum_{l=0}^{\infty} \oplus \mathcal{H}_l$$

$O(n)$ $\curvearrowright$ **irred.**

$\Downarrow$

$O(n) \curvearrowright L^2(S^{n-1})$ is **MF**

$\otimes$ -product rep.

$$SL_2(\mathbb{C}) \xrightarrow{\pi_k} S^k(\mathbb{C}^2) \quad (k = 0, 1, 2, \dots)$$

irred.

Eg. 6 (Clebsch-Gordan)

$$\pi_k \otimes \pi_l \simeq \pi_{k+l} \oplus \pi_{k+l-2} \oplus \dots \oplus \pi_{|k-l|}$$

MF

Notation

Highest weight

$$\lambda = (\lambda_1, \dots, \lambda_n) \in \mathbb{Z}^n, \lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n$$

$\Downarrow$

$$\pi_\lambda^{GL_n} \equiv \pi_\lambda : \text{irred. rep. of } GL_n(\mathbb{C})$$

Eg.

$$\lambda = (k, 0, \dots, 0) \quad \leftrightarrow \quad GL_n(\mathbb{C}) \hookrightarrow S^k(\mathbb{C}^n)$$

$$\lambda = (\underbrace{1, \dots, 1}_k, 0, \dots, 0) \quad \leftrightarrow \quad GL_n(\mathbb{C}) \hookrightarrow \wedge^k(\mathbb{C}^n)$$

$\otimes$ -product rep. (GL_n -case)

Eg.7 (Pieri's law)

$$\begin{aligned} & \pi(\lambda_1, \dots, \lambda_n) \otimes \pi(k, 0, \dots, 0) \\ & \simeq \bigoplus_{\substack{\mu_1 \geq \lambda_1 \geq \dots \geq \mu_n \geq \lambda_n \\ \sum(\mu_i - \lambda_i) = k}} \pi(\mu_1, \dots, \mu_n) \end{aligned}$$

$\uparrow$

MF as a GL_n -module.

$\otimes$ -product rep. (continued)

Eg. (counterexample)

$\pi_{(2,1,0)} \otimes \pi_{(2,1,0)}$ is NOT MF
as a $GL_3(\mathbb{C})$ -module.

$\otimes$ -product rep. (continued)

$$\lambda = (\underbrace{a, \dots, a}_p, \underbrace{b, \dots, b}_q) \in \mathbb{Z}^n, \quad a \geq b$$

Eg. 8. (Stembridge 2001, K-)

$\pi_\lambda \otimes \pi_\nu$ is MF as a $GL_n(\mathbb{C})$ -module if

1) $\min(a - b, p, q) = 1$ (and ν is any),

or

2) $\min(a - b, p, q) = 2$ and ν is of the form

$$\nu = (\underbrace{x \dots x}_{n_1}, \underbrace{y \dots y}_{n_2}, \underbrace{z \dots z}_{n_3}) \quad (x \geq y \geq z),$$

or

3) $\min(a - b, p, q) \geq 3$, $\star$ &

$$\min(x - y, y - z, n_1, n_2, n_3) = 1.$$

Eg.9 $(GL_n \downarrow GL_{n-1})$
 $\pi_{(\lambda_1, \dots, \lambda_n)}^{GL_n} | GL_{n-1}$
 $\simeq \bigoplus_{\lambda_1 \geq \mu_1 \geq \dots \geq \mu_{n-1} \geq \lambda_n} \pi_{(\mu_1, \dots, \mu_{n-1})}^{GL_{n-1}}$

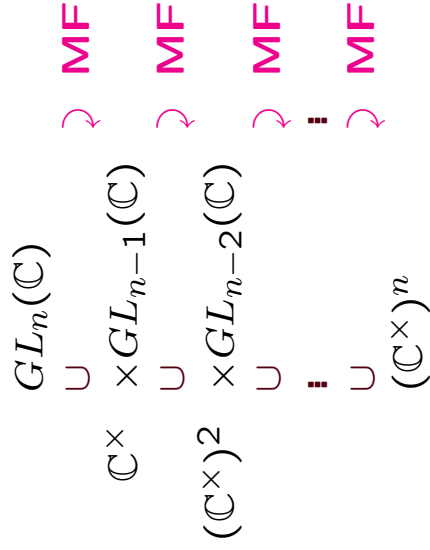
↑
MF as a $GL_{n-1}(\mathbb{C})$ -module

⇓ **Application**

Gelfand-Tsetlin basis
 $\pi_\lambda \hookrightarrow V$
 GL_n
 Find a ‘good’ basis of V

Remark. V is not necessarily MF
 as a $(\mathbb{C}^\times)^n$ -module.

Idea of Gelfand-Tsetlin basis



Infinite dimensional case (cf. Eg.9)

Eg.10. $(U(p, q) \downarrow U(p-1, q))$
 $\forall \pi$: irred. unitary rep. of $U(p, q)$
 with highest weight
 $\Rightarrow$ restriction $\pi|_{U(p-1, q)}$ is MF
 as a $U(p-1, q)$ -module

Eg.11. $(GL - GL \text{ duality})$

$$N = mn$$

$$\Rightarrow GL_m \times GL_n \hookrightarrow S(\mathbb{C}^N)$$

$$\quad \quad \quad \wr$$

$$\quad \quad \quad S(M(m, n; \mathbb{C}))$$

This rep. is MF

- Explicit formula $\cdots GL_m - GL_n$ duality

- Generalization

(Branching law of holo. disc.
 w.r.t. symmetric pair)

Hua-Kostant-Schmid- K-

each component $\cdots$ finite dim ∞ - dim

We recall:

$$\pi : G_{\text{group}} \rightarrow GL_{\mathbb{C}}(\mathcal{H})$$

Def. (naive)
 $(\pi, \mathcal{H})$ is MF **multiplicity-free**
 if $\dim \text{Hom}_G(\tau, \pi) \leq 1$
 $(\forall \tau: \text{irred. rep. of } G).$

Another generalization

Eg.12 (Kac's MF space)
 $S(\mathbb{C}^N)$ is still MF
 as a $GL_{\textcolor{red}{m}-1} \times GL_n$ module

Observation

$n \leq 1 \Leftrightarrow \text{End}(\mathbb{C}^n)$ is commutative.

$(\pi, \mathcal{H})$: unitary rep. of G

Def. $(\pi, \mathcal{H})$ is MF if

$\text{End}_G(\mathcal{H})$ is commutative.

More generally, (ϖ, W) : top. rep.

Def. (ϖ, W) is MF if any

unitary subrep. $(\pi, \mathcal{H})$ is MF

$(\exists G\text{-inj. cont. hom. } \mathcal{H} \hookrightarrow W)$

Eg. 2 (Fourier series expansion)

$$L^2(S^1) \simeq \sum_{n \in \mathbb{Z}}^{\oplus} \mathbb{C} e^{inx}$$

$$f(x) = \sum_{n \in \mathbb{Z}} a_n e^{inx}$$

e^{inx} (cf. centrifugal separator)

Translation ($\Rightarrow$ rep. of the group S^1)

$$f(\cdot) \mapsto f(\cdot - c) \quad (c \in S^1 \simeq \mathbb{R}/2\pi\mathbb{Z})$$

$S^1 \curvearrowright L^2(S^1)$ is MF

Symmetric Space

Eg.13 (Fourier transform)

$$L^2(\mathbb{R}) \simeq \int_{\mathbb{R}}^{\oplus} \mathbb{C} e^{i\zeta x} d\zeta$$

(direct integral of Hilbert sp.)

$$f(x) = \int_{\mathbb{R}} \tilde{f}(\zeta) e^{i\zeta x} d\zeta$$

Translation ($\Rightarrow$ regular representation on $L^2(\mathbb{R})$)

$$f(\cdot) \mapsto f(\cdot - c)$$

$\mathbb{R} \curvearrowright L^2(\mathbb{R})$ is “MF”

! continuous spectrum

Eg.14.

G/K : Riemannian Symm. Space

$\Rightarrow G \curvearrowright L^2(G/K)$ is MF.

Eg.15. (counter example)

G/H : Semisimple Symm. Space

$$\Rightarrow G \curvearrowright L^2(G/H)$$

is NOT always MF.

Eg.14-A. $G/K = \mathbf{SL}(n, \mathbb{R})/\mathbf{SO}(n)$

$$L^2(G/K) \simeq \int^{\oplus}_{\substack{\lambda_1 \geq \dots \geq \lambda_n \\ \sum \lambda_i = 0}} \mathcal{H}_{\lambda} \, \underline{\text{cont. spec.}} \, d\lambda$$

MF

$\mathcal{H}_{\lambda}$: ∞ -dim, irred. rep. of G

Eg.15-A. $G/H = \mathbf{SL}(n, \mathbb{R})/\mathbf{SO}(p, q)$

Multiplicity of most cont. spec.

in $L^2(G/H)$

$$= \frac{n!}{p!q!} > 1 \text{ if } p, q > 0.$$

$\Rightarrow$ NOT MF

Vector bundle case

Eg.15-B. $G/K = \mathbf{GL}(n, \mathbb{R})/\mathbf{O}(n)$

$$\mathcal{V}_{\tau} := G \times_K \tau \rightarrow G/K$$

τ : unitary rep. of K

$\Rightarrow G \curvearrowright L^2(\mathcal{V}_{\tau})$ unitary

$G \curvearrowright L^2(\mathcal{V}_{\tau})$ is NOT always MF.

but it is MF if $\tau \simeq \wedge^k(\mathbb{C}^n)$

$$(0 \leq k \leq n)$$

Examples of Multiplicity-Free reps

- Peter-Weyl theorem
 - Cartan-Helgason theorem
 - Branching laws: $GL_n \downarrow GL_{n-1}$, $O_n \downarrow O_{n-1}$
 - Clebsch-Gordan formula
 - Pieri's law
 - GL_m - GL_n duality
 - Plancherel formula for $L^2(G/K)$
(G/K : Riemannian symmetric spaces)
 - (Gelfand-Graev-Vershik) canonical representations
 - Hua-Kostant-Schmid K -type formula
 - (Kac) linear multiplicity-free spaces
 - (Panyushev) spherical nilpotent orbits
 - (Stembridge) multiplicity-free tensor product representations of GL_n
- etc.

Accordingly, **various techniques** can be applied in each MF case.

For example, one can

- 1) look for an open orbit of a Borel subgroup.
- 2) apply Littlewood-Richardson rules and variants.
- 3) use computer combinatorics.
- 4) employ the commutativity of the Hecke algebra.
- 5) apply Schur-Weyl duality and Howe duality.

§2 MF theorem multiplicity free

$\left\{ \begin{array}{l} H, K: \text{ Lie groups} \\ D: \text{ complex mfd.} \\ P \rightarrow D: H\text{-equiv. principal } K\text{-b'dle} \\ \mu: K \rightarrow GL_{\mathbb{C}}(V) \end{array} \right\} \quad \Downarrow$

Setting 1 $H\text{-equiv. holo.vector b'dle:}$

$$\mathcal{V} := P \times_K V \rightarrow D$$

$\Downarrow$

$$H^{\wedge} \mathcal{O}(D, \mathcal{V}) = \{\text{holo. sections}\}$$

Aim ...

To give a **simple principle**
 that explains the property MF
 of all these examples,
 and more.

Setting 2

P	$\hookrightarrow$	σ_1	difféo.
K	$\hookrightarrow$	σ_2	auto.
H	$\hookrightarrow$	σ_3	auto.

$$\left\{ \begin{array}{l} \sigma(h_p k) = \sigma(h) \sigma(p) \sigma(k) \\ \text{s.t.} \quad \bigcap_{H \subseteq P} H \subseteq K \\ \sigma \hookrightarrow D \simeq P/K \text{ anti-holomorphic} \end{array} \right.$$

Recall $H \hookrightarrow P \hookrightarrow K \xrightarrow{\quad} D$

$$\begin{array}{ccccc}
M & \cup & K & \xrightarrow{\mu} & GL_{\mathbb{C}}(V) \\
B & \cup & P & \xrightarrow{\quad} & D \\
H & \hookrightarrow & & &
\end{array}$$

$$\textcolor{blue}{B} \subset \textcolor{violet}{P}^\sigma := \{p \in P : \sigma(p) = p\}$$

$$M \equiv M_B := \{k \in K : bk \in Hb \text{ } (\forall b \in B)\}$$

Assumption

(a) $HBK = P$

(b) $\mu \mid M$ is MF

say, $\mu \mid_M \nu_i$

(c) $\mu \circ \sigma \simeq \mu^*$ as K -modules

$$\nu_i \circ \sigma \simeq \nu_i^* \text{ as } M\text{-modules } (\forall i)$$

Assumption

(a) HBK contains an interior point of P

(b) $\mu|_M$ is MF

say, $\mu|_M \simeq \bigoplus_{i=1}^l \nu_i$

(c) $\mu \circ \sigma \simeq \mu^*$ as K -modules

$\nu_i \circ \sigma \simeq \nu_i^*$ as M -modules ($\forall i$)

Point of Assumptions

- (a) $\dots$ base sp.
- (b) $\dots$ fiber
- (c) $\dots$ often automatic

Theorem (MF theorem)

Assume $\exists \sigma$ and $\exists B \subset P^\sigma$

satisfying (a) $\sim$ (c).

$\Rightarrow H \curvearrowright \mathcal{O}(D, \mathcal{V})$ is MF.

Point

- propagation of MF property

fiber MF $\Rightarrow$ sections **MF**

- geometry of base space

... ‘**visible action**’

$$\begin{array}{ccc} \hookrightarrow \mathcal{V} & & \\ H \hookrightarrow \downarrow & \Rightarrow H \hookrightarrow & \mathcal{O}(D, \mathcal{V}) \\ & D & \end{array}$$

Assumption (c)

$$\mu \circ \sigma \simeq \mu^* \quad \text{as } K\text{-modules}$$

holds if σ is a Weyl involution

i.e.

$$\begin{aligned} &\sigma \in \text{Aut}(K), \quad \sigma^2 = \text{id} \quad \text{s.t.} \\ &\sigma(g) = g^{-1} \quad \text{on some max torus of } K \end{aligned}$$

e.g. $K = U(n), \quad \sigma(g) = \bar{g}$

§3 Visible action

$H \overset{\text{holomorphic}}{\curvearrowright} D$ complex mfd, connected

Assumption (a): $HBK = P$
for some $B \subset P^\sigma$

$\Rightarrow N := P^\sigma K/K$ meets every
 H -orbit on $D := P/K$

$\Rightarrow H \overset{\curvearrowright}{\curvearrowright} D$ **visible**

(σ : involution $\Rightarrow N$: totally real)

Def. The action is (strongly) visible

if $\begin{array}{l} \exists D' \subset D \text{ open subset} \\ \exists \sigma \overset{\curvearrowright}{\curvearrowright} D \text{ anti-holomorphic} \\ \exists N \subset D \text{ totally real} \end{array}$
s.t. $\left\{ \begin{array}{l} \sigma|_N = \text{id} \\ N \text{ meets every } H\text{-orbit in } D' \\ \sigma \text{ stabilizes every } H\text{-orbit.} \end{array} \right.$

Assumption(a) $\doteq$ (strongly) visible action

Example of Visible actions

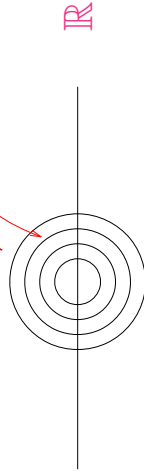
$$\mathbb{T} = \{a \in \mathbb{C} : |a| = 1\} \quad (\simeq S^1)$$

Eg.

$$\mathbb{T} \hookrightarrow \mathbb{C} \supset \mathbb{R}$$

$$\begin{array}{c} \cup \\ a \end{array} \quad z \mapsto az$$

$\mathbb{R}$ meets every $\mathbb{T}$ -orbit



$\Rightarrow$ $\mathbb{T}$ -action on $\mathbb{C}$ is visible.

$H \overset{\text{holomorphic}}{\curvearrowright} (D, J)$ complex mfd, connected

Def. Action is visible if

$$\exists D' \subset D, \quad \text{open}$$

$$\exists N \subset D \quad \text{s.t.} \quad \text{totally real}$$

$$\left\{ \begin{array}{l} N \text{ meets every } H\text{-orbit} \\ J_x(T_x N) \subset T_x(H \cdot x) \quad (x \in N) \end{array} \right.$$

$H \overset{\text{symplectic}}{\curvearrowright} (D, \omega)$ symplectic mfd

$H \overset{\text{isometric}}{\curvearrowright} (D, g)$ Riemannian mfd

Def. (Guillemin-Sternberg,
Huckleberry-Wurzbacher)

Action is coisotropic
(or multiplicity-free)

if

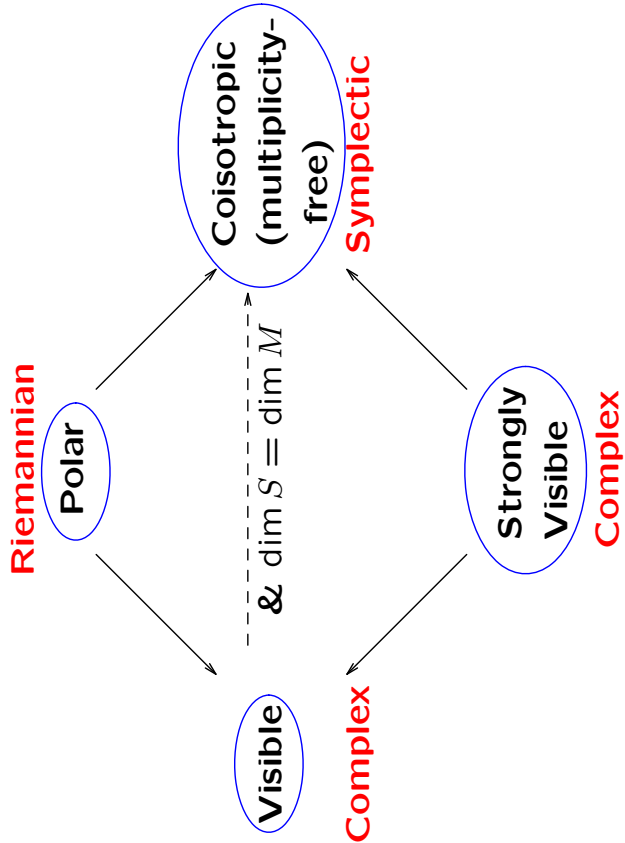
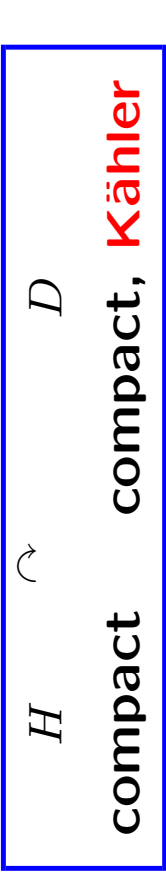
$$T_x(H \cdot x)^{\perp \omega} \subset T_x(H \cdot x) \quad (x \in D)$$

Def. (Podestà-Thorbergsson)

Action is polar if

$$\exists N \subset D \quad \text{s.t.}$$

$$\begin{cases} N \text{ meets every } H\text{-orbit.} \\ T_x N \perp T_x(H \cdot x) \quad (x \in N) \end{cases}$$



Examples of Visible actions

Eg. $\mathbb{T}^n \curvearrowright \mathbb{C}$ is visible.

$$\cup \mathbb{R} \quad \Downarrow$$

Eg. $\mathbb{T}^n \curvearrowright \mathbb{C}^n$ is visible.

$$\cup \mathbb{R}^n \quad \Downarrow$$

Eg. $\mathbb{T}^n \curvearrowright \mathbb{P}^{n-1}\mathbb{C}$ is visible.

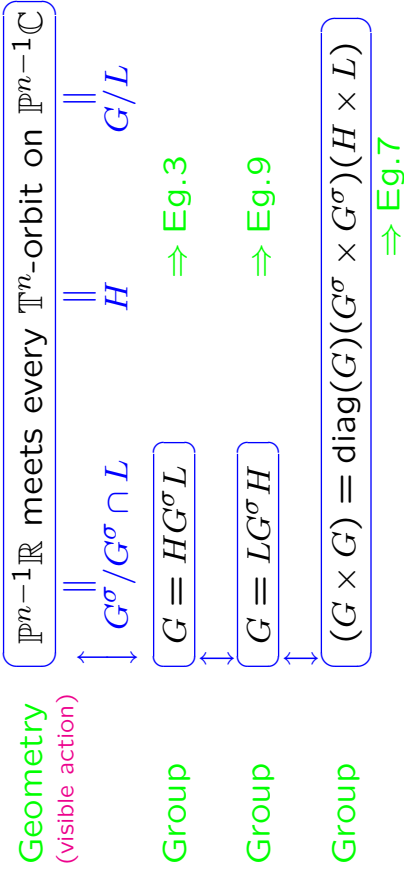
$$\cup \mathbb{P}^{n-1}\mathbb{R} \quad \Downarrow$$

Eg. $U(1) \times U(n-1) \curvearrowright \mathcal{B}_n$
(full flag variety) is visible.

Eg. $U(n) \curvearrowright \mathbb{P}^{n-1}\mathbb{C} \times \mathcal{B}_n$ is visible.

Understanding of visible actions

$$\left(\begin{array}{c} H \\ \cap \\ G \\ \cup \\ G^\sigma \end{array} \right) \cap \left(\begin{array}{c} L \\ \cap \\ \mathbb{T}^n \\ \cup \\ U(n) \\ \cup \\ O(n) \end{array} \right) \cap \left(\begin{array}{c} U(1) \times U(n-1) \end{array} \right)$$



$\Downarrow$ Theorem

Various kinds of MF result including

- (Fourier series) $\mathbb{T} \curvearrowright L^2(\mathbb{T})$ Eg.2
- (Taylor series) $\mathbb{T}^n \curvearrowright \mathcal{O}(\mathbb{C}^n)$ Eg.3
- $(GL_n \downarrow GL_{n-1})$ Restriction $\pi|_{GL_{n-1}}$ Eg.9
- (Pieri) $\pi \otimes S^k(\mathbb{C}^n)$ Eg.7
- (Kac) $GL_{m-1} \times GL_n \curvearrowright S(\mathbb{C}^{mn})$ Eg.12

Eg.

G/K compact symm. sp.

$\Rightarrow G \curvearrowright G_{\mathbb{C}}/K_{\mathbb{C}}$ is visible.

$\Downarrow$ Theorem

Eg.16. $G \curvearrowright L^2(G/K)$ is MF.

Eg.

G/K **non**-compact symm. sp.

$\Rightarrow G \curvearrowright \Omega \subset G_{\mathbb{C}}/K_{\mathbb{C}}$ is visible.
crown

$\Downarrow$ Theorem

Eg.16'. $G \curvearrowright L^2(G/K)$ is MF.

Eg.17. (vector bundle case)

$$\mathcal{V}_k = U(n) \times_{O(n)} \Lambda^k(\mathbb{C}^n) \rightarrow U(n)/O(n)$$

$U(n) \curvearrowright L^2(\mathcal{V}_k)$ is MF.

Eg.17'. (vector bundle case)

$$\mathcal{V}_k = GL(n, \mathbb{R}) \times_{O(n)} \Lambda^k(\mathbb{C}^n) \rightarrow U(n)/O(n)$$

$GL(n, \mathbb{R}) \curvearrowright L^2(\mathcal{V}_k)$ is MF.

G/K Hermitian symm. space

Eg.

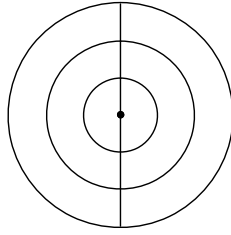
$$G = SL(2, \mathbb{R})$$

$$K = SO(2)$$

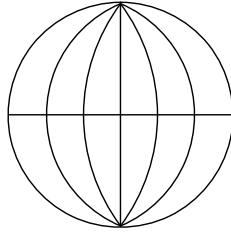
$$H = \left\{ \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} : a > 0 \right\}$$

$$G/K \simeq \{z \in \mathbb{C} : |z| < 1\}$$

$K \curvearrowright G/K$ visible $H \curvearrowright G/K$ visible



K -orbits



H -orbits

Theorem $H \subset G \supset K$

Assume $\begin{cases} G/K & \text{Hermitian symm. sp.} \\ (G, H) & \text{symmetric pair} \end{cases}$
 $\Rightarrow H \curvearrowright G/K$ is visible

$\Downarrow$ **Theorem**

Eg. 18 π_λ, π_μ : highest wt. modules
of scalar type

$\Rightarrow \pi_\lambda \otimes \pi_\mu$ is MF

Eg. 19 $\begin{cases} \pi_\lambda : & \text{highest wt. module} \\ & \text{of scalar type} \end{cases}$
 (G, H) : symmetric pair
 $\Rightarrow \pi_\lambda|_H$ is MF

Also, for finite dimensional case

↓ Theorem

Eg.20 (Okada, 1998)

$$\mathfrak{g}_{\mathbb{C}} = \mathfrak{gl}(n, \mathbb{C})$$

$$\lambda = (\underbrace{a, \dots, a}_p, \underbrace{b, \dots, b}_{n-p}) \in \mathbb{Z}^n, a \geq b$$

$\pi_{\lambda} \upharpoonright_{\mathfrak{h}_{\mathbb{C}}}$ is MF if

$$\mathfrak{h}_{\mathbb{C}} = \begin{cases} \mathfrak{gl}(k, \mathbb{C}) + \mathfrak{gl}(n-k, \mathbb{C}) & (1 \leq k \leq n) \\ \mathfrak{o}(n, \mathbb{C}) & (n : \text{even}) \\ \mathfrak{sp}(\frac{n}{2}, \mathbb{C}) & (n : \text{odd}) \end{cases}$$

G : non-compact, simple Lie gp.,
 G/K Hermitian

Eg. $SU(p, q), SO(n, 2), Sp(n, \mathbb{R}), SO^*(2n), E_6(-14), E_7(-25)$

$$\mathfrak{g}_{\mathbb{C}} = \mathfrak{k}_{\mathbb{C}} + \mathfrak{p}^+ + \mathfrak{p}^-$$

Def. $(\pi, V) \in \widehat{G}$ unitary highest wt rep

$$\Leftrightarrow \{v \in V^{\infty} : d\pi(X)v = 0 \ (\forall X \in \mathfrak{p}^+)\} \neq 0$$

$\mu \nearrow$
 $\widehat{K}$

Write

$$\pi = \pi^G(\mu) \quad (\mu \in \widehat{K})$$

Def. π : holomorphic discrete series

$$\Leftrightarrow \text{Hom}_G(\pi, L^2(G)) \neq 0$$

π : scalar type

$$\Leftrightarrow \dim \mu = 1$$

Def. (G, H) symmetric pair, holomorphic type

$\Leftrightarrow \exists \tau \in \text{Aut}(G), \tau^2 = \text{id} \quad \text{s.t.}$

$$\left\{ \begin{array}{l} (G^\tau)_0 \subset H \subset G^\tau \\ \tau \curvearrowright G/K \text{ holomorphic} \end{array} \right.$$

$$\begin{array}{c} \mathfrak{g}_{\mathbb{C}} = \mathfrak{k}_{\mathbb{C}} + \mathfrak{p}_+ + \mathfrak{p}_- \\ \cup \quad \cup \quad \cup \quad \cup \end{array}$$

$$\mathfrak{h}_{\mathbb{C}} = \mathfrak{g}_{\mathbb{C}}^\tau = \mathfrak{k}_{\mathbb{C}}^\tau + \mathfrak{p}_+^\tau + \mathfrak{p}_-^\tau$$

$$\begin{array}{ccc} \mathfrak{k}^\tau & \subset & \mathfrak{k}^\tau \\ \text{Cartan} \cap & \cap & \text{Cartan} \\ \mathfrak{k} & \subset & \mathfrak{k} \end{array}$$

$\{\nu_1, \dots, \nu_k\}$ maximal set of strongly orth. roots in $\Delta(\mathfrak{p}_+^{-\tau}, \mathfrak{k}^\tau)$

Note: $k = \mathbb{R}\text{-rank } G/H$

Def. (G, H) symmetric pair, holomorphic type

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Eg. $\tau = \theta, \quad H = K$

Eg. $G = Sp(n, \mathbb{R}) \quad (n=p+q)$

$$H = Sp(p, \mathbb{R}) \times Sp(q, \mathbb{R})$$

$$H = U(p, q)$$

$$H = U(n) \quad (= K)$$

cf. $H = GL(n, \mathbb{R})$ not holo. type

Theorem

$\left\{ \begin{array}{l} \pi^G(\mu) \in \hat{G} : \text{holo. disc. series,} \\ \text{scalar type} \\ (G, H) : \text{symmetric pair,} \\ \text{holomorphic type} \end{array} \right\}$



$$\pi^G(\mu)^H \simeq \sum_{\substack{a_1 \geq \dots \geq a_k \geq 0 \\ a_j \in \mathbb{N}}}^{\oplus} \pi^H \left(\mu|_{\mathfrak{t}^\tau} - \sum_{j=1}^k a_j \nu_j \right)$$

$H = K \quad \dots \quad \text{Schmid (1969)}$

$$GL(n, \mathbb{C}) \curvearrowright \mathcal{N} \subset \text{nilpotent orbit } M(n, \mathbb{C})$$

Eg.21 (Panyushev 1994)

$\mathcal{O}(\mathcal{N})$ is MF as $GL(n, \mathbb{C})$ -module

$\Leftrightarrow \mathcal{N} = \mathcal{N}_{(2^p, 1^{n-2p})} \quad (0 \leq p \leq \frac{n}{2})$

$$\mathcal{N}_{(2^p, 1^{n-2p})} \ni \begin{pmatrix} 0 & 1 & & & & \\ 0 & 0 & \ddots & & & \\ & & & p & & \\ & & & 0 & 1 & \\ & & & 0 & 0 & \\ & & & & & \underbrace{0 \dots 0}_{n-2p} \end{pmatrix}$$

Eg. $0 \leq 2p \leq n$

1) $U(n) \curvearrowright \mathcal{N}_{(2^p, 1^{n-2p})}$ is visible.

2) (Panyushev) $\mathcal{O}(\mathcal{N}_{2^p, 1^{n-2p}})$ is MF

$$\mathcal{N}_{(2^p, 1^{n-2p})} \ni \begin{pmatrix} \overbrace{\begin{smallmatrix} 0 & 1 & \dots & 1 \\ 0 & 0 & \dots & 0 \end{smallmatrix}}^p & \begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix} & \underbrace{\begin{smallmatrix} \dots & 0 \\ 0 & \dots & 0 \end{smallmatrix}}_{n-2p} \end{pmatrix}$$

Sketch of proof

$$G = U(n)$$

$$H = U(p) \times U(q) \quad (p+q = n)$$

$$G \times_H M(p, q; \mathbb{C}) \rightarrow \overline{\mathcal{N}_{(2^p, 1^{n-2p})}}$$

$H \curvearrowright M(p, q; \mathbb{C})$ visible

$\Rightarrow G \curvearrowright \mathcal{N}_{(2^p, 1^{n-2p})}$ visible

More examples of visible actions

Eg. $n_1 + n_2 + n_3 = p + q = n$

$Gr_p(\mathbb{R}^n)$ meets every orbit of
 $U(n_1) \times U(n_2) \times U(n_3) \curvearrowright Gr_p(\mathbb{C}^n)$
 ($\Rightarrow$ visible action)

$$\Leftrightarrow (U(n_1) \times U(n_2) \times U(n_3)) \times O(n) \times (U(p) \times U(q)) \twoheadrightarrow U(n)$$

$$\Leftrightarrow \min(n_1 + 1, n_2 + 1, n_3 + 1, p, q) \leq 2$$

⇓ Theorem

MF property of the following

- $GL_m \times GL_n \curvearrowright S(\mathbb{C}^{mn})$ **Eg.11**
- $GL_{m-1} \times GL_n \curvearrowright S(\mathbb{C}^{mn})$ **Eg.12**
- the Stembridge list of $\pi_\lambda \otimes \pi_\nu$ **Eg.8**
- $GL_n \downarrow (GL_p \times GL_q)$ **Eg.22**
- $GL_n \downarrow (GL_{n_1} \times GL_{n_2} \times GL_{n_3})$ **Eg.23**
- ∞ -dimensional versions

...

$\otimes$ -product rep. (continued)

$$\lambda = (\underbrace{a, \dots, a}_p, \underbrace{b, \dots, b}_q) \in \mathbb{Z}^n, \quad a \geq b$$

Eg. 8. (Stembridge 2001, K-)
 $\pi_\lambda \otimes \pi_\nu$ is MF as a $GL_n(\mathbb{C})$ -module if

1) $\min(a - b, p, q) = 1$ (and ν is any),
or

2) $\min(a - b, p, q) = 2$ and ν is of the form
 $\nu = (\underbrace{x \dots x}_{n_1}, \underbrace{y \dots y}_{n_2}, \underbrace{z \dots z}_{n_3}) \quad (x \geq y \geq z),$

or
3) $\min(a - b, p, q) \geq 3, \quad \&$
 $\min(x - y, y - z, n_1, n_2, n_3) = 1.$

Eg.22. $(GL_n \downarrow (GL_p \times GL_q)) \quad n = p + q$
 $\pi_{\underbrace{(x, \dots, x, y, \dots, y, z, \dots, z)}_{n_1} \underbrace{}_{n_2} \underbrace{}_{n_3}}^{GL_n} |_{GL_p \times GL_q}$ **is MF**
if $\min(p, q) \leq 2$
 or
if $\min(n_1, n_2, n_3, x - y, y - z) \leq 1$
(Kostant $n_3 = 0$; Krattenthaler 1998)

Eg.23. $(GL_n \downarrow (GL_{n_1} \times GL_{n_2} \times GL_{n_3}))$
 $n = n_1 + n_2 + n_3$
 $\pi_{\underbrace{(a, \dots, a, b, \dots, b)}_p \underbrace{}_q}^{GL_n} |_{GL_{n_1} \times GL_{n_2} \times GL_{n_3}}$ **is MF**
if $\min(n_1, n_2, n_3) \leq 1$
 or
if $\min(p, q, a - b) \leq 2$

Orbit method

$$\begin{array}{ccc} G & \xrightarrow{\text{Ad}} & \mathfrak{g}^* \supset \mathcal{O}^G \\ \cup & \text{Ad}^* \downarrow \text{pr} & \\ H & \xrightarrow{\text{Ad}} & \mathfrak{h}^* \supset \mathcal{O}^H \end{array}$$

$$\begin{array}{ccccc} H & \subset & G & \supset & K \\ \text{symm. pair} & & & & \text{Hermitian} \\ & & & & \text{symm. pair} \end{array}$$

$$\mathfrak{g} \supset \mathfrak{k} \supset \underline{\text{center}} \Leftrightarrow \underline{\mathbb{R} \cdot z} \subset \mathfrak{g}^*$$

$\nparallel$ 0

Theorem (Nasrin & K-)

If $\mathcal{O}^G \cap \mathbb{R} \cdot z \neq \emptyset$, then
 $\#(\mathcal{O}^G \cap \text{pr}^{-1}(\mathcal{O}^H))/H \leq 1$
 for any $\mathcal{O}^H$.