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Trigonometric Degeneration and Orbifold Wess-Zumino-Witten Model.

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Abstract

Trigonometric degeneration of the Baxter-Belavin elliptic r matrix is described by the degeneration of the twisted WZW model on elliptic curves. The spaces of conformal blocks and conformal coinvariants of the degenerate model are factorised into those of the orbifold WZW model.

1 Introduction

The Baxter-Belavin elliptic classical r matrix is a quasi-classical limit of the famous Baxter's elliptic R matrix [1]. It has a unique position in the classification of the classical r matrices by Belavin and Drinfeld [2].

Etingof [3] introduced the elliptic Knizhnik-Zamolodchikov (KZ) equation as a system of differential equations satisfied by the twisted trace of product of vertex operators (intertwining operators of representations) of affine Lie algebras $\widehat{sl}_N$. This system is formally obtained by replacing the rational r matrix in the ordinary KZ equation by the elliptic one. Kuroki and the author [4] constructed the *twisted Wess-Zumino-Witten (WZW) model* on elliptic curves and gave to Etingof's equations a geometric interpretation as flat connections.

As is well-known, the elliptic r matrix degenerates to the trigonometric r matrix when the elliptic modulus τ tends to $i\infty$. Hence it is natural to expect that the twisted WZW model becomes a degenerate model which is related to the trigonometric r matrix when the elliptic curve degenerates to a singular curve.

The purpose of this talk is to show that this is indeed the case and to analyse the degenerate WZW model, namely:

- The spaces of conformal blocks and conformal coinvariants of the twisted WZW model over the singular curve factorise into those of the orbifold WZW model ([6]).
- The sheaves of conformal blocks and conformal coinvariants are locally free even at the discriminant locus. These sheaves have integrable connections described by the trigonometric r matrices.

2 Twisted WZW model on elliptic curves

In this section we briefly review the twisted WZW model on elliptic curves. See [4] for details.

We fix an invariant inner product of $\mathfrak{g} = sl_N(\mathbb{C})$ by

$$(A|B) := \operatorname{tr}(AB) \quad \text{for } A, B \in \mathfrak{g}. \tag{1}$$

Define matrices β and γ by

$$\beta := \begin{pmatrix} 0 & 1 & & 0 \\ & 0 & \ddots & \\ & & \ddots & 1 \\ 1 & & & 0 \end{pmatrix}, \quad \gamma := \begin{pmatrix} 1 & & & 0 \\ & \varepsilon^{-1} & & \\ & & \ddots & \\ 0 & & & \varepsilon^{1-N} \end{pmatrix}, \tag{2}$$

where $\varepsilon = \exp(2\pi i/N)$. Then we have $\beta^N = \gamma^N = 1$ and $\gamma\beta = \varepsilon\beta\gamma$.

Let $E = E_\tau$ be the elliptic curve with modulus τ: $E_\tau := \mathbb{C}/\mathbb{Z} + \tau\mathbb{Z}$. We define the group bundle G^{tw} with fibre $G = SL_N(\mathbb{C})$ and its associated Lie algebra bundle $\mathfrak{g}^{\mathrm{tw}}$ with fibre $\mathfrak{g} = sl(\mathbb{C})$ over E by

$$G^{\mathrm{tw}} := (\mathbb{C} \times G)/\sim, \tag{3}$$
$$\mathfrak{g}^{\mathrm{tw}} := (\mathbb{C} \times \mathfrak{g})/\approx, \tag{4}$$

where the equivalence relations $\sim$ and $\approx$ are defined by

$$(z, g) \sim (z+1, \gamma g \gamma^{-1}) \sim (z+\tau, \beta g \beta^{-1}), \tag{5}$$
$$(z, A) \approx (z+1, \operatorname{Ad}\gamma(A)) \approx (z+\tau, \operatorname{Ad}\beta(A)). \tag{6}$$

(Because $\gamma\beta = \varepsilon\beta\gamma$, the group bundle G^{tw} is *not* a principal G-bundle.[1]) We shall use the same symbol for both a vector bundle and a locally free coherent sheaf consisting of its local holomorphic sections.

Let $J_{ab} = \beta^a \gamma^{-b}$, which satisfies

$$\operatorname{Ad}\gamma(J_{ab}) = \varepsilon^a J_{ab}, \qquad \operatorname{Ad}\beta(J_{ab}) = \varepsilon^b J_{ab}. \tag{7}$$

Global meromorphic sections of $\mathfrak{g}^{\mathrm{tw}}$ are linear combinations of $J_{ab} f(z)$ ($a, b = 0, \ldots, N-1$, $(a,b) \neq (0,0)$), where $f(z)$ is a meromorphic function with quasi-periodicity,

$$f(z+1) = \varepsilon^a f(z), \qquad f(z+\tau) = \varepsilon^b f(z). \tag{8}$$

For each point P on E, we define Lie algebras,

$$\mathfrak{g}^P := (\mathfrak{g}^{\mathrm{tw}} \otimes_{\mathcal{O}_E} \mathcal{K}_E)^\wedge_P, \tag{9}$$

where $\mathcal{K}_E$ is the sheaf of meromorphic functions on E and $(\cdot)^\wedge_P$ means the completion of the stalk at P with respect to the maximal ideal $\mathfrak{m}_P$ of $\mathcal{O}_{E,P}$, the stalk of the structure sheaf. The Lie algebra $\mathfrak{g}^P$ is isomorphic to the loop Lie algebra $\mathfrak{g}((z-z_0))$, where z_0 is the coordinate of P, but the isomorphism depends on choices of the coordinate z and the trivialisation of $\mathfrak{g}^{\mathrm{tw}}$. The subspace

$$\mathfrak{g}^P_+ := (\mathfrak{g}^{\mathrm{tw}})^\wedge_P \cong \mathfrak{g}[[z-z_0]] \tag{10}$$

of $\mathfrak{g}^P$ is a Lie subalgebra.

Let us fix mutually distinct points $Q_1, \ldots, Q_L$ on E whose coordinates are $z = z_1, \ldots, z_L$ and put $D := \{Q_1, \ldots, Q_L\}$. We shall also regard D as a divisor on E (i.e., $D = Q_1 + \cdots + Q_L$). The Lie algebra $\mathfrak{g}^D := \bigoplus_{i=1}^L \mathfrak{g}^{Q_i}$ has a 2-cocycle defined by

$$\mathrm{c_a}(A, B) := \sum_{i=1}^L \mathrm{c}_{\mathrm{a},i}(A_i, B_i), \qquad \mathrm{c}_{\mathrm{a},i}(A_i, B_i) := \operatorname{Res}_{Q_i}(dA_i | B_i), \tag{11}$$

[1]The bundle G^{tw} is a PGL_N principal bundle. Hurtubise and Markman (for example, see Ref. [7]) take this viewpoint, but we prefer to regard it as a group scheme over the elliptic curve by the reason we stated in §0 of Ref. [4].

where $A = (A_i)_{i=1}^L, B = (B_i)_{i=1}^L \in \mathfrak{g}^D$, Res_{Q_i} is the residue at Q_i and d is the exterior derivation. (The symbol "$\mathrm{c_a}$" stands for "Cocycle defining the Affine Lie algebra".) We denote the central extension of $\mathfrak{g}^D$ with respect to this cocycle by $\hat{\mathfrak{g}}^D$:

$$\hat{\mathfrak{g}}^D := \mathfrak{g}^D \oplus \mathbb{C}\hat{k},$$

where $\hat{k}$ is a central element. Explicitly the bracket of $\hat{\mathfrak{g}}^D$ is represented as

$$[A, B] = ([A_i, B_i]^\circ)_{i=1}^L \oplus \mathrm{c_a}(A, B)\hat{k} \quad \text{for } A, B \in \mathfrak{g}^D, \tag{12}$$

where $[A_i, B_i]^\circ$ are the natural bracket in $\mathfrak{g}^{Q_i}$. The Lie algebra $\hat{\mathfrak{g}}^P$ for a point P is nothing but the affine Lie algebra $\hat{\mathfrak{g}}$ of type $A_{N-1}^{(1)}$ (a central extension of the loop algebra $\mathfrak{g}((t-z)) = sl_N\big(\mathbb{C}((t-z))\big)$).

The affine Lie algebra $\hat{\mathfrak{g}}^{Q_i}$ can be regarded as a subalgebra of $\hat{\mathfrak{g}}^D$. The subalgebra $\mathfrak{g}_+^{Q_i}$ of $\mathfrak{g}^{Q_i}$ (cf. (10)) can be also regarded as a subalgebra of $\hat{\mathfrak{g}}^{Q_i}$ and $\hat{\mathfrak{g}}^D$.

Let $\mathfrak{g}_{\mathrm{out}}$ be the space of global meromorphic sections of $\mathfrak{g}^{\mathrm{tw}}$ which are holomorphic on E except at D:

$$\mathfrak{g}_{\mathrm{out}} := \Gamma(E, \mathfrak{g}^{\mathrm{tw}}(*D)). \tag{13}$$

There is a natural linear map from $\mathfrak{g}_{\mathrm{out}}$ into $\mathfrak{g}^D$ which maps a meromorphic section of $\mathfrak{g}^{\mathrm{tw}}$ to its germ at Q_i's. The residue theorem implies that this linear map is extended to a Lie algebra injection from $\mathfrak{g}_{\mathrm{out}}$ into $\hat{\mathfrak{g}}^D$, which allows us to regard $\mathfrak{g}_{\mathrm{out}}$ as a subalgebra of $\hat{\mathfrak{g}}^D$.

Definition 2.1. The space of *conformal coinvariants* $\mathrm{CC}_E(M)$ and that of *conformal blocks* $\mathrm{CB}_E(M)$ over E associated to $\hat{\mathfrak{g}}^{Q_i}$-modules M_i with the same level $\hat{k} = k$ are defined by

$$\mathrm{CC}_E(M) := M/\mathfrak{g}_{\mathrm{out}}M, \quad \mathrm{CB}_E(M) := \mathrm{Hom}_{\mathbb{C}}(M/\mathfrak{g}_{\mathrm{out}}M, \mathbb{C}), \tag{14}$$

where $M := \bigotimes_{i=1}^L M_i$. The module M_i is referred to as being inserted at the point Q_i.

3 Trigonometric twisted WZW model and orbifold WZW model

In this section we define the degeneration of the twisted WZW model in the previous section, §2, i.e., the trigonometric twisted WZW model on a singular "curve". (Or, exactly speaking, on a "stack", whatever it is.)

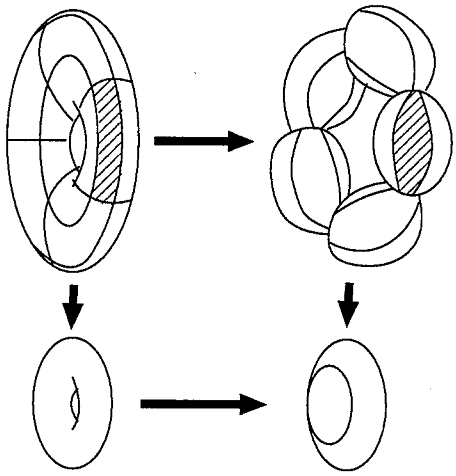

Figure 1: Degeneration of an elliptic curve and its N^2-covering.

When the modulus τ of the elliptic curve E_τ tends to $i\infty$, one of the cycle of the torus is pinched and we obtain a singular curve of the "croissant bread" shape. Such an object is constructed simply by identifying the north pole (∞) and the south pole (0) of the Riemann sphere $\mathbb{P}^1(\mathbb{C})$.

Unfortunately, the twisted bundle $\mathfrak{g}^{\mathrm{tw}}$ over E_τ does not simply degenerates to a bundle over such a singular curve. We need to take several steps:

1. Lift the bundle $\mathfrak{g}^{\mathrm{tw}}$ to a covering space of E_τ where the pull-back of $\mathfrak{g}^{\mathrm{tw}}$ become trivial. The covering is N^2-fold and the covering space is an elliptic curve isomorphic to E_τ.

2. Take such degeneration of the covering elliptic curve that it becomes the N^2-fold covering of the singular curve mentioned above. This covering space consists of N copies of $\mathbb{P}^1$ connected to form a ring.

3. The "degenerate bundle" $\mathfrak{g}^{\mathrm{tw}}$ over the singular curve is understood as the trivial bundle over the singular covering space constructed above with the cyclic group action together.

Let us denote the standard coordinate of $\mathbb{P}^1(\mathbb{C})$ by t. The cyclic group $C_N = \mathbb{Z}/N\mathbb{Z}$ acts as $t \mapsto \varepsilon^a t$ $(a \in C_N)$ and the quotient $E_{\mathrm{orb}} = \mathbb{P}^1/C_N$ is an orbifold. The degenerate elliptic curve E_{trig} is defined by identifying 0 and ∞ (i.e., the fixed points of the C_N action) of E_{orb}.

Not defining what the twisted bundle $\mathfrak{g}^{\mathrm{trig}}$ over E_{trig} and the twisted bundle $\mathfrak{g}^{\mathrm{orb}}$ over E_{orb} are, we directly define what their meromorphic sections are in the following way.

Definition 3.1. A $\mathfrak{g}$-valued meromorphic function $f(t)$ on $\mathbb{P}^1$ is a meromorphic section of $\mathfrak{g}^{\mathrm{orb}}$ if and only if it satisfies

$$f(\varepsilon t) = \operatorname{Ad}\gamma(f(t)). \tag{15}$$

The same $f(t)$ is a section of $\mathfrak{g}^{\mathrm{trig}}$ if and only if it satisfies

$$f(\infty) = \operatorname{Ad}\beta(f(0)), \tag{16}$$

in addition to (15). (The relations (16) and (15) are degenerate form of (6).)

In this and the next section we fix mutually distinct points $Q_1, \dots, Q_L$ ($Q_i \neq 0, \infty$) on the orbifold E_{orb} (modulo C_N-action). The algebras $\mathfrak{g}^{\mathrm{trig},Q_i}$, $\mathfrak{g}_+^{\mathrm{trig},Q_i}$, $\mathfrak{g}^{\mathrm{orb},Q_i}$ and $\mathfrak{g}_+^{\mathrm{orb},Q_i}$ are defined from $\mathfrak{g}^{\mathrm{trig}}$ and $\mathfrak{g}^{\mathrm{orb}}$ in the same way as $\mathfrak{g}^{Q_i}$ and $\mathfrak{g}_+^{Q_i}$ for the elliptic curve, (9) and (10). They are isomorphic to $\mathfrak{g}^{Q_i}$ and $\mathfrak{g}_+^{Q_i}$ since the definition is local.

For the orbifold WZW model, we need similar Lie algebras $\mathfrak{g}^{(0)}$, $\mathfrak{g}_+^{(0)}$, $\mathfrak{g}^{(\infty)}$, $\mathfrak{g}_+^{(\infty)}$ sitting at the singular points, 0 and ∞. Formally their definition is the same as (9) and (10) with P replaced by 0 and ∞ and E by E_{orb}. The mode expansion of $\mathfrak{g}^{(0)}$ has a different form from the usual loop algebra, namely,

$$\mathfrak{g}^{(0)} = \bigoplus_{\substack{a,b=0,\dots,N-1 \\ (a,b)\neq(0,0)}} \bigoplus_{m\in\mathbb{Z}} \mathbb{C} J_{a,b} \otimes t^{a+mN}. \tag{17}$$

The algebra $\mathfrak{g}^{(\infty)}$ has the same form with t and t^{-1} interchanged.

The central extension $\hat{\mathfrak{g}}^{(Q_i)}$ of $\mathfrak{g}^{Q_i}$ is defined in the same way as in §2 by using the cocycle (11). The central extension of $\mathfrak{g}^{(0)}$ and $\mathfrak{g}^{(\infty)}$ are defined essentially in the same manner. Difference is that we have to change the normalisation of the central element $\hat{k}$:

$$[A, B] = [A, B]^\circ \oplus \mathrm{c}_{\mathrm{a},0}(A, B)\hat{k}, \quad \mathrm{c}_{\mathrm{a},0}(A, B) := \frac{1}{N} \operatorname{Res}_{t=0}(\nabla A | B) \tag{18}$$

for $A, B \in \mathfrak{g}^{(0)}$, where $[A, B]^\circ$ is the natural loop-algebra bracket in $\mathfrak{g}^{(0)}$. The definition of $\hat{\mathfrak{g}}^{(\infty)}$ is exactly the same. We often denote $\mathfrak{g}^{(0)}$ and $\mathfrak{g}^{(\infty)}$ by $\mathfrak{g}_\gamma^{(0)}$

and $\mathfrak{g}_\gamma^{(\infty)}$, following §1 of Ref. [3], where $\hat{\mathfrak{g}}_\gamma$ is shown to be isomorphic to the ordinary affine Lie algebra $\hat{\mathfrak{g}}$. The suffix γ is put to emphasise the twisting of the algebra by γ, (15).

Let D be the divisor $Q_1 + \cdots + Q_L$ and $\tilde{D} = D + [0] + [\infty]$. The loop algebras $\mathfrak{g}^D$, $\mathfrak{g}^{\tilde{D}}$ and their central extensions are defined as before. The following Lie algebras are regarded as subalgebras of $\hat{\mathfrak{g}}^D$ and $\hat{\mathfrak{g}}^{\tilde{D}}$ respectively:

$$\mathfrak{g}_{\mathrm{out}}^{\mathrm{trig}} := \{f(t) : \text{global meromorphic section of } \mathfrak{g}^{\mathrm{trig}} \text{ with poles in } D\}. \tag{19}$$

$$\mathfrak{g}_{\mathrm{out}}^{\mathrm{orb}} := \{f(t) : \text{global meromorphic section of } \mathfrak{g}^{\mathrm{orb}} \text{ with poles in } \tilde{D}\}, \tag{20}$$

Definition 3.2. The space of *conformal coinvariants* $\mathrm{CC}_{\mathrm{trig}}(M)$ and that of *conformal blocks* $\mathrm{CB}_{\mathrm{trig}}(M)$ over E_{trig} associated to $\hat{\mathfrak{g}}^{Q_i}$-modules M_i with the same level $\hat{k} = k$ are defined by

$$\mathrm{CC}_{\mathrm{trig}}(M) := M/\mathfrak{g}_{\mathrm{out}}^{\mathrm{trig}} M, \quad \mathrm{CB}_{\mathrm{trig}}(M) := \mathrm{Hom}_{\mathbb{C}}(M/\mathfrak{g}_{\mathrm{out}}^{\mathrm{trig}} M, \mathbb{C}). \tag{21}$$

Similarly the space of *conformal coinvariants* $\mathrm{CC}_{\mathrm{orb}}(M)$ and that of *conformal blocks* $\mathrm{CB}_{\mathrm{orb}}(M)$ over E_{orb} are defined by

$$\mathrm{CC}_{\mathrm{orb}}(M) := M/\mathfrak{g}_{\mathrm{out}}^{\mathrm{orb}} M, \quad \mathrm{CB}_{\mathrm{orb}}(M) := \mathrm{Hom}_{\mathbb{C}}(M/\mathfrak{g}_{\mathrm{out}}^{\mathrm{orb}} M, \mathbb{C}). \tag{22}$$

In both cases $M := \bigotimes_{i=1}^L M_i$ and the module M_i is referred to as being inserted at the point Q_i.

4 Factorisation

The factorisation theorem, Theorem 4.2, says that $\mathrm{CC}_{\mathrm{trig}}$ (resp. $\mathrm{CB}_{\mathrm{trig}}$) is decomposed into a direct sum of those over $\mathrm{CC}_{\mathrm{orb}}$ (resp. $\mathrm{CB}_{\mathrm{orb}}$).

To state the theorem, we need the notion of the Verma module of $\hat{\mathfrak{g}}_\gamma$, i.e., $\hat{\mathfrak{g}}^{(0)}$ and $\hat{\mathfrak{g}}^{(\infty)}$. The Lie algebra $\hat{\mathfrak{g}}^{(0)}$ has a natural triangular decomposition:

$$\hat{\mathfrak{g}}^{(0)} = \mathfrak{n}_+^{(0)} \oplus \mathfrak{h}^{(0)} \oplus \mathfrak{n}_-^{(0)} \oplus \mathbb{C}\hat{k}, \tag{23}$$

where $\mathfrak{n}_+^{(0)}$ consists of $\mathfrak{g}$-valued power series in t with strictly positive powers while $\mathfrak{n}_-^{(0)}$ consists of series in t with negative powers. (See (17).) The component $\mathfrak{h}^{(0)}$ is the same as the Cartan subalgebra of $\mathfrak{g}$, which is spanned by $J_{0,b}$ ($b = 1, ..., N-1$). Since $\mathfrak{h}^{(0)}$ is canonically identified with the Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$, we often drop the superscript (0). The algebra $\hat{\mathfrak{g}}^{(\infty)}$ has the same decomposition but with opposite powers of t: $\mathfrak{n}_+^{(\infty)}$ is the subspace of $\mathfrak{g}^{(\infty)}$ with negative powers of t and $\mathfrak{n}_-^{(\infty)}$ is that with negative powers of t. To avoid repeating the same statements for $\hat{\mathfrak{g}}^{(0)}$ and $\hat{\mathfrak{g}}^{(\infty)}$, we identify them in the canonical way and denote the above decomposition as

$$\hat{\mathfrak{g}}_\gamma = \mathfrak{n}_{\gamma,+} \oplus \mathfrak{h}_\gamma \oplus \mathfrak{n}_{\gamma,-} \oplus \mathbb{C}\hat{k}, \tag{24}$$

and define $\hat{\mathfrak{b}}_\gamma := \mathfrak{n}_{\gamma,+} \oplus \mathfrak{h}_\gamma \oplus \mathbb{C}\hat{k}$.

Definition 4.1. Let λ is an element of $\mathfrak{h}_\gamma^* = \mathrm{Hom}_{\mathbb{C}}(\mathfrak{h}_\gamma, \mathbb{C})$ and k is an arbitrary complex number. The *Verma module* of $\hat{\mathfrak{g}}_\gamma$ of highest weight λ and level k is the $\hat{\mathfrak{g}}_\gamma$-module induced from the one-dimensional $\hat{\mathfrak{b}}_\gamma$-module:

$$M_\lambda := \mathrm{Ind}_{\hat{\mathfrak{b}}_\gamma}^{\hat{\mathfrak{g}}_\gamma} \mathbb{C}1_\lambda, \tag{25}$$

where 1_λ satisfies $\mathfrak{n}_{\gamma,+}1_\lambda = 0$, $H1_\lambda = \lambda(H)1_\lambda$ and $\hat{k}1_\lambda = k1_\lambda$. (See Ref. [3], (1.3).)

As the modules M_i inserted at Q_i, we take modules of the following type: for each $i = 1, \ldots, L$ there exists a finite dimensional $\mathfrak{g}$-module V_i which generates M_i over $\hat{\mathfrak{g}}^{Q_i}$ and $\hat{\mathfrak{g}}_+^{Q_i} V_i \subset V_i$.

Let us denote $V = V_1 \otimes \cdots \otimes V_L$ and the set of $\mathfrak{g}$-weights of V by $\mathrm{wt}(V)$. For $\lambda \in \mathfrak{h}^*$, we define

$$\begin{aligned} \tilde{\lambda} &:= -\lambda \circ (1 - \mathrm{Ad}\,\beta^{-1})^{-1} = \lambda \circ (1 - \mathrm{Ad}\,\beta)^{-1} \circ \mathrm{Ad}\,\beta, \\ \tilde{\lambda}' &:= -\lambda \circ (1 - \mathrm{Ad}\,\beta)^{-1}. \end{aligned} \tag{26}$$

They are well-defined since $1 - \mathrm{Ad}\,\beta^{\pm 1}$ is invertible on $\mathfrak{h} = \bigoplus_{b=1}^{N-1} \mathbb{C}J_{0,b}$.

Theorem 4.2. *Under the above assumption on M_i, $\mathrm{CC}_{\mathrm{trig}}(M)$ is finite dimensional and there is a linear isomorphism*

$$\mathrm{CC}_{\mathrm{trig}}(M) \cong \bigoplus_{\lambda \in \mathrm{wt}(V)} \mathrm{CC}_{\mathrm{orb}}(M_{\tilde{\lambda}} \otimes M \otimes M_{\tilde{\lambda}'}), \tag{27}$$

where $M_{\tilde{\lambda}}$ and $M_{\tilde{\lambda}'}$ are inserted at 0 and ∞ of E_{orb} respectively.

Example 4.3. The space of conformal coinvariants for Weyl modules and its factorisation is described as follows: Let V_i be a finite-dimensional $\mathfrak{g}$-module and regard them as $(\mathfrak{g}_+^{Q_i}\oplus\mathbb{C}\hat{k})$-module of level k as before. The *Weyl module* is the $\hat{\mathfrak{g}}^{Q_i}$-module induced up from V_i: $M(V_i) := \mathrm{Ind}_{\hat{\mathfrak{g}}_+^{Q_i}}^{\mathfrak{g}^{Q_i}\oplus\mathbb{C}\hat{k}} V_i$. We have

$$\mathrm{CC}_{\mathrm{trig}}(M(V)) \cong V, \qquad \mathrm{CC}_{\mathrm{orb}}(M_{\bar{\lambda}} \otimes M(V) \otimes M_{\bar{\lambda}'}) \cong V_\lambda. \tag{28}$$

Namely, the factorisation (27) for the Weyl modules $M_i = M(V_i)$ is the same as the weight space decomposition of V, $V = \bigoplus_{\lambda\in\mathrm{wt}(V)} V_\lambda$.

5 Locally freeness of the sheaf of conformal coinvariants

The singular curve E_{trig} appears when the modulus τ of an elliptic curve E_τ tends to $i\infty$ and accordingly the space of conformal coinvariants $\mathrm{CC}_{\mathrm{trig}}$ on E_{trig} is a "limit" of CC_{E_τ}.

The precise statement is as follows: Let $S = \{(q; Q_1, \dots, Q_L)\}$ be a base space of a family of L-pointed elliptic curves with a singular fibre at $q = 0$, i.e., when $q \neq 0$, the point $(q; Q_1, \dots, Q_L) \in S$ corresponds to the elliptic curve E_τ ($q = \exp(2\pi i\tau)$) and L points $Q_1, \dots, Q_L$ on it. When $q = 0$, $(q; Q_1, \dots, Q_L)$ corresponds to E_{trig} with L points $Q_1, \dots, Q_L$. We restrict ourselves to such a family that Q_i's do not collide with each other nor with the singular point of E_{trig}. Let M_i's be the $\hat{\mathfrak{g}}$ modules as in Theorem 4.2.

Theorem 5.1. *There is a finite rank locally free $\mathcal{O}_S$-sheaf $\mathcal{CC}(M)$ on S, the fibre of which over $(q; Q_1, \dots, Q_L)$ is $\mathrm{CC}_{E_\tau}(M)$ when $q = \exp(2\pi i\tau)$ and $\mathrm{CC}_{\mathrm{trig}}(M)$ when $q = 0$.*

This is the twisted version of the factorisation theorem in [5].

The locally freeness of the sheaf $\mathcal{CC}$ over $S \setminus \{q = 0\}$ is proved in [4]. In fact $\mathcal{CC}$ is first constructed as a coherent $\mathcal{O}_S$-sheaf. Then thanks to the existence of the elliptic KZ connection it is endowed with the D-module structure, which implies the locally freeness. The restriction of $\mathcal{CC}$ on the discriminant locus $S_0 = \{q = 0\} \subset S$ has the trigonometric KZ connection, hence is a locally free $\mathcal{O}_{S_0}$-sheaf.

When M_i's are the Weyl modules, Theorem 5.1 is easily proved. When M_i's are quotients of the Weyl modules, the proof is more involved. The idea of the proof is the same as that of [5]: use Theorem 4.2 and the completion of $\mathcal{CC}$ along S_0.

This proof says that the flat section of the trigonometric KZ connection is extended to that of the elliptic KZ connection. This fact might be a clue to description (e.g., integral representation) of solutions of the elliptic KZ equation even in the case for the Weyl modules.

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