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Affine Analogue of the Jack Polynomials and the Dunkl Operators

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1. Introduction

The Jack polynomials are a family of symmetric polynomials which can be formulated for every classical root system $A_n, B_n, C_n, \ldots$. They not only have many interesting combinatorial formulas **[S]** but also appear in various mathematical subjects: the theory of spherical functions **[He][HO][O93]**, the quantum many-body problem such as Calogero-Sutherland model **[vDV]**, the representation theory of the Virasoro algeba **[AMOS][MY]** and the homology of the Hilbert schemes **[N]**.

By considering the Jack polynomials in the framework of the representation theory of certain Hecke algebras, many formulas such as the inner product identity, can be proved in a unified fashion for all root systems **[O95]**. This point of view also gives us deep insight into other properties of the Jack polynomials such as the differential operators whose eigenfunctions are given by the Jack polynomials.

The Jack polynomials along with this Hecke algebraic structure admits a nice q-deformation, that is the theory of the celebrated Macdonald polynomials **[C][M]**.

The aim of our research is to construct a thoery of this kind to the affine analogue of the Jack and Macdonald polynomials introduced by Kirillov Jr. and Etingof **[EK]**. We will talk about some recent results in this direction.

2. Jack Polynomials

Since the time of the talk is limited, we summarize some facts about the Jack polynomials here for those who are not familiar with them. (For simplicity we do not consider them under full generality.)

2.1. Jack polynomials. Let $\mathfrak{h}, \mathfrak{h}^*$ be a vetor space over $\mathbb{C}$ and the dual respectively, $\langle\, , \rangle$ be the pairing, $R \subset \mathfrak{h}^*$ be a reduced irreducible root system of $r := \mathrm{rank}R = \dim \mathfrak{h}^*$, and $R^\vee$ be the coroot system. We use standard notations: $(\, ,)$ be the inner product of both $\mathfrak{h}$ and $\mathfrak{h}^*$, $\{\alpha_i | 1 \leq i \leq r\}$ be a set of simple roots, θ be the highest root, R_+ be the set of positive roots w.r.t. this $\{\alpha_i\}$, Q be the root lattice and $Q_+ = \{\sum c_i\alpha_i | c_i \in \mathbb{Z}_+\}$, $\alpha_i^\vee, Q^\vee, \cdots$ for those of $R^\vee$, $\lambda_i, \lambda_i^\vee$ be the fundamental weights and the fundamental coweights: $\langle \lambda_i, \alpha_j^\vee\rangle = \langle \lambda_i^\vee, \alpha_j\rangle = \delta_{i,j}$, $P, P^\vee$ and $\mathbb{C}[P], \mathbb{C}[P^\vee]$ be the weight and the coweight lattice and their group algebras, P_+ be the set of dominant weghts, W be the Weyl group, $\rho = \frac{1}{2}\sum_{\alpha \in R_+} \alpha$.

Let $k \in \mathbb{N}$ and introduce an operator $M_k \in \mathrm{End}(\mathbb{C}[P]^W)$

$$M_k = \Delta + 2k \sum_{\alpha \in R^+} \frac{1}{e^\alpha - 1}\partial_\alpha + 2k\partial_\rho$$

where the Laplacian Δ and derivatives $\partial_\beta\ (\beta \in \mathfrak{h}^*)$ are defined as

$$\Delta e^\lambda = (\lambda, \lambda)e^\lambda, \qquad \partial_\beta e^\lambda = (\beta, \lambda)e^\lambda \qquad \text{for } \lambda \in P.$$

Note that this operator is W-invariant

$$w \circ M_k \circ w^{-1} = M_k.$$

The set of monomial symmetric functions $\{\, m_\lambda = \sum_{\mu \in W\lambda} e^\mu \,|\, \lambda \in P_+ \,\}$ form a $\mathbb{C}$-basis of $\mathbb{C}[P]^W$ where $W\lambda$ is the W-orbit of λ. As generators, $\mathbb{C}[P]^W = \mathbb{C}[m_{\lambda_1}, \cdots, m_{\lambda_r}]$ and m_{λ_i} are algebraically independent.

The Jack polynomial $J_{\lambda,k}$ is labeled by a dominant weight $\lambda \in P_+$ and k. It is the unique element of $\mathbb{C}[P]^W$ satisfying the following conditions (Usually we fix k and write J_λ)

(1) $J_\lambda = m_\lambda + \sum_{\mu < \lambda} c_{\lambda,\mu} m_\mu \qquad c_{\lambda,\mu} \in \mathbb{C}$,

(2) $M_k J_\lambda = (\lambda, \lambda + 2k\rho)J_\lambda$.

Here $<$ is the dominance order: $\mu < \lambda \Leftrightarrow \lambda - \mu \in Q_+$.

2.2. Graded Hecke algebra. The Hecke algebra which provides a neat framework for the Jack polynomials is a graded Hecke algebra $\bar{\mathcal{H}}$. The algebra $\bar{\mathcal{H}}$ is the unique associative algebra over $\mathbb{C}$ with the following properties:

(1) As a $\mathbb{C}$-vector space, $\bar{\mathcal{H}} = S(\mathfrak{h}) \otimes \mathbb{C}[W]$.

(2) $S(\mathfrak{h}) \to \bar{\mathcal{H}},\ p \mapsto p \otimes e$ and $\mathbb{C}[W] \to \bar{\mathcal{H}},\ w \mapsto 1 \otimes w$ are algebra homomorphisims.

(3) $(p \otimes e) \cdot (1 \otimes w) = p \otimes w \quad$ for ${}^\forall p \in S(\mathfrak{h})\ {}^\forall w \in W$.

(4) $(1 \otimes s_i) \cdot (\xi \otimes e) - (s_i(\xi) \otimes e) \cdot (1 \otimes s_i) = k_i \langle \xi, \alpha_i \rangle$ for ${}^\forall \xi \in \mathfrak{h} \quad 0 \leq i \leq n$

The algebra $\bar{\mathcal{H}}$ acts on $\mathbb{C}[P]$. The action of $w \in W$ is the natural extension of the action on P. The crucial step is to find a realization of $\xi \in \mathfrak{h}$ in $\mathrm{End}(\mathbb{C}[P])$ and this is given by the so-called Dunkl operator

$$T_\xi = \partial_\xi + k \sum_{\alpha \in R_+} \langle \xi, \alpha \rangle \frac{1}{e^\alpha - 1}(1 - s_\alpha) + k\langle \rho, \xi \rangle.$$

Under this action of $\bar{\mathcal{H}}$, the group algebra $\mathbb{C}[P]$ is decomposed into

$$\mathbb{C}[P] = \bigoplus_{\lambda \in P_+} V_\lambda$$

as a $\bar{\mathcal{H}}$-module. The submodule V_λ is ispmophic to a certain induced module and there exists a positive definite Hermitian form on V_λ such that adjoint are given by $\xi^* = \bar{\xi}$ (complex conjugate) and $w^* = w^{-1}$.

2.3. Inner product formula. There is a Hermitian inner product $(\,,\,)_k$ also on $\mathbb{C}[P]$ which is defined as follows. Let T be the torus $\mathfrak{h}/Q^\vee$ and dt be the normalized Haar measure. We may regard each e^λ, $\lambda \in P$ as a character of T by the rule $e^\lambda(\xi) = e^{2\pi i \langle \lambda, \xi \rangle}$. For $f, g \in \mathbb{C}[P]$, the inner product is given by

$$(f, g)_k = \int_T f(t)\overline{g(t)}\delta_k(t)dt,$$

$$\delta_k = \prod_{\alpha \in R_+} \left| e^{\alpha/2} - e^{-\alpha/2} \right|^{2k_\alpha}.$$

It can be shown that this $(\,,\,)_k$ coincides with the inner product on V_λ and the structure of V_λ as a $\bar{\mathcal{H}}$-module enables us to calculate the following inner product formula

$$(1) \qquad (J_\lambda, J_\mu)_k = \delta_{\lambda,\mu} \prod_{i=1}^{k-1} \prod_{\alpha \in R_+} \frac{\langle \lambda + k\rho, \alpha^\vee \rangle + i}{\langle \lambda + k\rho, \alpha^\vee \rangle - i}.$$

2.4. Commuting operators. Let $\mathfrak{R}$ be the algebra generated by the elements $\frac{1}{e^\alpha - 1}$, $\alpha \in R_+$. The differential operators which commute with M_k, is W-invariant and has coefficients in $\mathfrak{R}$ form a commutative algebra isomorphic to the center $\mathcal{Z}(\bar{\mathcal{H}})$. It can be shown that $\mathcal{Z}(\bar{\mathcal{H}}) \simeq S(\mathfrak{h})^W$. Since $S(\mathfrak{h})^W$ is isomorphic to the polynomial ring with $\dim \mathfrak{h}$ generators, this gives a clear picture about the commuting operators and J_λ is the simultaneous eigenfunction.

3. Affine Jack polynomials

3.1. Affine Jack polynomials. Like the Jack polynomials, the affine Jack polynomials are defined for every root system of an untwisted affine Lie algebra $X_r^{(1)}$ where $X = A, B, C, D, G, F, E$. So let us fix an affine Lie algebra $X_r^{(1)}$. We use the notations in the previous section for the root system of X_r e.g., the Cartan subalgebra and the set of roots of X_r are $\mathfrak{h}$ and R respectively. For the affine case we adopt the following notations. Let $\hat{\mathfrak{h}} = \mathfrak{h} \oplus \mathbb{C}c \oplus \mathbb{C}d$ and $\hat{\mathfrak{h}}^* = \mathfrak{h}^* \oplus \mathbb{C}\delta \oplus \mathbb{C}\Lambda_0$ be the Cartan subalgebra of $X_r^{(1)}$ and its dual. The canonical pairing $\langle\,,\,\rangle$ between $\hat{\mathfrak{h}}$ and $\hat{\mathfrak{h}}^*$ is given by $\langle \Lambda_0, \mathfrak{h}\oplus\mathbb{C}d\rangle = \langle \delta, \mathfrak{h}\oplus\mathbb{C}c\rangle = 0, \langle \delta, d\rangle = 1, \langle \Lambda_0, c\rangle = 1$. Bilinear non-degenerate symmetric form on $\hat{\mathfrak{h}}^*$ is given by $(\Lambda_0, \delta) = 1, (\Lambda_0, \mathfrak{h}^*) = (\delta, \mathfrak{h}^*) = (\Lambda_0, \Lambda_0) = (\delta, \delta) = 0$ and on $\hat{\mathfrak{h}}$ by $(c, d) = 1, (c, \mathfrak{h}) = (d, \mathfrak{h}) = (c, c) = (d, d) = 0$. This pairing and the bilinear form are required to coincide with those of X_r on $\mathfrak{h}$ and $\mathfrak{h}^*$. Let $\widehat{R} = \{\hat{\alpha} = \alpha + n\delta \,|\, \alpha \in R, n \in \mathbb{Z} \text{ or } \alpha = 0, n \in \mathbb{Z} \setminus \{0\}\}$ be the affine root system. Fix the polarization by taking the set of positive roots as $\widehat{R}^+ = \{\hat{\alpha} = \alpha + n\delta \in \widehat{R} \,|\, n > 0 \text{ or } n = 0, \alpha \in R^+\}$. Simple roots are given by $\alpha_0 = -\theta + \delta, \alpha_1, \ldots, \alpha_r$. The affine root lattice and its positive part are $\hat{Q} = \bigoplus_{i=0}^r \mathbb{Z}\alpha_i$, $\hat{Q}^+ = \bigoplus_{i=0}^r \mathbb{Z}_+\alpha_i$. The affine weight lattice is given by $\widehat{P} = P \oplus \mathbb{Z}\delta \oplus \mathbb{Z}\Lambda_0 \subset \hat{\mathfrak{h}}^*$ and the set of dominant weights is $\widehat{P}^+ = \{\hat{\lambda} \in \hat{\mathfrak{h}}^* \,|\, \langle \hat{\lambda}, \alpha_i^\vee\rangle \in \mathbb{Z}_+, i = 0, \ldots, r\}$. For $K \in \mathbb{Z}$, we define $\widehat{P}_K^+ = \{\lambda + n\delta + K\Lambda_0 \,|\, \lambda \in P_K^+, n \in \mathbb{Z}\}$ where $P_K^+ = \{\lambda \in P^+ \,|\, (\lambda, \theta) \leq K\}$. The fundamental weights are denoted by Λ_i and $\langle \Lambda_i, \alpha_i^\vee\rangle = \delta_{i,j}$, $\langle \Lambda_i, \Lambda_0\rangle = 0$. The affine Weyl group is $W^a = W \ltimes Q^\vee$, the reflection w.r.t. $\alpha + n\delta$ is denoted by $s_{\alpha+n\delta}$ and the elements from $Q^\vee$-part are denoted by $t_{\alpha^\vee}$. The affine analogue of ρ is $\hat{\rho} = \sum_{0=1}^{N-1} \Lambda_i$.

As an affine analogue of the monomial symmetric function, we consider the orbit sum

$$m_{\hat{\lambda}} = \sum_{\hat{\mu} \in W^a \hat{\lambda}} e^{\hat{\mu}}.$$

Since the classical theta function associated to $\hat{\lambda} \in \widehat{P}_K$ is

$$\Theta_{\hat{\lambda}} = e^{-\frac{(\hat{\lambda}, \hat{\lambda})}{2K}\delta} \sum_{\alpha \in Q} e^{t_{\alpha^\vee}(\hat{\lambda})},$$

m_λ is essentially an W-invariant theta function. Set $p = e^{-\delta}$, let $K \in \mathbb{Z}_+$ and consider the space

$$A_K^{W^a} = \begin{cases} \mathbb{C}\left((p)\right) & K = 0 \\ \bigoplus_{\lambda \in P_K^+} \mathbb{C}\left((p)\right) m_{\lambda + K\Lambda_0} & K \in \mathbb{N}, \end{cases}$$

$$A^{W^a} = \bigoplus_{K \in \mathbb{Z}_+} A_K^{W^a}.$$

It will be seen that this is the affine analogue of the ring of the symmetric polynomial $\mathbb{C}[P]^W$. The affine analogue of $\mathbb{C}[P]$ which is A, should be some completion of $\mathbb{C}[\widehat{P}]$ but yet to be determined. (In [**EK**], the space A is considered but their A seems to big to us.)

Now we can introduce the affine Jack polynomials. Introduce an operator $\widehat{M}_k \in \mathrm{End}(A_K^{W^a})$

$$\widehat{M}_k = \widehat{\Delta} + 2k \sum_{\hat{\alpha} \in \widehat{R}^+} \frac{1}{e^{\hat{\alpha}} - 1} \partial_{\hat{\alpha}} + 2k \partial_{\hat{\rho}}$$

where

$$\widehat{\Delta} e^{\hat{\lambda}} = (\hat{\lambda}, \hat{\lambda}) e^{\hat{\lambda}}, \qquad \partial_{\hat{\xi}} e^{\hat{\lambda}} = (\hat{\xi}, \hat{\lambda}) e^{\hat{\lambda}} \qquad \text{for } \hat{\lambda} \in \widehat{P},\ \hat{\beta} \in \hat{\mathfrak{h}}^*.$$

Remark Consider

$$\begin{aligned} \widehat{L}_k &= \hat{\delta}^k \circ (\widehat{M}_k + k^2(\hat{\rho}, \hat{\rho})) \circ \hat{\delta}^{-k} \\ &= \widehat{\Delta} - k(k-1) \sum_{\substack{\alpha \in R_+ \\ n \in \mathbb{Z}}} \frac{p^n e^{\alpha}}{(1 - p^n e^{\alpha})^2} (\alpha, \alpha) \end{aligned}$$

where $\hat{\delta}$ is the affine Weyl denominator. Since $\widehat{\Delta} = \Delta + 2\partial_\delta \partial_{\Lambda_0}$, this $\widehat{L}_k$ is nothing but the Calogero-Sutherland model with elliptic potential, i.e., the Olshanetsky-Perelomov model on the space s.t. $\partial_\delta = 0$. Unfortunately because of the difference in the range of parameters, the following diagonalization of $\widehat{M}_k$ is not applicable to the diagonalization of the Olshanetsky-Perelomov model.

PROPOSITION 3.1.

(1) $\widehat{M}_k$ is $\widehat{W}$-invariant: $w \circ \widehat{M}_k \circ w^{-1} = \widehat{M}_k$.

(2) $\widehat{M}_k$ preserves $A_K^{W^a}$ and triangular w.r.t. the dominance order:

$$\widehat{M}_k m_{\hat{\lambda}} = (\hat{\lambda}, \hat{\lambda} + 2k\hat{\rho}) m_{\hat{\lambda}} + \sum_{\substack{\hat{\mu} < \hat{\lambda} \\ \hat{\mu} \in \widehat{P}_K^+}} c_{\hat{\lambda}\hat{\mu}} m_{\hat{\mu}}. \quad c_{\hat{\lambda}\hat{\mu}} \in \mathbb{C}.$$

Here the dominance order is defined as for the classical case: $\hat{\mu} < \hat{\lambda} \Leftrightarrow \hat{\lambda} - \hat{\mu} \in \widehat{Q}^+$.

The affine Jack polynomial $\widehat{J}_{\hat{\lambda},k}$ is a certain eigenfunction of this $\widehat{M}_k$ labeled by $\hat{\lambda} \in \widehat{P}^+$ and k. Again we usually fix k and write $\widehat{J}_{\hat{\lambda}}$.

DEFINITION 3.2. *The affine Jack polynomial $\widehat{J}_{\hat{\lambda}}$, $\hat{\lambda} \in \widehat{P}^+$ is the element of $A_K^{\widehat{W}}$ defined by the following conditions:*

(1) $\widehat{J}_{\hat{\lambda}} = m_{\hat{\lambda}} + \sum_{\hat{\mu} < \hat{\lambda}} c_{\hat{\lambda},\hat{\mu}} m_{\hat{\mu}}$,

(2) $\widehat{M}_k \widehat{J}_{\hat{\lambda}} = (\hat{\lambda}, \hat{\lambda} + 2k\hat{\rho}) \widehat{J}_{\hat{\lambda}}$.

For lower k, $\widehat{J}_{\hat{\lambda}}$ is something we are familiar with:

$$\widehat{J}_{\hat{\lambda}} = \begin{cases} m_{\hat{\lambda}} & k = 0 \\ \mathrm{ch}_{L_{\hat{\lambda}}} & k = 1 \end{cases}$$

where $\mathrm{ch}_{L_{\hat\lambda}}$ is the character of the irreducible highest weight representation of the highest weight $\hat\lambda$.
Remark The affine Jack polynomial is not a polynomial but a kind of theta function. We use this terminology following [**EK**].

To see analytic aspects of this theory, let us consider the domain $Y = \{\,\hat\xi \in \hat{\mathfrak{h}}_{\mathbb{C}} \,|\, \mathrm{Re}\langle\delta,\hat\xi\rangle > 0\,\}$ where $\hat{\mathfrak{h}}_{\mathbb{C}} = \mathbb{C}\otimes\hat{\mathfrak{h}}$. Then any monomial $e^{\hat\lambda}$ can be regarded as a function on this Y by $e^{\hat\lambda}(\hat\xi) = \exp\langle\hat\lambda,\hat\xi\rangle$. Under this identification, it is known that $m_{\hat\lambda}$ and $\mathrm{ch}_{L_{\hat\lambda}}$ converge to anaytic functions on Y. A merit of regarding $m_{\hat\lambda}$ and $\mathrm{ch}_{L_{\hat\lambda}}$ as functions on Y is that we can consider an action of $SL_2(\mathbb{Z})$ on them. Under a certain normaliztion, it can be shown that the linear span of $m_{\hat\lambda}$ or $\mathrm{ch}_{L_{\hat\lambda}}$ for $\hat\lambda \in \widehat{P}_K$ is invariant under the action of $SL_2(\mathbb{Z})$ [**KP**]. Since the set $\{\,\widehat{J}_{\hat\lambda} \,|\, \hat\lambda \in \widehat{P}_K\,\}$ is just another basis of this space, it gives rise to another representation of $SL_2(\mathbb{Z})$ which is a projective representation. For the detail, see [**EK**].

In [**Ha**], we studied various kinds of explicit expressions for the affine Jack polynomials for the case of $A_1^{(1)}$. Main results are: (1) An integral formula for $\widehat{J}_{\hat\lambda}$ in terms of an elliptic Selberg integral, (2) Explicit expression for the projective representation of $SL_2(\mathbb{Z})$, (3) Expressions for $\widehat{J}_{\hat\lambda}$ in terms of $\mathrm{ch}_{L_{\hat\lambda}}$ and the Dedekind's η-fucntion for $K = 1, 2$.

3.2. What to be done. One of the aims of our research is to study the affine Jack polynomials from the representation theory of a certain affine Hecke algebra and to give the inner product formula as was done to the Jack polynomials. Since the extension of the Jack polynomials to the affine case is so straightforward, we expect that the relevant Hecke algebra $\mathcal{H}$ would be the one defined similarly as $\bar{\mathcal{H}}$: as a $\mathbb{C}$-vector space $\mathcal{H} = S(\hat{\mathfrak{h}})\otimes\mathbb{C}[W^a]$ and so on. The first step is to construct the Dunkl operators but at the moment of writing this article we have only partial results. Hopefully, this will be finished until the symposium!

It is not difficult to speculate the inner product formula. A naive guess from (1) is

$$(\widehat{J}_{\hat\lambda}, \widehat{J}_{\hat\mu})_k = \prod_{i=1}^{k-1}\prod_{\hat\alpha\in\widehat{R}_+} \frac{\langle\hat\lambda + k\hat\rho, \hat\alpha^\vee\rangle + i}{\langle\hat\lambda + k\hat\rho, \hat\alpha^\vee\rangle - i}$$

but the right hand side apparently diverges. Using

$$\frac{1}{\Gamma(z)} = ze^{\gamma}\prod_{n=1}^{\infty}\left(1+\frac{z}{n}\right)e^{-z/n},$$

we can renormalize this divergence by hand as

$$(2)\qquad (\widehat{J}_{\hat\lambda}, \widehat{J}_{\hat\mu})_k = \delta_{\hat\lambda,\hat\mu}(J_\lambda, J_\lambda)_k \prod_{i=1}^{k-1}\left(\frac{\Gamma\left(-\frac{i}{\kappa}\right)}{\Gamma\left(\frac{i}{\kappa}\right)}\right)^r$$

$$\times \prod_{i=1}^{k-1} \prod_{\alpha\in R_+} \frac{\Gamma\left(\frac{\langle\lambda+k\rho,\alpha^\vee\rangle-i}{\kappa}\right)\Gamma\left(\frac{-\langle\lambda+k\rho,\alpha^\vee\rangle-i}{\kappa}\right)}{\Gamma\left(\frac{\langle\lambda+k\rho,\alpha^\vee\rangle+i}{\kappa}\right)\Gamma\left(\frac{-\langle\lambda+k\rho,\alpha^\vee\rangle+i}{\kappa}\right)} \tag{3}$$

where $\kappa = \langle\hat{\lambda}, c\rangle + kh^\vee$.

3.3. Relation to the KZB heat equation. The KZB (Knizhnik-Zamolodchikov-Bernard) equation is the equation satisfied by the correlation functions of the conformal field theory on a torus, which is a generalization of the KZ-equation of the confomal field theory on $\mathbb{C}P^1$. And the KZB heat equation is the equation which is compatible with the KZB equation and contains differentiation w.r.t. the elliptic modulus. It is known that the general solution to this KZB system is given by the trace of certain intertwiners of $A_n^{(1)}$-modules ,i.e., vertex operators.

When the root system is of type $X = A$, there is an interesting connection to the KZB heat equation [**EK**]. In this case $\widehat{J}_{\hat{\lambda}}$ equals to the trace of a vertex operator divided by $\hat{\delta}^{k-1}$ hence essentially the simplest solution to the KZB system. The equation (2) of Definiteion 3.2 is equivalent to the KZB heat equation. (In this single vertex operator case, the KZB equation is trivial.)

This viewpoint gives us some knowledge about the inner product. In [**Ki**], the inner product on the space of conformal blocks is studied. It is shown that the conjecture for the inner product in 3.2 is valid up to constant for the case of $A_1^{(1)}$. Also shown is the existence and $SL_2(\mathbb{Z})$-invariance of the inner product for $A_n^{(1)}$.

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