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Vandermonde-type determinants and Laplacians of categories

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Abstract

In this note we prove a certain minor determinant formula of the Laplacians $\Delta_{\mathbf{K}}$ of categories, especially the category **Ab** of abelian groups. In order to achieve this, we establish Vandermonde-type determinant formulas whose entries are indexed by partitions.

1 Introduction

Let $\mathbf{K}$ be a given category. We denote by $\mathrm{Ob}(\mathbf{K})$ the set of objects of $\mathbf{K}$, $\mathrm{Mor}(\mathbf{K})$ the set of all morphisms in $\mathbf{K}$ and $\mathrm{Mor}_{\mathbf{K}}(X,Y)$ the set of all morphisms from X to Y. Analogous to the Laplacian (or the adjacency matrix) of an oriented graph, the **Laplacian** $\Delta_{\mathbf{K}}$ of a given category $\mathbf{K}$ is introduced in [KuST] by

$$\Delta_{\mathbf{K}} := (\#\mathrm{Mor}_{\mathbf{K}}(X,Y))_{X,Y\in \mathrm{Ob}_o(\mathbf{K})} \tag{1.1}$$

where $\mathrm{Ob}_o(\mathbf{K})$ denotes the subset of $\mathrm{Ob}(\mathbf{K})$ such that the cardinality $\#\mathrm{Mor}_{\mathbf{K}}(X,Y)$ is finite for every pair (X,Y) of elements in $\mathrm{Ob}_o(\mathbf{K})$.

Our main interest lies on the analysis of the spectral zeta function

$$\zeta_{\Delta_{\mathbf{K}}}(s) := \sum_{\lambda\in \mathrm{Spec}(\Delta_{\mathbf{K}})} \lambda^{-s} \tag{1.2}$$

of the Laplacian $\Delta_{\mathbf{K}}$. (In [Ki], we discuss the spectral zeta function for the cases of $\mathbf{K} = \mathbf{PB}, \mathrm{Mod}(\mathbb{F}_q)$.) The following problem is fundamental.

Problem 1.1 ([KuST, KuW]). Is the Laplacian $\Delta_{\mathbf{K}}$ of a given 'nice' category $\mathbf{K}$ **positive**?

In [Ki] we show that the Laplacian $\Delta_{\mathbf{K}}$ is positive if $\mathbf{K}$ is a **totally ordered category**, in which case representation theory of ordered categories works effectively, by proving that every principal minor determinant of $\Delta_{\mathbf{K}}$ is positive.

In this note we study the following problem which states more accurate information on the principal minor determinants of the Laplacian $\Delta_{\mathbf{K}}$.

Problem 1.2. Let $\mathbf{K}$ be a category. Assume that $\mathrm{Ob}(\mathbf{K})$ is partially ordered, say $\mathrm{Ob}(\mathbf{K}) = \{X_\lambda \mid \lambda \in \Sigma\}$ for some partially ordered set Σ, and there exists an injection from X_α to X_β if $\alpha \leq \beta$. When the equality

$$\det \Delta_{\mathbf{K}}(\lambda) = \prod_{\nu \leq \lambda} \# \mathrm{Aut}_{\mathbf{K}} X_\nu \tag{1.3}$$

holds?

We remark that the equality (1.3) readily implies the positivity of the Laplacian $\Delta_{\mathbf{K}}$ if $\mathrm{Ob}(\mathbf{K})$ is **totally ordered.**

The purpose of this note is to prove the equality (1.3) for several concrete categories, especially the category $\mathbf{Ab}$ consisting of abelian groups. This result does not imply the positivity of the Laplacian $\Delta_{\mathbf{Ab}}$, but it is regarded as a weaker version of positivity.

In order to prove (1.3) for $\mathbf{Ab}$, we establish more general identity which is thought to be a generalization of the Vandermonde determinant formula (Theorem 3.3). This formula has distinguished interest of its own right. The desired equality (1.3) is obtained by a specialization of variables.

2 Examples of totally ordered cases

In this section we give two examples of totally ordered categories.

2.1 The category $\mathrm{Mod}(\mathbb{F}_q)$ of $\mathbb{F}_q$-modules

Fix a prime power q. Denote by $\mathbb{F}_q$ the finite field consisting of q elements. Let $\mathrm{Mod}(\mathbb{F}_q)$ be the category of $\mathbb{F}_q$-modules. An object of $\mathrm{Mod}(\mathbb{F}_q)$ is a n-dimensional vector space $\mathbb{F}_q^n$ over the finite field $\mathbb{F}_q$, and morphisms are $\mathbb{F}_q$-linear operators. The group $\mathrm{Aut}_{\mathrm{Mod}(\mathbb{F}_q)} \mathbb{F}_q^k$ is nothing but the general linear group $GL_k(\mathbb{F}_q)$. The number of morphisms from $\mathbb{F}_q^m$ to $\mathbb{F}_q^n$ is given by q^{mn}. We put $\Delta_{\mathrm{Mod}(\mathbb{F}_q)}(N) = (q^{ij})_{0 \leq i,j \leq N}$, the principal N-minor of the Laplacian Δ.

Proposition 2.1. *The category* $\mathrm{Mod}(\mathbb{F}_q)$ *satisfies the equality* (1.3).

Proof. In fact, we have

$$\begin{aligned} \det \Delta_{\mathrm{Mod}(\mathbb{F}_q)}(N) = \det(q^{ij})_{0 \leq i,j \leq N} &= \prod_{0 \leq i < j \leq N} (q^j - q^i) \\ &= \prod_{k=0}^{N} \prod_{j=1}^{k} (q^k - q^{k-j}) = \prod_{k=0}^{N} \# GL_k(\mathbb{F}_q). \end{aligned}$$

$\square$

Remark 2.1. We see that **Set** = "Mod($\mathbb{F}_1$)", the category of all sets and all maps, also satisfies the equality (1.3). The proof is almost same as above (q^i is replaced by i). Note that Mod($\mathbb{F}_q$) is (totally) ordered category and the Laplacian is symmetric, but **Set** is not.

2.2 The category PB of partial bijections

Let **PB** be the category of partial bijections, which is originally introduced to study the representation of full symmetric group $\mathfrak{S}_\omega$ of all bijections on $\mathbb{Z}_+ = \{1, 2, 3, \dots\}$ (see [Ne]). Let us recall the definition of **PB**. An object in **PB** is a finite set $[n] = \{1, 2, \dots, n\}$. A morphism from $[m]$ to $[n]$ is given by a **partial bijection**, that is, the triplet $(\varphi, D_\varphi, R_\varphi)$, where $D_\varphi \subset [m]$ and $R_\varphi \subset [n]$ have the same cardinality and $\varphi : D_\varphi \to R_\varphi$ is a bijection. For given two morphisms $\varphi : [l] \to [m]$ and $\psi : [m] \to [n]$, the composition $\psi\varphi : [l] \to [n]$ of φ and ψ is defined to be a partial bijection from $D_{\psi\varphi} := \varphi^{-1}(R_\varphi \cap D_\psi)$ to $R_{\psi\varphi} := \psi(R_\varphi \cap D_\psi)$. We note that **PB** belongs to the class of **totally ordered categories** (see [Ne]).

The group $\mathrm{Aut}_{\mathbf{PB}}\,[n]$ is nothing but the symmetric group $\mathfrak{S}_n$ of degree n. It is easy to check that the number of morphisms are given by

$$d_{mn} := \#\mathrm{Mor}_{\mathbf{PB}}([m], [n]) = \sum_{k=0}^{\min(m,n)} \binom{m}{k}\binom{n}{k} k!. \tag{2.1}$$

Denote by $\Delta_{\mathbf{PB}}(N) = (d_{mn})_{0 \le m,n \le N}$ the principal N-minor determinant of the Laplacian $\Delta_{\mathbf{PB}}$.

Proposition 2.2. *The category* **PB** *satisfies the equality* (1.3).

Proof. Notice that $\Delta_{\mathbf{PB}}(N)$ is decomposed into the product of three matrices as

$$\Delta_{\mathbf{PB}}(N) = \left(\binom{i}{j}\right)_{0 \le i,j \le N} \mathrm{diag}(0!, 1!, \dots, N!) \left(\binom{j}{i}\right)_{0 \le i,j \le N},$$

where $\binom{i}{j}$ is the binomial coefficient. Here we regard $\binom{i}{j} = 0$ if $i < j$. Since the first (resp. the third) matrix in the right hand side is lower (resp. upper) unitriangular, we have

$$\det \Delta_{\mathbf{PB}}(N) = \det \mathrm{diag}(0!, 1!, \dots, N!) = \prod_{k=0}^{N} k!,$$

which is the desired equality. $\square$

Remark 2.2. Here we give a counter example of (1.3). Let us take a subcategory $\mathbf{PB}(M)$ of **PB** whose objects are given by $\mathrm{Ob}(\mathbf{PB}(M)) = \{[Mk] \mid k = 0, 1, 2, \dots\}$. Then $\mathbf{PB}(M)$ is still a totally ordered category but its Laplacian $\Delta_{\mathbf{PB}(M)}$ does not satisfy (1.3). For instance, if we take $M = 2$ and $N = 1$, then we see that

$$\det \Delta_{\mathbf{PB}(2)}(1) = 4 \neq 2 = \#\mathrm{Aut}_{\mathbf{PB}(2)}\,[0] \#\mathrm{Aut}_{\mathbf{PB}(2)}\,[2].$$

3 The category Ab of abelian groups

Let **Ab** be the category of abelian groups and group homomorphisms. Ob(**Ab**) is partially ordered by inclusion $\leq$. (We denote by $H \leq G$ if H is a subgroup of G.) For a given finite abelian group G, the principal G-minor of $\Delta_{\mathbf{Ab}}$ is given by $\Delta_{\mathbf{Ab}}(G) := (\#\mathrm{Hom}(H, K))_{H,K \leq G}$, where H and K runs over all subgroups of G up to isomorphism. The purpose of this section is to prove the equality (1.3) for the category **Ab**:

Theorem 3.1. *The category* **Ab** *satisfies the equality* (1.3). *Namely, we have*

$$\det \Delta_{\mathbf{Ab}}(G) = \prod_{H \leq G} \#\mathrm{Aut}\, H \tag{3.1}$$

for any finite abelian group G.

Remark 3.1. Since Ob(**Ab**) is **not** totally ordered, the theorem above does not imply immediately that **every** principal minor of $\Delta_{\mathbf{Ab}}$ is positive.

We prove Theorem 3.1 in the following subsections.

3.1 Reduction to p-group cases

Let us denote by $C(n) := \mathbb{Z}/n\mathbb{Z}$ the cyclic group of order n. For two abelian groups $G_1 = \prod_{i=1}^{r} C(m_i)$ and $G_2 = \prod_{j=1}^{s} C(n_j)$, the number $\#\mathrm{Hom}(G_1, G_2)$ of homomorphisms from G_1 to G_2 is given by

$$\#\mathrm{Hom}(G_1, G_2) = \prod_{i=1}^{r} \prod_{j=1}^{s} (m_i, n_j),$$

where (a, b) is the greatest common divisor of a and b. First we see that we can divide the discussion into p-components.

Lemma 3.2. *Let $G = \prod_{p:\mathrm{prime}} G_p$ be the direct product decomposition into Sylow p-groups G_p. Then we have*

$$\Delta_{\mathbf{Ab}}(G) = \bigotimes_{p:\mathrm{prime}} \Delta_{\mathbf{Ab}}(G_p). \tag{3.2}$$

Here $\otimes$ denotes the Kronecker product of matrices.

Proof. For given finite abelian groups G_1 and G_2, we decompose them into cyclic groups as

$$G_1 = \prod_{p:\mathrm{prime}} \prod_{i} C(p^{\lambda_i(p)}), \quad G_2 = \prod_{q:\mathrm{prime}} \prod_{j} C(q^{\mu_j(q)}).$$

Then the number of homomorphisms from G_1 to G_2 is given by

$$\begin{aligned}\#\mathrm{Hom}(G_1, G_2) &= \prod_{p,q:\text{prime}} \prod_{i,j} (p^{\lambda_i(p)}, q^{\mu_j(q)}) = \prod_{p:\text{prime}} \prod_{i,j} (p^{\lambda_i(p)}, p^{\mu_j(p)}) \\ &= \prod_{p:\text{prime}} \#\mathrm{Hom}((G_1)_p, (G_2)_p),\end{aligned}$$

which clearly means the desired conclusion. □

By virtue of this lemma, we have only to check the validity of (3.1) when G is a finite abelian p-group for each prime number p.

3.2 Vandermonde-type identities

We set several conventions on partitions which are necessary in below. A finite sequence $\lambda = (\lambda_i)_{i=1}^r$ of positive integers is called a partition of length r if it is weakly decreasing. We denote by $l(\lambda)$ the length of λ. The empty sequence $\emptyset$ is regarded as a partition of length 0. For a given partition $\lambda = (\lambda_i)_{i=1}^r$, we put $m_k(\lambda) := \#\{i \mid \lambda_i = k\}$, $\alpha(\lambda) := \#\{i \mid \lambda_i = \lambda_1\}$, $|\lambda| := \sum_{i=1}^r \lambda_i$, and $n(\lambda) := \sum_{i=1}^r (i-1)\lambda_i$. We notice that $l(\lambda) = \sum_{k\geq 1} m_k(\lambda)$. We also regard λ as a weakly increasing function on $\mathbb{R}$ by

$$\lambda(t) := \sum_{i=1}^r \min\{t, \lambda_i\}.$$

We notice that $\emptyset(t) \equiv 0$. We also remark that $\lambda(\lambda_1) = |\lambda|$.

When two partitions $\lambda = (\lambda_i)_{i=1}^r$ and $\mu = (\mu_j)_{j=1}^s$ are given, we construct a sequence $\mu(\lambda)$ of non-negative integers of length r by

$$\mu(\lambda) := (\mu(\lambda_i))_{i=1}^r.$$

By definition we see that $\emptyset(\lambda) = (0^r) = (0, \ldots, 0)$ and $\mu(\emptyset) = \emptyset$.

Let $\lambda = (\lambda_i)_{i=1}^r$ and $\mu = (\mu_j)_{j=1}^s$ be given partitions. We define $\lambda \subset \mu$ if and only if $r \leq s$ and $\lambda_k \leq \mu_k$ for every $k \leq r$. When $\lambda \subset \mu$, we say λ is a subpartition of μ.

Let $\{x_0, x_1, x_2, \ldots\}$ be a countable set of commuting variables. For a given finite sequence $\lambda = (\lambda_i)_{i=1}^r$ of nonnegative integers, we put $x^\lambda := \prod_{i=1}^r x_{\lambda_i} = \prod_{k\geq 0} x_k^{m_k(\lambda)}$. We also put $x^\emptyset := 1$. We introduce a polynomial function

$$\mathcal{A}_\lambda(x) := \prod_{k\geq 1} \prod_{j=1}^{m_k(\lambda(\lambda))} (x_k - x_{k-j})$$

associated with a partition λ. It is easy to see that $\mathcal{A}_\lambda(x)$ is a homogeneous polynomial of degree $l(\lambda)$.

Example 3.1. If $\lambda = (3,3,2,1,1)$, then $|\lambda| = 10$, $\alpha(\lambda) = 2$, $\lambda(\lambda) = (10,10,8,5,5)$ and $\mathcal{A}_\lambda(x) = (x_5 - x_4)(x_5 - x_3)(x_8 - x_7)(x_{10} - x_9)(x_{10} - x_8)$.

We prove the following Vandermonde-type determinant formula.

Theorem 3.3. *We have*

$$\det(x^{\mu(\lambda)})_{\lambda,\mu\subset\nu} = \prod_{\lambda\subset\nu} \mathcal{A}_\lambda(x) \tag{3.3}$$

for any given partition ν.

We give the proof of the theorem above in the next section.

Remark 3.2. (1) This Vandermonde formula (3.3) is essentially due to [BCR]. The author would thank Sho Matsumoto for telling him this fact.
(2) The formula (3.3) seems to hold even if we replace the order $\subset$ by the lexicographic order or the dominance order of partitions. We notice that the validity of (3.3) with respect to the lexicographic order implies the positivity of the Laplacian $\Delta_{\mathbf{Ab}}$.

Example 3.2. We see two extreme cases of $\nu = (n)$ and $\nu = (1^n)$. When $\nu = (n)$, then the subpartitions of ν are given by $\emptyset$ and (k) for $1 \le k \le n$. Since $(j)((i)) = (\min\{i,j\})$ for $1 \le i,j \le n$, the equality (3.3) reads

$$\det(x^{(j)((i))})_{0\le i,j\le n} = \det\begin{pmatrix} 1 & 1 & 1 & \dots & 1 & 1 \\ x_0 & x_1 & x_1 & \dots & x_1 & x_1 \\ x_0 & x_1 & x_2 & \dots & x_2 & x_2 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ x_0 & x_1 & x_2 & \dots & x_{n-1} & x_{n-1} \\ x_0 & x_1 & x_2 & \dots & x_{n-1} & x_n \end{pmatrix} = \prod_{k=1}^{n}(x_k - x_{k-1}).$$

When $\nu = (1^n)$, then the subpartitions of ν are given by $\emptyset$ and (1^k) for $1 \le k \le n$. Since $(1^j)((1^i)) = (j^i)$ for $1 \le i,j \le n$, the equality (3.3) reads

$$\det(x^{(1^j)((1^i))})_{0\le i,j\le n} = \det(x_j^i)_{0\le i,j\le n} = \prod_{0\le i<j\le n}(x_j - x_i),$$

which is nothing but the Vandermonde determinant formula.

3.3 Application of the determinant formula

Every finite abelian p-group can be written in the form $G_\lambda := \prod_{i=1}^r C(p^{\lambda_i})$ for some partition λ, which is said to be of type λ. It is clear that $G_\lambda \subset G_\mu$ if and only if $\lambda \subset \mu$. The number of homomorphisms from G_λ to G_μ is given by

$$\#\mathrm{Hom}(G_\lambda, G_\mu) = \prod_{i=1}^{r}\prod_{j=1}^{s} p^{\min\{\lambda_i,\mu_j\}} = \prod_{i=1}^{r} p^{\mu(\lambda_i)}.$$

The cardinality of the group $\operatorname{Aut} G_\lambda$ for a finite abelian p-group G_λ of type λ is known as follows (see, e.g., [TH]).

Lemma 3.4. *Let $\lambda = (\lambda_i)_{i=1}^r$ be a partition and let G_λ be a finite abelian p-group of type λ. Then*

$$\#\operatorname{Aut} G_\lambda = p^{|\lambda|+2n(\lambda)} \prod_{k\geq 1} \prod_{j=1}^{m_k(\lambda)} (1-p^{-j}). \tag{3.4}$$

If we put $x_i = p^i$, then it is immediate to check that $x^{\mu(\lambda)} = \#\operatorname{Hom}(G_\lambda, G_\mu)$. Moreover we have the

Lemma 3.5. *We have*

$$\#\operatorname{Aut} G_\lambda = \mathcal{A}_\lambda(1, p, p^2, \ldots). \tag{3.5}$$

Proof. Notice that the equality

$$|\lambda| + 2n(\lambda) = |\lambda(\lambda)| = \sum_{k\geq 1} k m_k(\lambda(\lambda))$$

holds for a given diagram $\lambda = (\lambda_1, \ldots, \lambda_{l(\lambda)})$. We have then

$$\begin{aligned}\#\operatorname{Aut} G_\lambda &= \prod_{k\geq 1} \prod_{j=1}^{m_k(\lambda(\lambda))} p^k \times \prod_{k\geq 1} \prod_{j=1}^{m_k(\lambda(\lambda))} (1-p^{-j}) \\ &= \prod_{k\geq 1} \prod_{j=1}^{m_k(\lambda(\lambda))} (p^k - p^{k-j}) = \mathcal{A}_\lambda(1, p, p^2, \ldots)\end{aligned}$$

as we desired. □

Hence we have

$$\Delta_{\mathbf{Ab}}(G) = \prod_{H\leq G} \#\operatorname{Aut} H \tag{3.6}$$

for any finite abelian p-group G. Eventually, this equality and Lemma 3.2 imply Theorem 3.1 as follows.

Proof of Theorem 3.1. Let G be a finite abelian group. For every prime number p, we denote by G_p the Sylow p-subgroup of G, and by n_p the number of subgroups of G_p. We put $N := \prod_{p:\text{prime}} n_p$, which is a finite product.

For any subgroup K of G_p, the number of subgroups H of G such that its Sylow p-subgroup H_p is given by K is N/n_p. Thus we have

$$\begin{aligned}\prod_{H \leq G} \#\mathrm{Aut}\, H &= \prod_{H \leq G} \prod_{p:\text{prime}} \#\mathrm{Aut}\, H_p = \prod_{p:\text{prime}} \prod_{H_p \leq G_p} (\#\mathrm{Aut}\, H_p)^{N/n_p} \\ &= \prod_{p:\text{prime}} \det \Delta_{\mathbf{Ab}}(G_p)^{N/n_p} = \det \left(\bigotimes_{p:\text{prime}} \Delta_{\mathbf{Ab}}(G_p) \right) \\ &= \det \Delta_{\mathbf{Ab}}(G).\end{aligned}$$

This completes the proof of the theorem. □

Remark 3.3. The category $\mathbf{Gr}$ of groups and group homomorphisms is not an ordered category, and its Laplacian $\Delta_{\mathbf{Gr}}$ is no longer symmetric. However, the equality (1.3) seems to be valid for $\mathbf{Gr}$. Here we present the first non-trivial example of (1.3). Let us take the symmetric group $\mathfrak{S}_3$ of degree 3. The subgroups of $\mathfrak{S}_3$ (up to isomorphism) is given by $C(1)$, $C(2)$, $C(3)$ and $\mathfrak{S}_3$. The $\mathfrak{S}_3$-principal minor $\Delta_{\mathbf{Gr}}(\mathfrak{S}_3)$ of the Laplacian $\Delta_{\mathbf{Gr}}$ is given by

$$\Delta_{\mathbf{Gr}}(\mathfrak{S}_3) = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 1 & 4 \\ 1 & 1 & 3 & 2 \\ 1 & 2 & 1 & 10 \end{pmatrix},$$

and we directly check that

$$\det \Delta_{\mathbf{Gr}}(\mathfrak{S}_3) = 12 = \#\mathrm{Aut}\, C(1) \#\mathrm{Aut}\, C(2) \#\mathrm{Aut}\, C(3) \#\mathrm{Aut}\, \mathfrak{S}_3.$$

3.4 Example

In stead of giving the proof of the determinant formula (3.3), we present an example of a calculation of the determinant in (3.3) for the sake of intuitive understanding for the proof of the formula.

Let us observe the case where $\nu = (2, 2, 1)$. We arrange the entries of $(x^{\mu(\lambda)})$ according to the order

$$\emptyset, (1), (1, 1), (1, 1, 1), (2), (2, 1), (2, 1, 1), (2, 2), (2, 2, 1).$$

The corresponding polynomials $\mathcal{A}_\lambda(x)$ are given as follows.

λ	$\mathcal{A}_\lambda(x)$
$\emptyset$	1
(1)	$(x_1 - x_0)$
$(1,1)$	$(x_2 - x_0)(x_1 - x_0)$
$(1,1,1)$	$(x_3 - x_0)(x_3 - x_1)(x_3 - x_2)$
(2)	$(x_2 - x_1)$
$(2,1)$	$(x_3 - x_2)(x_2 - x_1)$
$(2,1,1)$	$(x_4 - x_3)(x_3 - x_1)(x_3 - x_2)$
$(2,2)$	$(x_4 - x_2)(x_4 - x_3)$
$(2,2,1)$	$(x_5 - x_3)(x_5 - x_4)(x_3 - x_2)$

The determinant in (3.3) is written as

$$\det\left(\begin{array}{c|ccc|ccccc}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ \hline
x_0 & x_1 & x_2 & x_3 & x_1 & x_2 & x_3 & x_2 & x_3 \\
x_0^2 & x_1^2 & x_2^2 & x_3^2 & x_1^2 & x_2^2 & x_3^2 & x_2^2 & x_3^2 \\
x_0^3 & x_1^3 & x_2^3 & x_3^3 & x_1^3 & x_2^3 & x_3^3 & x_2^3 & x_3^3 \\ \hline
x_0 & x_1 & x_2 & x_3 & x_2 & x_3 & x_4 & x_4 & x_5 \\
x_0^2 & x_1^2 & x_2^2 & x_3^2 & x_2x_1 & x_3x_2 & x_4x_3 & x_4x_2 & x_5x_3 \\
x_0^3 & x_1^3 & x_2^3 & x_3^3 & x_2x_1^2 & x_3x_2^2 & x_4x_3^2 & x_4x_2^2 & x_5x_3^2 \\
x_0^2 & x_1^2 & x_2^2 & x_3^2 & x_2^2 & x_3^2 & x_4^2 & x_4^2 & x_5^2 \\
x_0^3 & x_1^3 & x_2^3 & x_3^3 & x_2^2x_1 & x_3^2x_2 & x_4^2x_3 & x_4^2x_2 & x_5^2x_3
\end{array}\right)$$

where the vertical and horizontal lines indicate the blockwise decomposition according to the first row. We see that if we subtract μ^C-column from μ-column for every non-empty $\mu \subset \nu$, then the blocks in the upper triangular area vanish as follows:

$$\det\left(\begin{array}{c|ccc|cccc}
1 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\ \hline
x_0 & x_1 - x_0 & x_2 - x_0 & x_3 - x_0 & 0 & 0 & \cdots & 0 \\
x_0^2 & x_1^2 - x_0^2 & x_2^2 - x_0^2 & x_3^2 - x_0^2 & 0 & 0 & \cdots & 0 \\
x_0^3 & x_1^3 - x_0^3 & x_2^3 - x_0^3 & x_3^3 - x_0^3 & 0 & 0 & \cdots & 0 \\ \hline
x_0 & x_1 - x_0 & x_2 - x_0 & x_3 - x_0 & x_2 - x_1 & x_3 - x_2 & \cdots & x_5 - x_3 \\
x_0^2 & x_1^2 - x_0^2 & x_2^2 - x_0^2 & x_3^2 - x_0^2 & x_2x_1 - x_1^2 & x_3x_2 - x_2^2 & \cdots & x_5x_3 - x_3^2 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
x_0^3 & x_1^3 - x_0^3 & x_2^3 - x_0^3 & x_3^3 - x_0^3 & x_2^2x_1 - x_1^3 & x_3^2x_2 - x_2^3 & \cdots & x_5^2x_3 - x_3^3
\end{array}\right).$$

Thus we see that the determinant $\det(x^{\mu(\lambda)})_{\lambda,\mu\subset\nu}$ decomposes into the product of two determinants $\det(x^{\mu(\lambda)} - x^{\mu^C(\lambda)})_{\lambda_1=\mu_1=1}$ and $\det(x^{\mu(\lambda)} - x^{\mu^C(\lambda)})_{\lambda_1=\mu_1=2}$ (determinants of the $(1,1)$- and $(2,2)$-blocks). It is easily seen that the first determinant $\det(x^{\mu(\lambda)} - x^{\mu^C(\lambda)})_{\lambda_1=\mu_1=1}$ is the usual Vandermonde determinant, which is also written as

$$(x_1 - x_0)(x_2 - x_1)(x_2 - x_0)(x_3 - x_2)(x_3 - x_1)(x_3 - x_0) = \prod_{\substack{\lambda\subset\nu \\ \lambda_1=1}} \mathcal{A}_\lambda(x).$$

The second determinant is calculated as

$$
\begin{aligned}
&\det\begin{pmatrix}
x_2-x_1 & x_3-x_2 & x_4-x_3 & x_4-x_2 & x_5-x_3\\
x_2x_1-x_1^2 & x_3x_2-x_2^2 & x_4x_3-x_3^2 & x_4x_2-x_2^2 & x_5x_3-x_3^2\\
x_2x_1^2-x_1^3 & x_3x_2^2-x_2^3 & x_4x_3^2-x_3^3 & x_4x_2^2-x_2^3 & x_5x_3^2-x_3^3\\
x_2^2-x_1^2 & x_3^2-x_2^2 & x_4^2-x_3^2 & x_4^2-x_2^2 & x_5^2-x_3^2\\
x_2^2x_1-x_1^3 & x_3^2x_2-x_2^3 & x_4^2x_3-x_3^3 & x_4^2x_2-x_2^3 & x_5^2x_3-x_3^3
\end{pmatrix}\\
=&(x_2-x_1)(x_3-x_2)(x_4-x_3)(x_4-x_2)(x_5-x_3)\\
&\times\det\left(\begin{array}{c|cc|cc}
1 & 1 & 1 & 1 & 1\\
\hline
x_1 & x_2 & x_3 & x_2 & x_3\\
x_1^2 & x_2^2 & x_3^2 & x_2^2 & x_3^2\\
\hline
x_1+x_2 & x_2+x_3 & x_3+x_4 & x_2+x_4 & x_3+x_5\\
x_1(x_1+x_2) & x_2(x_2+x_3) & x_3(x_3+x_4) & x_2(x_2+x_4) & x_3(x_3+x_5)
\end{array}\right).
\end{aligned}
$$

Here the appearing linear factors are

$$
(x_2-x_1)(x_3-x_2)(x_4-x_3)(x_4-x_2)(x_5-x_3)=\prod_{\substack{\lambda\subset\nu\\ \lambda_1=2}}\mathcal{A}_\lambda(x;1).
$$

The last determinant is calculated as

$$
\begin{aligned}
&\det\left(\begin{array}{c|cc|cc}
1 & 0 & 0 & 0 & 0\\
\hline
\star & x_2-x_1 & x_3-x_1 & 0 & 0\\
\star & x_2^2-x_1^2 & x_3^2-x_1^2 & 0 & 0\\
\hline
\star & \star & \star & x_4-x_3 & x_5-x_4\\
\star & \star & \star & x_2(x_4-x_3) & x_3(x_5-x_4)
\end{array}\right)\\
&=\det\begin{pmatrix} x_2-x_1 & x_3-x_1\\ x_2^2-x_1^2 & x_3^2-x_1^2\end{pmatrix}\det\begin{pmatrix} x_4-x_3 & x_5-x_4\\ x_2(x_4-x_3) & x_3(x_5-x_4)\end{pmatrix}\\
&=(x_2-x_1)(x_3-x_1)(x_4-x_3)(x_5-x_4)\det\begin{pmatrix}1 & 1\\ x_1+x_2 & x_1+x_3\end{pmatrix}\det\begin{pmatrix}1 & 1\\ x_2 & x_3\end{pmatrix}\\
&=(x_2-x_1)(x_3-x_1)(x_4-x_3)(x_5-x_4)(x_3-x_2)^2
\end{aligned}
$$

where the vertical and horizontal lines indicate the blockwise decomposition according to the second row. The linear factors appearing in the equality above are

$$
\begin{aligned}
(x_2-x_1)(x_3-x_1)(x_4-x_3)(x_5-x_4)&=\prod_{\substack{\lambda\subset\nu\\ \lambda_1=2\\ l(\lambda)\geq 2}}\mathcal{A}_\lambda(x;2),\\
(x_3-x_2)^2&=\prod_{\substack{\lambda\subset\nu\\ \lambda_1=2\\ l(\lambda)=3}}\mathcal{A}_\lambda(x;3).
\end{aligned}
$$

Combining these calculations above, we have

$$\begin{aligned}\det(x^{\mu(\lambda)})_{\lambda,\mu\subset(2,2,1)} =&(x_5-x_4)(x_5-x_3)(x_4-x_3)^2(x_4-x_2)(x_3-x_2)^4\\ &\times(x_3-x_1)^2(x_3-x_0)(x_2-x_1)^3(x_2-x_0)(x_1-x_0)\\ =&\prod_{\lambda\subset(2,2,1)}\mathcal{A}_\lambda(x).\end{aligned}$$

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