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Local cohomology modules of the Fourier transform of A-hypergeometric systems of Cohen-Macaulay type

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1 Introduction

Let $A = \{\boldsymbol{a}_1 \dots \boldsymbol{a}_n\} \in \mathbb{Z}^d$ be integral vectors and $\boldsymbol{\beta} \in K^n$. To these data, Gel'fand, Zelevinskii, and Kapranov [9] associate a system of partial differential equations (denote by $M_A(\boldsymbol{\beta})$) called an A-hypergeometric system. They determined the characteristic cycles [9], and proved the irreducibility of the monodromy representation for nonresonant parameters [8] in the Cohen-Macaulay case; Adolphson [1] proved that an A-hypergeometric system is holonomic, and the holonomic rank of an A-hypergeometric system equals the volume of the convex hull of $\boldsymbol{a}_1 \dots \boldsymbol{a}_n \in \mathbb{Z}^d$ in the semi-nonresonant case ; Hotta [10] proved that an A-hypergeometric system is a regular homogeneous holonomic system under the homogenuity condition (see [1]); Saito [14] and Saito-Traves [15] classified the parameters according to the D-isomophism classes of their corresponding A-hypergeometric systems in combinatorial terms; and many authers studied other D-invariants of the A-hypergeometric systems.

We easily see that the Fourier transform of an A-hypergeometric system $\mathcal{F}(M_A(\boldsymbol{\beta}))$ has support in the affine toric variety determined by A. In general, an algebraic torus canonically acts on an affine toric variety. The orbits of the action can be described in the faces of the convex (rational) polyhedral cone $\mathbb{Q}_{\geq 0}A$ generated by A, and there exists a one-to-one correspondece between the orbits and the faces (see [7]). Furthermore, it is known that the solution sheaf of $\mathcal{F}(M_A(\boldsymbol{\beta}))$ has an algebraic stratification composed of the orbits (see [9],[10]). In this talk, we examine the structure of the algebraic local cohomology modules of $\mathcal{F}(M_A(\boldsymbol{\beta}))$ with respect to each orbit (or face) for an arbitrary parameter $\boldsymbol{\beta}$ in the Cohen-Macaulay case. As a result, we see that the local cohomology module with respect to a face σ is isomorphic to a direct sum of the external tensor products of the D-module generated by the Dirac's delta functions and the algebraic local cohomology modules of $\mathcal{F}(M_{A\cap\sigma}(\boldsymbol{\beta}-\boldsymbol{\lambda}))$ for some shifted parameters $\boldsymbol{\beta}-\boldsymbol{\lambda}$. We see that, for each face, the structure is completely controled by the equivalence class of the lattice $\mathbb{Z}A$ generated by A determined by the face. As an application of this result, we expect that we can analize the structure of the solution sheaf of $\mathcal{F}(M_A(\boldsymbol{\beta}))$. Moreover, Hotta and Kashiwara [11] gave a relation between the solution of a homogeneous coherent D-module and that of its Fourier transform. We say that A satisfies the homogenuity condition if there exists an integaral vector $\boldsymbol{c}\in\mathbb{Z}^d$ such that

$$\boldsymbol{c}A = {}^t(1,1,\cdots,1). \tag{1}$$

It is well-known that $M_A(\boldsymbol{\beta})$ is a homogeneous regular holonomic D-module under the homogenuity condition (see [10]). Hence we expect that we can also give an approach to analyzing the structure of the solution sheaf of the original system $M_A(\boldsymbol{\beta})$ and its monodromy representations in detail.

2 Preliminaries

2.1 A-hypergeometric system $M_A(\boldsymbol{\beta})$

Let K be a field of characteristic zero, and let

$$D = K\langle \boldsymbol{x},\boldsymbol{\partial}\rangle = K\langle x_1,\ldots,x_n,\partial_1,\ldots,\partial_n\rangle$$

be the n-th Weyl algebra over K. Let $A=(\boldsymbol{a}_1\ldots\boldsymbol{a}_n)=(a_{ij})$ be a $d\times n$ integer matrix of rank d. We denote by $I_A(\boldsymbol{\partial})$ the toric ideal in $K[\boldsymbol{\partial}]$, that is

$$I_A(\boldsymbol{\partial}) := \langle \boldsymbol{\partial}^{\boldsymbol{u}}-\boldsymbol{\partial}^{\boldsymbol{v}} \mid A\boldsymbol{u}=A\boldsymbol{v}, \boldsymbol{u},\boldsymbol{v}\in\mathbb{N}^n\rangle \subset K[\boldsymbol{\partial}].$$

Let $\boldsymbol{\beta} = {}^t(\beta_1,\ldots,\beta_d)\in K^d$ be a parameter (column) vector. We denote by $H_A(\boldsymbol{\beta})$ the left ideal of D generated by $I_A(\boldsymbol{\partial})\cup\{\sum_{j=1}^n a_{ij}x_j\partial_j-\beta_i \mid i=1,\ldots,d\}$, and call the quotient module $M_A(\boldsymbol{\beta}) := D/H_A(\boldsymbol{\beta})$ the A-hypergeometric system with parameter $\boldsymbol{\beta}$. Adolphson [1] proved that $M_A(\boldsymbol{\beta})$ is a holonomic D-module.

We consider the Fourier transform of $M_A(\boldsymbol{\beta})$ (denote by $\mathcal{F}M_A(\boldsymbol{\beta})$), that is

$$\begin{aligned}\mathcal{F}M_A(\boldsymbol{\beta}) &= D/\mathcal{F}^{-1}H_A(\boldsymbol{\beta})\\ &= D/D\langle I_A(\boldsymbol{x}), \textstyle\sum_{j=1}^n a_{ij}\partial_j x_j+\beta_i \mid i=1,\ldots,d\rangle,\end{aligned}$$

where $\mathcal{F}$ is the automorphism of D defined by $\mathcal{F}(x_j) = \partial_j, \mathcal{F}(\partial_j) = -x_j$, and $I_A(\boldsymbol{x}) := \mathcal{F}^{-1} I_A(\boldsymbol{\partial})$. It is well-known that a module M over the Weyl algebra is holonomic if and only if $\mathcal{F}M$ is holonomic, and homogeneous regular holonomic if and only if $\mathcal{F}M$ is homogeneous regular holonomic. Here we say that a coherent D-module is homogeneous, if for any $m \in M$, $K[\sum_{j=1}^n x_j \partial_j]m$ is a finite dimensional K-vector space (see [10], [11]). Moreover, it is clear that $\mathcal{F}M_A(\boldsymbol{\beta})$ has support in the affine toric variety:

$$\mathrm{V}(I_A(\boldsymbol{x})) := \{\boldsymbol{x} \in k^n | \ f(\boldsymbol{x}) = 0, \ \text{ for all } f \in I_A(\boldsymbol{x})\}.$$

Namely,

$$\mathrm{Supp}(\mathcal{F}M_A(\boldsymbol{\beta})) \subset \mathrm{V}(I_A(\boldsymbol{x})). \tag{2}$$

2.2 Affine semigroup algebra $K[\mathbb{N}A]$

Let us recall affine semigroup algebras. We denote by A the set $\{\boldsymbol{a}_1, \dots, \boldsymbol{a}_n\}$ as well, and by $\mathbb{N}A$ the affine semigroup generated by A. Let $K[\mathbb{N}A]$ be the affine semigroup algebra of $\mathbb{N}A$, that is

$$K[\mathbb{N}A] = \bigoplus_{\boldsymbol{\lambda} \in \mathbb{N}A} K\boldsymbol{t}^{\boldsymbol{\lambda}} \subset K[t_1^{\pm}, \dots, t_d^{\pm}]$$

as K-vector spaces, where $\boldsymbol{t}^{\boldsymbol{\lambda}} := t_1^{\lambda_1} \cdots t_d^{\lambda_d}$, and the multiplication of $K[\mathbb{N}A]$ is defined by $\boldsymbol{t}^{\boldsymbol{\lambda}} \cdot \boldsymbol{t}^{\boldsymbol{\lambda}'} := \boldsymbol{t}^{\boldsymbol{\lambda}+\boldsymbol{\lambda}'}$. We recall that the toric ideal $I_A(\boldsymbol{x})$ is nothing but the kernel of a K-algebra epimorphism

$$\phi_A : K[\boldsymbol{x}] \longrightarrow K[\mathbb{N}A] \ , \ \phi_A(x_j) := \boldsymbol{t}^{\boldsymbol{a}_j}$$

We regard $K[\mathbb{N}A]$ as a $K[\boldsymbol{x}]$-module through the map ϕ_A, that is $f \cdot g := \phi_A(f)g \ \ (f \in K[\boldsymbol{x}], g \in K[\mathbb{N}A])$.

Next we recall a localization of $K[\mathbb{N}A]$. We denote by $\mathbb{Q}_{\geq 0}A$ the cone $\{\sum_{j=1}^n c_j \boldsymbol{a}_j \mid c_j \in \mathbb{Q}_{\geq 0}A\}$. Let $\mathcal{S}_A$ be the set of faces of $\mathbb{Q}_{\geq 0}A$, $\mathbb{Z}A$ the group generated by A, and KA the K-vector space generated by A. For a face $\sigma \in \mathcal{S}_A$, we denote by $\mathbb{Z}(A \cap \sigma)$ the $\mathbb{Z}$-submodule of $\mathbb{Z}A$ generated by $A \cap \sigma$, and $K(A \cap \sigma)$ the K-subspace of KA generated by $A \cap \sigma$. We agree that $\mathbb{Z}(A \cap \sigma) = K(A \cap \sigma) = 0$ when $A \cap \sigma = \varnothing$. For each $\sigma \in \mathcal{S}_A$, we consider the multiplicatively closed sets S_σ defined by $S_\sigma := \{\boldsymbol{t}^{\boldsymbol{\lambda}} \mid \boldsymbol{\lambda} \in \mathbb{N}(A \cap \sigma)\}$ and its preimage $T_\sigma := \phi_A^{-1}(S_\sigma)$. We denote by $K[\mathbb{N}A]_\sigma$ (resp. $K[\boldsymbol{x}]_\sigma$) the localization of $K[\mathbb{N}A]$ (resp. $K[\boldsymbol{x}]$) with respect to S_σ (resp. T_σ). Let $\overline{\phi_{A,\sigma}}$ be the natural morphism induced by ϕ_A. Then we have

$$\begin{aligned} K[\boldsymbol{x}]_\sigma / K[\boldsymbol{x}]_\sigma I_A(\boldsymbol{x}) \overset{\overline{\phi_A}}{\simeq} K[\mathbb{N}A]_\sigma &= K[\mathbb{N}A + \mathbb{Z}(A \cap \sigma)] \\ &= \bigoplus_{\boldsymbol{\lambda} \in \mathbb{N}A + \mathbb{Z}(A \cap \sigma)} K\boldsymbol{t}^{\boldsymbol{\lambda}}. \end{aligned} \tag{3}$$

2.3 Orbits of a canonical action of an algebraic torus on an affine toric variety

To discuss algebraic local cohomology modules of $\mathcal{F}M_A(\boldsymbol{\beta})$, we recall the orbits of the canonical action of the algebraic torus $(K^\times)^d$, defined by

$$(K^\times)^d \times \mathrm{V}(I_A(\boldsymbol{x})) \ni (\boldsymbol{t}, x_1, \dots, x_n) \mapsto (\boldsymbol{t}^{\boldsymbol{a}_1} x_1, \dots, \boldsymbol{t}^{\boldsymbol{a}_n} x_n) \in \mathrm{V}(I_A(\boldsymbol{x})). \quad (4)$$

The orbits of the action are the very sets that we consider. In this subsection, we briefly recall a relation between the faces of $\mathbb{Q}_{\geq 0}A$ and the orbits (cf. [7], [10]).

For $\sigma \in \mathcal{S}_A$, we denote by I_σ the subset $\{j \mid \boldsymbol{a}_j \in \sigma\}$ of $\{1, \dots, n\}$, and put

$$X_\sigma := \{\boldsymbol{x} \in K^n \mid x_j \neq 0 \ (j \in I_\sigma), x_j = 0 \ (j \notin I_\sigma)\} \cap \mathrm{V}(I_A(\boldsymbol{x})). \quad (5)$$

It is well-known that $\{X_\sigma \mid \sigma \in \mathcal{S}_A\}$ is the sets of the orbits. Namely X_σ is invariant under the action of $(K^\times)^d$ for each $\sigma \in \mathcal{S}_A$, and

$$\mathrm{V}(I_A(\boldsymbol{x})) = \coprod_{\sigma \in \mathcal{S}_A} X_\sigma, \text{disjoint union.}$$

Moreover, there exists an order preserving one to one correspondence among $\mathcal{S}_A$, $\{I_\sigma \mid \sigma \in \mathcal{S}_A\}$, and $\{\overline{X_\sigma} \mid \sigma \in \mathcal{S}_A\}$, where $\overline{X_\sigma}$ is the Zariski closure of X_σ in $\mathrm{V}(I_A(\boldsymbol{x}))$. Namely,

$$\begin{array}{ccccc} \mathcal{S}_A & \leftrightarrow & \{I_\sigma \mid \sigma \in \mathcal{S}_A\} & \leftrightarrow & \{\overline{X_\sigma} \mid \sigma \in \mathcal{S}_A\}, \\ \sigma & \longleftrightarrow & I_\sigma & \longleftrightarrow & \overline{X_\sigma} \end{array}$$

2.4 Algebraic local cohomology and Čech complex

We consider the algebraic local cohomology modules with respect to each of the orbits described in Subsection 2.3. In [[5], Chapter 6], the argument and results of the local cohomology modules are limited to the minimum face $\{0\}$, provided that $\mathbb{Q}_{\geq 0}A$ is a strongly convex cone. Thus we improve some results of the local cohomology modules in [[5], Chapter 6] for any face. We fix a face $\sigma \in \mathcal{S}_A$. Put

$$L_\sigma^p := \bigoplus_{\substack{\sigma \prec \tau \in \mathcal{S}_A \\ \dim \tau - \dim \sigma = p}} K[\boldsymbol{x}]_\tau \quad (p = 0, \dots, d_\sigma := \dim \mathbb{Q}_{\geq 0}A - \dim \sigma),$$

and define $f^p : L_\sigma^p \to L_\sigma^{p+1}$ by specifying its component $f_{\tau',\tau} : K[\boldsymbol{x}]_{\tau'} \to K[\boldsymbol{x}]_\tau$ to be $\begin{cases} 0 & \text{if } \tau' \not\prec \tau \\ \varepsilon(\tau', \tau)\mathrm{nat} & \text{if } \tau' \prec \tau, \end{cases}$ where ε is an incidence function on $\mathcal{S}_A$ and nat is the natural inclusion. It is clear that

$$\boldsymbol{L}_\sigma^\bullet : 0 \longrightarrow L_\sigma^0 \xrightarrow{f^0} L_\sigma^1 \xrightarrow{f^1} \cdots \xrightarrow{f^{d_\sigma -1}} L_\sigma^{d_\sigma} \longrightarrow 0 \tag{6}$$

is a complex as in [[5], Section 6.2].

We generalize the statement [[5], Theorem 6.2.5].

Theorem 2.1 *Let M be a $K[\boldsymbol{x}]$-module with support in $\mathrm{V}(I_A(\boldsymbol{x}))$. Then, for all $\sigma \in \mathcal{S}_A$ and for all p, we have*

$$H^p_{[X_\sigma]}(M) \simeq H^p(\boldsymbol{L}_\sigma^\bullet \otimes_{K[\boldsymbol{x}]} M). \tag{7}$$

Furthermore, if M has a D-module structure, then the tensor product of the D-modules L_σ^p and M has the natural D-module structure, and (7) holds as D-modules.$\square$

We call the complex $\boldsymbol{L}_\sigma^\bullet \otimes_{K[\boldsymbol{x}]} M$ the Čech complex of M associated to $\sigma \in \mathcal{S}_A$.

Corollary 2.2 *For all $\sigma \in \mathcal{S}_A$ and all k, we have*

$$H^k_{[X_\sigma]}(\mathcal{F}M_A(\boldsymbol{\beta})) \simeq H^k(\boldsymbol{L}_\sigma^\bullet \otimes_{K[\boldsymbol{x}]} \mathcal{F}M_A(\boldsymbol{\beta})) \tag{8}$$

as D-modules.$\square$

3 Main theorem

Assume A to be Cohen-Macaulay, that is, $K[\mathbb{N}A]$ is a Cohen-Macaulay algebra, and fix a face $\sigma \in \mathcal{S}_A$. We compute the algebraic local cohomology modules $H^k_{[X_\sigma]}(\mathcal{F}M_A(\boldsymbol{\beta}))$. First we consider the following set:

$$P_\sigma := \mathbb{Z}A \setminus \bigcup_{\tau \succ \sigma, facet} (\mathbb{N}A + \mathbb{Z}(A \cap \tau)).$$

Then P_σ plays an important part for the structure of the local cohomology in fact. Note that, for $\boldsymbol{\lambda} \in P_\sigma$, $r_\sigma := \mathrm{rank}_K(A \cap \sigma) \leqq \mathrm{rank}_K([A \cap \sigma; \boldsymbol{\beta} - \boldsymbol{\lambda}])$, where $[A \cap \sigma; \boldsymbol{\beta} - \boldsymbol{\lambda}]$ means the enlarged coefficient matrix of $A \cap \sigma$ with $\boldsymbol{\beta} - \boldsymbol{\lambda}$. The vanishing of the local cohomology completely depends on whether or not r_σ is equal to $\mathrm{rank}_K([A \cap \sigma; \boldsymbol{\beta} - \boldsymbol{\lambda}])$. Second it is clear that, for any $\tau \succ \sigma$, $\mathbb{N}A + \mathbb{Z}(A \cap \tau)$ contains $\mathbb{Z}(A \cap \sigma)$. Hence we may define a relation $\sim$ on P_σ by

$$\boldsymbol{\lambda} \sim \boldsymbol{\lambda}' \iff \boldsymbol{\lambda} - \boldsymbol{\lambda}' \in \mathbb{Z}(A \cap \sigma).$$

It is clear that this relation $\sim$ is an equivalence relation. We denote the coset by $\widetilde{E}_\sigma$, and fix a system of representatives $\{\boldsymbol{\lambda}\}$ of $\widetilde{E}_\sigma$. Let $\widetilde{E}_\sigma(\boldsymbol{\beta};\{\boldsymbol{\lambda}\}) := \{\boldsymbol{\lambda} \mid \mathrm{rank}(\sigma;\boldsymbol{\beta}-\boldsymbol{\lambda}) = r_\sigma\}$. To simplify the description, we denote $K\langle x_j, \partial_j | j \in I_\sigma\rangle$ and $K\langle x_j, \partial_j | j \notin I_\sigma\rangle$ by $D_{(\sigma)}$ and $D_{(\sigma^c)}$ respectively. Similarly, we denote $K[x_j | j \in I_\sigma]$ and $K[x_j | j \notin I_\sigma]$ by $K[\boldsymbol{x}_\sigma]$ and $K[\boldsymbol{x}_{\sigma^c}]$ respectively. Recall that we can identify D (resp. $K[\boldsymbol{x}]$) with the external product $D_{(\sigma^c)} \boxtimes_K D_{(\sigma)}$ (resp. $K[\boldsymbol{x}_{\sigma^c}] \boxtimes_K K[\boldsymbol{x}_\sigma]$). Under these identifications, we obtain the following theorem.

Theorem 3.1 *Assume that A is Cohen-Macaulay, and fix a system of representatives $\{\boldsymbol{\lambda}\}$ of $\widetilde{E}_\sigma$. Then we have*

$$
\begin{aligned}
&H^k_{[X_\sigma]}(\mathcal{F}M_A(\boldsymbol{\beta})) \\
&\quad \simeq \bigoplus_{\boldsymbol{\lambda} \in \widetilde{E}_\sigma(\boldsymbol{\beta};\{\boldsymbol{\lambda}\})} \frac{D_{(\sigma^c)}}{D_{(\sigma^c)}\langle \boldsymbol{x}_{\sigma^c}\rangle} \boxtimes (\Gamma_{[X_{\mathbf{Q}_{\geqq 0}(A\cap\sigma)}]}(\mathcal{F}M_{A\cap\sigma}(\boldsymbol{\beta}-\boldsymbol{\lambda}))^{\oplus\binom{d_\sigma}{d_\sigma-k}}. \square
\end{aligned} \tag{9}
$$

We easily see that, if $\boldsymbol{\lambda} \simeq \boldsymbol{\lambda}'$, then we have

$$
\begin{aligned}
\mathrm{rank}_K(A\cap\sigma;\boldsymbol{\beta}-\boldsymbol{\lambda}) &= \mathrm{rank}_K(A\cap\sigma;\boldsymbol{\beta}-\boldsymbol{\lambda}'), \\
\Gamma_{[X_{\mathbf{Q}_{\geqq 0}(A\cap\sigma)}]}(\mathcal{F}M_{A\cap\sigma}(\boldsymbol{\beta}-\boldsymbol{\lambda})) &\simeq \Gamma_{[X_{\mathbf{Q}_{\geqq 0}(A\cap\sigma)}]}(\mathcal{F}M_{A\cap\sigma}(\boldsymbol{\beta}-\boldsymbol{\lambda}')).
\end{aligned}
$$

Hence, it is clear that the statement (9) is independent of the choice of representative for Λ up to isomorphism. Thus we write $\Gamma_{[X_{\mathbf{Q}_{\geq 0}(A\cap\sigma)}]}(\mathcal{F}M_{A\cap\sigma}(\boldsymbol{\beta}-\Lambda))$, $\mathrm{rank}(\sigma;\boldsymbol{\beta}-\Lambda)$, and $\widetilde{E}_\sigma(\boldsymbol{\beta})$, instead of $\Gamma_{[X_{\mathbf{Q}_{\geq 0}(A\cap\sigma)}]}(\mathcal{F}M_{A\cap\sigma}(\boldsymbol{\beta}-\boldsymbol{\lambda}))$, $\mathrm{rank}_K(A\cap\sigma;\boldsymbol{\beta}-\boldsymbol{\lambda})$, and $\widetilde{E}_\sigma(\boldsymbol{\beta};\{\boldsymbol{\lambda}\})$ respectively. We finally obtain the main theorem.

Theorem 3.2 (Main theorem) *Assume that A is Cohen-Macaulay. For all k and all $\sigma \in \mathcal{S}_A$, we have*

$$
\begin{aligned}
&H^k_{[X_\sigma]}(\mathcal{F}M_A(\boldsymbol{\beta})) \\
&\quad \simeq \bigoplus_{\Lambda \in \widetilde{E}_\sigma(\boldsymbol{\beta})} \frac{D_{(\sigma^c)}}{D_{(\sigma^c)}\langle \boldsymbol{x}_{\sigma^c}\rangle} \boxtimes (\Gamma_{[X_{\mathbf{Q}_{\geqq 0}(A\cap\sigma)}]}(\mathcal{F}M_{A\cap\sigma}(\boldsymbol{\beta}-\Lambda)))^{\oplus\binom{d_\sigma}{d_\sigma-k}}.
\end{aligned} \tag{10}
$$

In particular, $\widetilde{E}_\sigma(\boldsymbol{\beta}) = \varnothing$ if and only if $R\Gamma_{[X_\sigma]}(\mathcal{F}M_A(\boldsymbol{\beta})) = 0$. $\square$

4 Examples

We provide some examples of Cohen-Macaulay type for computation of the algebraic local cohomology modules.

Example 1 Let $d=1$ and $A=(1)$. Then we easily see the following:

$$\mathbb{Q}_{\geq 0}A=\{p\in\mathbb{Q}|\ p\geq 0\},\ \mathcal{S}_A=\{\mathbb{Q}_{\geq 0},\{0\}\},\ I_A(x)=0,\ \mathrm{V}(I_A(x))=K.$$

Note that the orbits X_σ corresponding to $\sigma\in\mathcal{S}_A$ are given by;

$$X_{\{0\}}=\{0\},X_{\mathbb{Q}_{\geq 0}}=K^\times=\{x\in K|\ x\neq 0\}.$$

We examine $\widetilde{E}_\sigma(\boldsymbol{\beta})$ corresponding to $\sigma\in\mathcal{S}_A$. First, for $\sigma=\{0\}$, we have

$$P_{\{0\}}=\mathbb{Z}\setminus\mathbb{N},\ \widetilde{E}_{\{0\}}=\mathbb{Z}\setminus\mathbb{N}.$$

Note that $r_{\{0\}}$ is equal to 0. Hence we have

$$\widetilde{E}_{\{0\}}(\beta)=\begin{cases}\{\beta\} & \beta\in\mathbb{Z}\setminus\mathbb{N},\\ \varnothing & \text{otherwise.}\end{cases}$$

Second, for $\mathbb{Q}_{\geq 0}$, we have

$$P_{\mathbb{Q}_{\geq 0}}=\mathbb{Z},\ \widetilde{E}_{\mathbb{Q}_{\geq 0}}=\{0\}.$$

Note that $r_{\mathbb{Q}_{\geq 0}}$ is equal to d. Hence we have $\widetilde{E}_{\mathbb{Q}_{\geq 0}}(\beta)=\{0\}$ for an arbitrary parameter β.

The following table summalizes the results of the calculation.

$\mathcal{S}(A)$	$\{0\}$	$\mathbb{Q}_{\geq 0}$
orbits $X_{(\cdot)}$	$\{0\}$	$\{x\neq 0\}$
$\widetilde{E}_{(\cdot)}\neq\varnothing$	$\beta\in\mathbb{Z}\setminus\mathbb{N}$	arbitrary

Example 2 $A=\begin{pmatrix}1&1&1\\0&1&2\end{pmatrix}$

$\mathcal{S}(A)$	$\{0\}$	$\mathbb{Q}_{\geq 0}\begin{pmatrix}1\\0\end{pmatrix}$
orbits $X_{(\cdot)}$	$\{0\}$	$\{x_1\neq 0,x_2=x_3=0\}$
$\widetilde{E}_{(\cdot)}\neq\varnothing$	$2\beta_1-\beta_2,\beta_2\in\mathbb{Z}\setminus\mathbb{N}$	$\beta_2\in\mathbb{Z}\setminus\mathbb{N}$
$\mathcal{S}(A)$	$\mathbb{Q}_{\geq 0}\begin{pmatrix}1\\2\end{pmatrix}$	$\mathbb{Q}_{\geq 0}A$
orbits $X_{(\cdot)}$	$\{x_1=x_2=0,x_3\neq 0\}$	$\{x_2^2-x_1x_3=0,x_1x_2x_3\neq 0\}$
$\widetilde{E}_{(\cdot)}\neq\varnothing$	$2\beta_1-\beta_2\in\mathbb{Z}\setminus\mathbb{N}$	arbitrary

Note that

$$|\widetilde{E}_{\mathbb{Q}_{\geqq 0}{}^t(1,0)}(\boldsymbol{\beta})| = 1 \iff \beta_2 = -1, -2, \cdots,$$
$$|\widetilde{E}_{\mathbb{Q}_{\geqq 0}{}^t(1,2)}(\boldsymbol{\beta})| = 1 \iff 2\beta_1 - \beta_2 = -1, -2, \cdots,$$
$$|\widetilde{E}_{\{0\}}(\boldsymbol{\beta})| = 1 \iff 2\beta_1 - \beta_2, \beta_2 = -1, -2, \cdots.$$

Example 3 $A = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$: not homogeneous.

$\mathcal{S}(A)$	$\{0\}$	$\mathbb{Q}_{\geqq 0}\begin{pmatrix} 2 \\ 0 \end{pmatrix}$
orbits $X_{(\cdot)}$	$\{0\}$	$\{x_1 \neq 0, x_2 = x_3 = 0\}$
$\widetilde{E}_{(\cdot)} \neq \varnothing$	$\beta_1, \beta_2 \in \mathbb{Z} \setminus \mathbb{N}$	$\beta_2 \in \mathbb{Z} \setminus \mathbb{N}_{>0}$
$\mathcal{S}(A)$	$\mathbb{Q}_{\geqq 0}\begin{pmatrix} 0 \\ 1 \end{pmatrix}$	$\mathbb{Q}_{\geqq 0}A$
orbits $X_{(\cdot)}$	$\{x_1 = x_2 = 0, x_3 \neq 0\}$	$\{x_2^2 - x_1 x_3^2 = 0, x_1 x_2 x_3 \neq 0\}$
$\widetilde{E}_{(\cdot)} \neq \varnothing$	$\beta_1 \in \mathbb{Z} \setminus \mathbb{N}$	arbitrary

Note that

$$|\widetilde{E}_{\mathbb{Q}_{\geq 0}{}^t(2,0)}(\boldsymbol{\beta})| = 1 \iff \beta_2 = 0,$$
$$|\widetilde{E}_{\mathbb{Q}_{\geq 0}{}^t(2,0)}(\boldsymbol{\beta})| = 2 \iff \beta_2 = -1, -2, \cdots,$$
$$|\widetilde{E}_{\mathbb{Q}_{\geq 0}{}^t(0,1)}(\boldsymbol{\beta})| = 1 \iff \beta_1 = -1, -2, \cdots,$$
$$|\widetilde{E}_{\{0\}}(\boldsymbol{\beta})| = 2 \iff \beta_1, \beta_2 = -1, -2, \cdots.$$

Example 4 $A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -1 \end{pmatrix}$

Put $a_1 = {}^t(1,0,0), a_2 = {}^t(0,1,0), a_3 = {}^t(0,0,1), a_4 = {}^t(0,1,-1)$.

$\mathcal{S}(A)$	$\{0\}$	$\mathbb{Q}_{\geq 0}a_1$
orbits $X_{(\cdot)}$	$\{0\}$	$\{x_1 \neq 0, x_2 = x_3 = x_4 = 0\}$
$\widetilde{E}_{(\cdot)} \neq \varnothing$	$\beta_1, \beta_2, \beta_1 + \beta_2, \beta_2 + \beta_3 \in \mathbb{Z} \setminus \mathbb{N}$	$\beta_2, \beta_2 + \beta_3 \in \mathbb{Z} \setminus \mathbb{N}$
$\mathcal{S}(A)$	$\mathbb{Q}_{\geq 0}a_2$	$\mathbb{Q}_{\geq 0}a_3$
orbits $X_{(\cdot)}$	$\{x_2 \neq 0, x_1 = x_3 = x_4 = 0\}$	$\{x_3 \neq 0, x_1 = x_2 = x_4 = 0\}$
$\widetilde{E}_{(\cdot)} \neq \varnothing$	$\beta_1, \beta_1 + \beta_3 \in \mathbb{Z} \setminus \mathbb{N}$	$\beta_1, \beta_2 \in \mathbb{Z} \setminus \mathbb{N}$
$\mathcal{S}(A)$	$\mathbb{Q}_{\geq 0}a_4$	$\mathbb{Q}_{\geq 0}(a_1, a_3)$
orbits $X_{(\cdot)}$	$\{x_4 \neq 0, x_1 = x_2 = x_3 = 0\}$	$\{x_1x_3 \neq 0, x_2 = x_4 = 0\}$
$\widetilde{E}_{(\cdot)} \neq \varnothing$	$\beta_1 + \beta_3, \beta_2 + \beta_3 \in \mathbb{Z} \setminus \mathbb{N}$	$\beta_2 \in \mathbb{Z} \setminus \mathbb{N}$
$\mathcal{S}(A)$	$\mathbb{Q}_{\geq 0}(a_2, a_3)$	$\mathbb{Q}_{\geq 0}(a_2, a_4)$
orbits $X_{(\cdot)}$	$\{x_2x_3 \neq 0, x_1 = x_4 = 0\}$	$\{x_2x_4 \neq 0, x_1 = x_3 = 0\}$
$\widetilde{E}_{(\cdot)} \neq \varnothing$	$\beta_1 \in \mathbb{Z} \setminus \mathbb{N}$	$\beta_1 + \beta_3 \in \mathbb{Z} \setminus \mathbb{N}$
$\mathcal{S}(A)$	$\mathbb{Q}_{\geq 0}(a_1, a_4)$	$\mathbb{Q}_{\geq 0}A$
orbits $X_{(\cdot)}$	$\{x_1x_4 \neq 0, x_2 = x_3 = 0\}$	$\{x_1x_2x_3x_4 \neq 0, x_1x_2 - x_3x_4 = 0\}$
$\widetilde{E}_{(\cdot)} \neq \varnothing$	$\beta_2 + \beta_3 \in \mathbb{Z} \setminus \mathbb{N}$	arbitrary

Note that for any $\sigma \in \mathcal{S}(A)$, $\widetilde{E}_{(\sigma)} \neq \varnothing$ implies $|\widetilde{E}_\sigma(\boldsymbol{\beta})| = 1$.

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