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On the homomorphisms between scalar generalized Verma modules

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In my talk, I discuss the existence problem of homomorphisms between generalized Verma modules, which are induced from one dimensional representations (such generalized Verma modules are called scalar, cf. [Boe 1985]). My main result is the classification of the homomorphisms between scalar generalized Verma modules with respect to the maximal parabolic subalgebras. The contents of my talk is taken from my paper: "The homomorphisms between scalar generalized Verma modules associated to maximal parabolic subalgebras," math.RT/0309454.

§ 1. History

A sufficient condition for the existence of the homomorphisms between Verma modules is given by [Verma 1968]. Bernstein, I. M. Gelfand, and S. I. Gelfand proved the condition of Verma is also a necessary condition. ([Bernstein-Gelfand-Gelfand 1975])

Later, Lepowsky studied the problem for the generalized Verma modules. In particular, Lepowsky ([Lepowsky 1975a]) solved the existence problem of nontrivial homomorphisms between scalar generalized Verma modules associated to the parabolic subalgebras which are the complexifications of the minimal parabolic subalgebras of real rank one simple Lie algebras (so-called the real rank one case).

Lepowsky also obtained a sufficient condition for the existence of the homomorphisms between scalar generalized Verma modules associated to the complexification of the minimal parabolic subalgebras of (not necessarily rank one) real semisimple Lie algebras. His condition is quite similar to that of Verma and he conjectured it is also a sufficient condition in the setting of complexified minimal parabolic algebras ([Lepowsky 1975b]).

Boe ([Boe 1985]) solved the existence problem in the case of parabolic subalgebras whose nilradical is commutative (so-called the Hermitian symmetric case).

The existence problem for maximal parabolic algebras is, in principle, reduced to the Kazdhan-Lusztig algorithm. Casian and Collingwood ([Casian-Collingwood 1987]) proposed a direct method of computing the Kazdhan-Lusztig data involving the generalized Verma modules. Applying the Kazdhan-Lusztig algorithm works very well in some cases. The structures of the (not necessarily scalar) generalized Verma modules in the real rank one case and in the Hermitian symmetric case are studied precisely ([Boe-Collingwood 1985], [Boe-Enright-Shelton 1988], [Collingwood-Irving-Shelton 1988]). Boe and Collingwood ([Boe-Collingwood 1990]) studied the case that all the (not necessarily scalar) generalized Verma modules are multiplicity free. The cases treated by Boe and Collingwood is more general than the real rank one case and the Hermitian symmetric case (so-called the multiplicity free case). In that case, they studied the structures of the (not necessarily scalar) generalized Verma modules precisely. In particular, they solved the existence problem of the non-trivial homomorphisms for regular integral infinitesimal characters in the multiplicity free case.

I surmise it is not easy to give the explicit answer to the existence problem by the Kazdhan-Lusztig algorithm in the general setting. Our central dogma is "*Consider the most singular parameter, then everything turns to be easy.*" Our approach to the problem consists of the following three main ingredients.

(1) The translation principle

The translation principle has a long history. In [Vogan 1988], Vogan proposed an idea on translation principle in order to establish the irreducibility of a discrete series representation of a semisimple symmetric space in some case. His idea is extremely useful for the study of the existence problem. Depending Vogan's idea, we formulated a version of translation principle in [Matumoto 1993] Proposition 2.2.3. In some cases, this enable us to reduce the existence of a non-trivial homomorphism to the most singular case in which the problem is often trivial.

(2) Jantzen's irreducibility criterion

Applying a version of the translation principle, we can often reduce the nonexistence of a non-trivial homomorphism to the irreducibility of a particular generalized Verma module. In [Jantzen 1977], Jantzen gave a sufficient and necessary condition for the irreducibility of a generated Verma module. His result is extremely useful for our purpose and we can establish the nonexistence of nontrivial homomorphisms in many cases.

(3) The Kazdhan-Lusztig theory ([Kazdhan-Lusztig 1979], [Brylinski-Kashiwara 1981], [Beilinson-Bernstein 1993])

Although we do not compute Kazdhan-Lusztig polynomials, the existence of the Kazdhan-Lusztig algorithm plays an important role in our approach. The point is that the definition of the Kazdhan-Lusztig polynomials only depends on Coxeter systems. In some cases, this enable us to reduce the problem to that of a different maximal parabolic subalgebra of a different simple Lie algebra, which is easier than the original problem.

In the most of the cases we can solve the existence problem by the above three ideas. However, in some cases we need extra arguments.

§ 2. Notations and Preliminaries

2.1 General notations

In this article, we use the following notations and conventions.

As usual we denote the complex number field, the real number field, the ring of (rational) integers, and the set of non-negative integers by $\mathbb{C}$, $\mathbb{R}$, $\mathbb{Z}$, and $\mathbb{N}$ respectively. $\frac{1}{2}\mathbb{N}$ means the set $\left\{ \frac{n}{2} \middle| n \in \mathbb{N}\right\}$, and $\frac{1}{2}+\mathbb{N}$ means the set $\left\{ \frac{1}{2}+n \middle| n \in \mathbb{N}\right\}$. We denote by $\emptyset$ the empty set. For any (non-commutative) $\mathbb{C}$-algebra R, "ideal" means "2-sided ideal", "R-module" means "left R-module", and sometimes we denote by 0 (resp. 1) the trivial R-module $\{0\}$ (resp. $\mathbb{C}$). Often, we identify a (small) category and the set of its objects. Hereafter "dim" means the dimension as a complex vector space, and "$\otimes$" (resp. Hom) means the tensor product over $\mathbb{C}$ (resp. the space of $\mathbb{C}$-linear mappings), unless we specify. For a complex vector space V, we denote by V^*

the dual vector space. For $a, b \in \mathbb{C}$, "$a \leqslant b$" means that $a, b \in \mathbb{R}$ and $a \leqslant b$. We denote by $A - B$ the set theoretical difference. $\mathrm{card}A$ means the cardinality of a set A.

2.2 Notations for reductive Lie algebras

Let $\mathfrak{g}$ be a complex reductive Lie algebra, $U(\mathfrak{g})$ the universal enveloping algebra of $\mathfrak{g}$, and $\mathfrak{h}$ a Cartan subalgebra of $\mathfrak{g}$. We denote by Δ the root system with respect to $(\mathfrak{g}, \mathfrak{h})$. We fix some positive root system Δ^+ and let Π be the set of simple roots. Let W be the Weyl group of the pair $(\mathfrak{g}, \mathfrak{h})$ and let $\langle\ ,\ \rangle$ be a non-degenerate invariant bilinear form on $\mathfrak{g}$. For $w \in W$, we denote by $\ell(w)$ the length of w as usuall. We also denote the inner product on $\mathfrak{h}^*$ which is induced from the above form by the same symbols $\langle\ ,\ \rangle$. For $\alpha \in \Delta$, we denote by s_α the reflection in W with respect to α. We denote by w_0 the longest element of W. For $\alpha \in \Delta$, we define the coroot $\check{\alpha}$ by $\check{\alpha} = \frac{2\alpha}{\langle\alpha,\alpha\rangle}$, as usual. We call $\lambda \in \mathfrak{h}^*$ is dominant (resp. anti-dominant), if $\langle\lambda, \check{\alpha}\rangle$ is not a negative (resp. positive) integer, for each $\alpha \in \Delta^+$. We call $\lambda \in \mathfrak{h}^*$ regular, if $\langle\lambda, \alpha\rangle \neq 0$, for each $\alpha \in \Delta$. We denote by P the integral weight lattice, namely $\mathsf{P} = \{\lambda \in \mathfrak{h}^* \mid \langle\lambda, \check{\alpha}\rangle \in \mathbb{Z}\ \text{ for all }\ \alpha \in \Delta\}$. If $\lambda \in \mathfrak{h}^*$ is contained in P, we call λ an integral weight. We define $\rho \in \mathsf{P}$ by $\rho = \frac{1}{2}\sum_{\alpha\in\Delta^+}\alpha$. Put $\mathfrak{g}_\alpha = \{X \in \mathfrak{g} \mid \forall H \in \mathfrak{h}\ \ [H, X] = \alpha(H)X\}$, $\mathfrak{u} = \sum_{\alpha\in\Delta^+}\mathfrak{g}_\alpha$, $\mathfrak{b} = \mathfrak{h} + \mathfrak{u}$. Then $\mathfrak{b}$ is a Borel subalgebra of $\mathfrak{g}$. We denote by Q the root lattice, namely $\mathbb{Z}$-linear span of Δ. We also denote by Q^+ the linear combination of Π with non-negative integral coefficients. For $\lambda \in \mathfrak{h}^*$, we denote by W_λ the integral Weyl group. Namely,

$$W_\lambda = \{w \in W \mid w\lambda - \lambda \in \mathsf{Q}\}.$$

We denote by Δ_λ the set of integral roots.

$$\Delta_\lambda = \{\alpha \in \Delta \mid \langle\lambda, \check{\alpha}\rangle \in \mathbb{Z}\}.$$

It is well-known that W_λ is the Weyl group for Δ_λ. We put $\Delta_\lambda^+ = \Delta^+ \cap \Delta_\lambda$. This is a positive system of Δ_λ. We denote by Π_λ the set of simple roots for Δ_λ^+ and denote by Φ_λ the set of reflection corresponding to the elements in Π_λ. So, $(W_\lambda, \Phi_\lambda)$ is a Coxeter system. We denote by Q_λ the integral root lattice, namely $\mathsf{Q}_\lambda = \mathbb{Z}\Delta_\lambda^+$ and put $\mathsf{Q}_\lambda^+ = \mathbb{N}\Pi_\lambda$.

Next, we fix notations for a parabolic subalgebra (which contains $\mathfrak{b}$). Hereafter, through this article we fix an arbitrary subset Θ of Π. Let $\bar{\Theta}$ be the set of the elements of Δ which are written by linear combinations of elements of Θ over $\mathbb{Z}$. Put $\mathfrak{a}_\Theta = \{H \in \mathfrak{h} \mid \forall\alpha \in \Theta\ \ \alpha(H) = 0\}$, $\mathfrak{l}_\Theta = \mathfrak{h} + \sum_{\alpha\in\bar{\Theta}}\mathfrak{g}_\alpha$, $\mathfrak{n}_\Theta = \sum_{\alpha\in\Delta^+\setminus\bar{\Theta}}\mathfrak{g}_\alpha$, $\mathfrak{p}_\Theta = \mathfrak{l}_\Theta + \mathfrak{n}_\Theta$. Then $\mathfrak{p}_\Theta$ is a parabolic subalgebra of $\mathfrak{g}$ which contains $\mathfrak{b}$. Conversely, for an arbitrary parabolic subalgebra $\mathfrak{p} \supseteq \mathfrak{b}$, there exists some $\Theta \subseteq \Pi$ such that $\mathfrak{p} = \mathfrak{p}_\Theta$. We denote by W_Θ the Weyl group for $(\mathfrak{l}_\Theta, \mathfrak{h})$. W_Θ is identified with a subgroup of W generated by $\{s_\alpha \mid \alpha \in \Theta\}$. We denote by w_Θ the longest element of W_Θ. Using the invariant non-degenerate bilinear form $\langle\ ,\ \rangle$, we regard $\mathfrak{a}_\Theta{}^*$ as a subspace of $\mathfrak{h}^*$. It is known that there is a unique nilpotent (adjoint) orbit (say $\mathcal{O}_{\mathfrak{p}_\Theta}$) whose intersection with $\mathfrak{n}_\Theta$ is Zarisky dense in $\mathfrak{n}_\Theta$. $\mathcal{O}_{\mathfrak{p}_\Theta}$ is called the Richardson orbit with respect to $\mathfrak{p}_\Theta$. We denote by $\bar{\mathcal{O}}_{\mathfrak{p}_\Theta}$ the closure of $\mathcal{O}_{\mathfrak{p}_\Theta}$ in $\mathfrak{g}$. Put $\rho_\Theta = \frac{1}{2}(\rho - w_\Theta\rho)$ and $\rho^\Theta = \frac{1}{2}(\rho + w_\Theta\rho)$. Then, $\rho^\Theta \in \mathfrak{a}_\Theta{}^*$.

2.3 Generalized Verma modules

Define

$$\mathsf{P}_\Theta^{++} = \{\lambda \in \mathfrak{h}^* \mid \forall\alpha \in \Theta\ \ \langle\lambda, \check{\alpha}\rangle \in \{1, 2, ...\}\}$$
$$^\circ\mathsf{P}_\Theta^{++} = \{\lambda \in \mathfrak{h}^* \mid \forall\alpha \in \Theta\ \ \langle\lambda, \check{\alpha}\rangle = 1\}$$

We easily have

$$^\circ\mathsf{P}_\Theta^{++} = \{\rho_\Theta + \mu \mid \mu \in \mathfrak{a}_\Theta^*\}.$$

For $\mu \in \mathfrak{h}^*$ such that $\mu + \rho \in \mathsf{P}_\Theta^{++}$, we denote by $\sigma_\Theta(\mu)$ the irreducible finite-dimensional $\mathfrak{l}_\Theta$-representation whose highest weight is μ. Let $E_\Theta(\mu)$ be the representation space of $\sigma_\Theta(\mu)$. We define a left action of $\mathfrak{n}_\Theta$ on $E_\Theta(\mu)$ by $X \cdot v = 0$ for all $X \in \mathfrak{n}_\Theta$ and $v \in E_\Theta(\mu)$. So, we regard $E_\Theta(\mu)$ as a $U(\mathfrak{p}_\Theta)$-module.

For $\mu \in \mathsf{P}_\Theta^{++}$, we define a generalized Verma module ([Lepowsky 1977]) as follows.

$$M_\Theta(\mu) = U(\mathfrak{g}) \otimes_{U(\mathfrak{p}_\Theta)} E_\Theta(\mu - \rho).$$

For all $\lambda \in \mathfrak{h}^*$, we write $M(\lambda) = M_\emptyset(\lambda)$. $M(\lambda)$ is called a Verma module. For $\mu \in \mathsf{P}_\Theta^{++}$, $M_\Theta(\mu)$ is a quotient module of $M(\mu)$. Let $L(\mu)$ be the unique highest weight $U(\mathfrak{g})$-module with the highest weight $\mu - \rho$. Namely, $L(\mu)$ is a unique irreducible quotient of $M(\mu)$. For $\mu \in \mathsf{P}_\Theta^{++}$, the canonical projection of $M(\mu)$ to $L(\mu)$ is factored by $M_\Theta(\mu)$.

$\dim E_\Theta(\mu - \rho) = 1$ if and only if $\mu \in {}^\circ\mathsf{P}_\Theta^{++}$. If $\mu \in {}^\circ\mathsf{P}_\Theta^{++}$, we call $M_\Theta(\mu)$ a scalar generalized Verma module.

For a finitely generated $U(\mathfrak{g})$-module V, we denote by $\mathrm{Dim}(V)$ (resp. $c(V)$) the Gelfand-Krillov dimesion (resp. the multiplicity) of V. (See [Vogan 1978]). We easily see $\mathrm{Dim}(M_\Theta(\mu)) = \dim \mathfrak{n}_\Theta$ and $c((M_\Theta(\mu)) = \dim E_\Theta(\mu - \rho)$.

The following result is one of the fundamental results on the existence problem of homomorphisms between scalar generalized Verma modules.

Theorem 2.3.1. *([Lepowsky 1976])*
Let $\mu, \nu \in {}^\circ\mathsf{P}_\Theta^{++}$.
(1) $\dim \mathit{Hom}_{U(\mathfrak{g})}(M_\Theta(\mu), M_\Theta(\nu)) \leqslant 1$.
(2) *Any non-zero homomoorphism of $M_\Theta(\mu)$ to $M_\Theta(\nu)$ is injective.*

Hence, the existence problem of homomorphisms between scalar generalized Verma modules is reduce to the following problem.

Problem Let $\mu, \nu \in {}^\circ\mathsf{P}_\Theta^{++}$. When is $M_\Theta(\mu) \subseteq M_\Theta(\nu)$?

2.4 Notations for maximal parabolic subalgebras

Hereafter we fix $\alpha \in \Pi$. Put $\Theta^\alpha = \Pi - \{\alpha\}$. $\mathfrak{a}_{\Theta^\alpha}$ is one-dimensional and spanned by ρ^{Θ^α}. Moreover, we have

$$^\circ\mathsf{P}_{\Theta^\alpha}^{++} = \{\rho_{\Theta^\alpha} + t\rho^{\Theta^\alpha} \mid t \in \mathbb{C}\}.$$

We denote by ω_α the fundamental weight corresponding to α. For any $\beta \in \Theta^\alpha = \Pi - \{\alpha\}$, we have $\langle \beta, \rho^{\Theta^\alpha} \rangle = 0$. Hence there exists some $d_\alpha \in \mathbb{R}$ such that $d_\alpha \omega_\alpha = \rho^{\Theta^\alpha}$. Since $2\rho^{\Theta^\alpha}$ is integral, we have $d_\alpha \in \frac{1}{2}\mathbb{N}$.

Fir simplicity, for $t \in \mathbb{C}$, we write $M_{\Theta^\alpha}[t]$ for $M_{\Theta^\alpha}(\rho_{\Theta^\alpha} + t\omega_\alpha)$. We easily have:

Lemma 2.4.1. *Let s and t be distinct complex numbers such that $M_{\Theta^\alpha}[s] \subseteq M_{\Theta^\alpha}[t]$. Then, $s = -t$.*

§ 3. Classical Lie algebras

Throughout this section, n means a positive integer such that $n \geqslant 2$ (resp. $n \geqslant 4$) whenever we consider the simple Lie algebra of the type B_n or C_n (resp. D_n).

3.1 The root systems

We retain the notations in §1 and §2.

($\mathbf{B_n}$ type) We consider the root system Δ for $\mathfrak{g} = \mathfrak{so}(2n+1, \mathbb{C})$. Then we can choose an orthonormal basis $e_1, ..., e_n$ of $\mathfrak{h}^*$ such that

$$\Delta = \{\pm e_i \pm e_j \mid 1 \leqslant i < j \leqslant n\} \cup \{\pm e_i \mid 1 \leqslant i \leqslant n\}.$$

We choose a positive system as follows.

$$\Delta^+ = \{e_i \pm e_j \mid 1 \leqslant i < j \leqslant n\} \cup \{e_i \mid 1 \leqslant i \leqslant n\}.$$

If we put $\alpha_i = e_i - e_{i+1} \quad (1 \leqslant i < n)$ and $\alpha_n = e_n$, then $\Pi = \{\alpha_1, ..., \alpha_n\}$.

For simplicity, we write Θ^k for Θ^{α_k} for $1 \leqslant k \leqslant n$. Then $\mathfrak{l}_{\Theta^k}$ is isomorphic to $\mathfrak{gl}(k, \mathbb{C}) \times \mathfrak{so}(2(n-k)+1, \mathbb{C})$.

($\mathbf{C_n}$ type)

We consider the root system Δ for $\mathfrak{g} = \mathfrak{sp}(n, \mathbb{C})$. Then we can choose an orthonormal basis $e_1, ..., e_n$ of $\mathfrak{h}^*$ such that

$$\Delta = \{\pm e_i \pm e_j \mid 1 \leqslant i < j \leqslant n\} \cup \{\pm 2e_i \mid 1 \leqslant i \leqslant n\}.$$

We choose a positive system as follows.

$$\Delta^+ = \{e_i \pm e_j \mid 1 \leqslant i < j \leqslant n\} \cup \{2e_i \mid 1 \leqslant i \leqslant n\}.$$

If we put $\alpha_i = e_i - e_{i+1} \quad (1 \leqslant i < n)$ and $\alpha_n = 2e_n$, then $\Pi = \{\alpha_1, ..., \alpha_n\}$.

For simplicity, we write Θ^k for Θ^{α_k} for $1 \leqslant k \leqslant n$. Then $\mathfrak{l}_{\Theta^k}$ is isomorphic to $\mathfrak{gl}(k, \mathbb{C}) \times \mathfrak{sp}(n-k, \mathbb{C})$.

($\mathbf{D_n}$ type)

We consider the root system Δ for $\mathfrak{g} = \mathfrak{so}(2n, \mathbb{C})$. Then we can choose an orthonormal basis $e_1, ..., e_n$ of $\mathfrak{h}^*$ such that

$$\Delta = \{\pm e_i \pm e_j \mid 1 \leqslant i < j \leqslant n\}.$$

We choose a positive system as follows.

$$\Delta^+ = \{e_i \pm e_j \mid 1 \leqslant i < j \leqslant n\}.$$

If we put $\alpha_i = e_i - e_{i+1} \quad (1 \leqslant i < n)$ and $\alpha_n = e_{n-1} + e_n$, then $\Pi = \{\alpha_1, ..., \alpha_n\}$.

For simplicity, we write Θ^k for Θ^{α_k} for $1 \leqslant k \leqslant n$. Since the case of $k = n-1$ is essentially same as the case of $k = n$, since $\mathfrak{p}_{\Theta^{n-1}}$ and $\mathfrak{p}_{\Theta^n}$ are conjugate under an automorphism of $\mathfrak{g}$. So, when we consider the type D case, we omitt the case of $k = n-1$.

Then, $\mathfrak{l}_{\Theta^k}$ is isomorphic to $\mathfrak{gl}(k, \mathbb{C}) \times \mathfrak{so}(2(n-k), \mathbb{C})$.

3.2 Statements of the main results

Here, we describe the existence of homomorphisms between scalar generalized Verma modules with respect to the maximal parabolic subalgebras of classical Lie algebras. For simple Lie algebras of type A, the answer is given in [Boe 1985]. So, we treat the case of the types of B,C, and D.

Theorem 3.2.1. *(B_n-type)*
Consider the case of $\mathfrak{g} = \mathfrak{so}(2n+1, \mathbb{C})$. Let k be a positive integer such that $k \leqslant n$.
(1) We consider the case that $3k < 2n+1$. Then, $M_{\Theta^k}[-t] \subseteq M_{\Theta^k}[t]$ if and only if $t \in \mathbb{N}$.
(2) We consider the case that $3k \geqslant 2n+1$.
(2a) Assume k is odd and $k \neq n$. Then, $M_{\Theta^k}[-t] \subseteq M_{\Theta^k}[t]$ if and only if $t \in \frac{1}{2}\mathbb{N}$.
(2b) Assume n is odd. Then, $M_{\Theta^n}[-t] \subseteq M_{\Theta^n}[t]$ if and only if $t \in \mathbb{N}$.
(2c) Assume k is even. Then, $M_{\Theta^k}[-t] \subseteq M_{\Theta^k}[t]$ if and only if $t = 0$.

Remark The case of $k = 1$ is due to [Lepowsky 1975a]. The cases of $k = 1, 2, 3, n$ are in the multiplicity free case in [Boe-Collingwood 1990].

Theorem 3.2.2. *(C_n-type)*
Consider the case of $\mathfrak{g} = \mathfrak{sp}(n, \mathbb{C})$. Let k be a positive integer such that $k \leqslant n$.
(1) We consider the case that $3k \leqslant 2n$.
(1a) Assume k is odd. Then, $M_{\Theta^k}[-t] \subseteq M_{\Theta^k}[t]$ if and only if $t = 0$.
(1b) Assume k is even. Then, $M_{\Theta^k}[-t] \subseteq M_{\Theta^k}[t]$ if and only if $t \in \frac{1}{2}\mathbb{N}$.
(2) We consider the case that $3k > 2n$. Then, $M_{\Theta^k}[-t] \subseteq M_{\Theta^k}[t]$ if and only if $t \in \mathbb{N}$.

Remark The case of $k = 2$ is due to [Lepowsky 1975a]. The case of $k = n$ is due to [Boe 1985]. The cases of $k = 1, 2, 3, n$ are in the multiplicity free case in [Boe-Collingwood 1990].

Theorem 3.2.3. *(D_n-type)*
Consider the case of $\mathfrak{g} = \mathfrak{so}(2n, \mathbb{C})$. Let k be a positive integer such that $k \leqslant n-2$ or $k = n$.
(1) We consider the case that $3k < 2n$. Then, $M_{\Theta^k}[-t] \subseteq M_{\Theta^k}[t]$ if and only if $t \in \mathbb{N}$.
(2) We consider the case that $3k \geqslant 2n$.
(2a) Assume k is odd. Then, $M_{\Theta^k}[-t] \subseteq M_{\Theta^k}[t]$ if and only if $t = 0$.
(2b) Assume k is even and $k \neq n$. Then, $M_{\Theta^k}[-t] \subseteq M_{\Theta^k}[t]$ if and only if $t \in \frac{1}{2}\mathbb{N}$.
(2c) Assume n is even. Then, $M_{\Theta^n}[-t] \subseteq M_{\Theta^n}[t]$ if and only if $t \in \mathbb{N}$.

Remark The case of $k = 1$ is due to [Lepowsky 1975a]. The case of $k = n$ is due to [Boe 1985]. The cases of $k = 1, 2, n$ are in the multiplicity free case in [Boe-Collingwood 1990].

We give proofs of the above theorems in the subsquent sections.

§ 4. Exceptional algebras

As in §3, we write $\Theta^k = \Pi - \{\alpha_k\}$, $\omega_k = \omega_{\alpha_k}$, $c_k = c_{\alpha_k}$, $d_k = d_{\alpha_k}$, etc.

4.1 G_2

In the regular integral case, homomorphisms between (not necessarily scalar) generalized Verma modules are classified for G_2 in [Boe-Collingwood 1990].

I imagine in the case of G_2 the classification of the homomorphisms between scalar generalized Verma modules is well-known, but I would like to state the result for the completeness.

Let $\mathfrak{g}$ be a simple Lie algebra of the type G_2. Then, we may write $\Pi = \{\alpha_1, \alpha_2\}$ such that $\langle\alpha_1, \alpha_1\rangle > \langle\alpha_2, \alpha_2\rangle$.

Theorem 4.1.1. *Let $\mathfrak{g}$ be a simple Lie algebra of the type G_2 and let k be 1 or 2. Then, $M_{\Theta^k}[-t] \subseteq M_{\Theta^k}[t]$ if and only if $t \in \mathbb{N}$.*

4.2 F_4

For the simple algebra of the type F_4, [Boe-Collingwood] treated the regular integral case. The half-integral case is somewhat easier, but I would like to mention the results for the completeness.

We consider the root system Δ for a simple Lie algebra $\mathfrak{g}$ of the type F_4. (For example, see [Knapp 2002] p691.) We can choose an orthonormal basis $e_1, ..., e_4$ of $\mathfrak{h}^*$ such that

$$\Delta = \{\pm e_i \pm e_j \mid 1 \leqslant i < j \leqslant 4\} \cup \{\pm e_i \mid 1 \leqslant i \leqslant 4\} \cup \left\{\frac{1}{2}(\pm e_1 \pm e_2 \pm e_3 \pm e_4)\right\}.$$

We choose a positive system as follows.

$$\Delta^+ = \{e_i \pm e_j \mid 1 \leqslant i < j \leqslant 4\} \cup \{e_i \mid 1 \leqslant i \leqslant 4\} \cup \left\{\frac{1}{2}(e_1 \pm e_2 \pm e_3 \pm e_4)\right\}.$$

Put $\alpha_1 = \frac{1}{2}(e_1 - e_2 - e_3 - e_4)$, $\alpha_2 = e_4$, $\alpha_3 = e_3 - e_4$, and $\alpha_4 = e_2 - e_3$. Then, $\Pi = \{\alpha_1, ..., \alpha_4\}$.

$$1 \;-\; 2 \;\Leftarrow\; 3 \;-\; 4$$

Since $\mathsf{P} = \mathsf{Q}$ in this case, we have $c_k = \frac{1}{2}$ for $1 \leqslant k \leqslant 4$.

The result is:

Theorem 4.2.1. *((1) is due to [Lepowsky 1975a].)*

(1) $M_{\Theta^1}[-t] \subseteq M_{\Theta^1}[t]$ *if and only if* $t \in \frac{1}{2}\mathbb{N}$.
(2) $M_{\Theta^2}[-t] \subseteq M_{\Theta^2}[t]$ *if and only if* $t \in \mathbb{N}$.
(3) $M_{\Theta^3}[-t] \subseteq M_{\Theta^3}[t]$ *if and only if* $t \in \mathbb{N}$.
(4) $M_{\Theta^3}[-t] \subseteq M_{\Theta^3}[t]$ *if and only if* $t \in \frac{1}{2}\mathbb{N}$.

4.3 E_6

We consider the root system Δ for a simple Lie algebra $\mathfrak{g}$ of the type E_6. Put $\kappa = \frac{1}{2\sqrt{3}}$. We can choose an orthonormal basis $e_1, ..., e_6$ of $\mathfrak{h}^*$ such that

$$\begin{aligned}\Delta = &\{e_i - e_j \mid 1 \leqslant i, j \leqslant 6, i \neq j\} \\ &\cup \left\{\pm \sum_{i=1}^{6} \left(\kappa + (-1)^{n(i)}\frac{1}{2}\right) e_i \,\middle|\, n(i) = 1 \text{ or } n(i) = 0 \text{ for } 1 \leqslant i \leqslant 6, \sum_{i=1}^{6} n(i) = 3\right\} \\ &\cup \left\{\pm 2\kappa \sum_{i=1}^{6} e_i\right\}.\end{aligned}$$

We choose a positive system as follows.

$$\Delta^+ = \{e_i - e_j \mid 1 \leqslant i < j \leqslant 6\}$$
$$\cup \left\{ \sum_{i=1}^{6} \left(\kappa + (-1)^{n(i)} \frac{1}{2} \right) e_i \,\middle|\, n(i) = 1 \text{ or } n(i) = 0 \text{ for } 1 \leqslant i \leqslant 6, \sum_{i=1}^{6} n(i) = 3 \right\}$$
$$\cup \left\{ 2\kappa \sum_{i=1}^{6} e_i \right\}.$$

Put $\alpha_i = e_i - e_{i+1}$ $(1 \leqslant i \leqslant 5)$ and $\alpha_6 = \sum_{i=1}^{3} \left(\kappa - \frac{1}{2}\right) e_i + \sum_{i=4}^{6} \left(\kappa + \frac{1}{2}\right) e_i$. Then, $\Pi = \{\alpha_1, ..., \alpha_6\}$.

$$\begin{array}{ccccccccc} 1 & - & 2 & - & 3 & - & 4 & - & 5 \\ & & & & | & & & & \\ & & & & 6 & & & & \end{array}$$

Since the Dynkin diagram of E_6 has a symmetry, the cases of Θ^4 and Θ^5 are similar to Θ^2 and Θ^1, respectively. So, we only consider $\Theta^1, \Theta^2, \Theta^3$, and Θ^6.

We have:

Theorem 4.3.1. *((1) is due to [Boe 1985].)*

(1) $M_{\Theta^1}[-t] \subseteq M_{\Theta^1}[t]$ *if and only if* $t = 0$.

(2) $M_{\Theta^2}[-t] \subseteq M_{\Theta^2}[t]$ *if and only if* $t = 0$.

(3) $M_{\Theta^3}[-t] \subseteq M_{\Theta^3}[t]$ *if and only if* $t \in \mathbb{N}$.

(4) $M_{\Theta^6}[-t] \subseteq M_{\Theta^3}[t]$ *if and only if* $t \in \mathbb{N}$.

4.4 $\mathbf{E_7}$

Let $\mathfrak{g}$ be a simple Lie algebra of the type E_7.

We fix an orthonormal basis $e_1, ..., e_8$ in $\mathbb{R}^8$. We identify $\mathfrak{h}^*$ with $\{v \in \mathbb{R}^8 \mid \langle v, e_1 - e_2 \rangle = 0\}$ so that

$$\Delta = \{\pm(e_1 + e_2)\} \cup \{\pm e_i \pm e_j \mid 3 \leqslant i < j \leqslant 8\}$$
$$\cup \left\{ \pm\frac{1}{2} \left(e_1 + e_2 + \sum_{i=3}^{8} (-1)^{n(i)} e_i \right) \,\middle|\, n(i) \text{ is either } 0 \text{ or } 1 \text{ for } 3 \leqslant i \leqslant 8 \text{ and } \textstyle\sum_{i=3}^{8} n(i) \text{ is even.} \right\}$$

We choose a positive system as follows.

$$\Delta^+ = \{(e_1 + e_2)\} \cup \{e_i \pm e_j \mid 3 \leqslant i < j \leqslant 8\}$$
$$\cup \left\{ \frac{1}{2} \left(e_1 + e_2 + \sum_{i=3}^{8} (-1)^{n(i)} e_i \right) \,\middle|\, n(i) \text{ is either } 0 \text{ or } 1 \text{ for } 3 \leqslant i \leqslant 8 \text{ and } \textstyle\sum_{i=3}^{8} n(i) \text{ is even.} \right\}$$

Put $\alpha_i = e_{i+2} - e_{i+3}$ for $1 \leqslant i \leqslant 5$, $\alpha_6 = e_7 + e_8$, and $\alpha_7 = \frac{1}{2}(e_1 + e_2 - e_3 - e_4 - e_5 - e_6 - e_7 - e_8)$. Then, $\Pi = \{\alpha_1, ..., \alpha_7\}$ is the set of simple roots in Δ^+.

$$\begin{array}{ccccccccccc} 1 & - & 2 & - & 3 & - & 4 & - & 6 & - & 7 \\ & & & & & & | & & & & \\ & & & & & & 5 & & & & \end{array}$$

We have:

Theorem 4.4.1. *(The case of $k = 1$ is due to [Boe 1985].)*

(1) Assume $k \in \{1, 3, 5, 6, 7\}$. Then, $M_{\Theta^k}[-t] \subseteq M_{\Theta^k}[t]$ if and only if $t \in \mathbb{N}$.

(2) Assume $k \in \{2, 4\}$. Then, $M_{\Theta^k}[-t] \subseteq M_{\Theta^k}[t]$ if and only if $t \in \frac{1}{2}\mathbb{N}$.

Remark In the case of $k = 7$, the non-existence of the homomorphism is proved in [Boe-Collingwood 1990] for the regular integral case.

4.5 $\mathbf{E_8}$

We fix an orthonormal basis $e_1, ..., e_8$ in $\mathfrak{h}^*$ such that

$$\Delta = \{\pm e_i \pm e_j \mid 1 \leqslant i < j \leqslant 8\}$$
$$\cup \left\{ \pm\frac{1}{2}\left(\sum_{i=1}^{8}(-1)^{n(i)}e_i\right) \middle| n(i) \text{ is either 0 or 1 for } 3 \leqslant i \leqslant 8 \text{ and } \textstyle\sum_{i=3}^{8} n(i) \text{ is odd.} \right\}$$

We choose a positive system as follows.

$$\Delta^+ = \{e_i \pm e_j \mid 1 \leqslant i < j \leqslant 8\}$$
$$\cup \left\{ \frac{1}{2}\left(e_1 + \sum_{i=2}^{8}(-1)^{n(i)}e_i\right) \middle| n(i) \text{ is either 0 or 1 for } 2 \leqslant i \leqslant 8 \text{ and } \textstyle\sum_{i=2}^{8} n(i) \text{ is odd.} \right\}$$

Put $\alpha_i = e_{i+1} - e_{i+2}$ for $1 \leqslant i \leqslant 5$, $\alpha_7 = e_7 + e_8$, and $\alpha_7 = \frac{1}{2}(e_1 - e_2 - e_3 - e_4 - e_5 - e_6 - e_7 - e_8)$. Then, $\Pi = \{\alpha_1, ..., \alpha_8\}$ is the set of simple roots in Δ^+.

$$\begin{array}{ccccccccccccc} 1 & - & 2 & - & 3 & - & 4 & - & 5 & - & 7 & - & 8 \\ & & & & & & & & | & & & & \\ & & & & & & & & 6 & & & & \end{array}$$

We have:

Theorem 4.5.1.

(1) Assume $k \in \{1, 2, 4, 6, 8\}$. Then, $M_{\Theta^k}[-t] \subseteq M_{\Theta^k}[t]$ if and only if $t \in \mathbb{N}$.

(2) Assume $k \in \{3, 5, 7\}$. Then, $M_{\Theta^k}[-t] \subseteq M_{\Theta^k}[t]$ if and only if $t \in \frac{1}{2}\mathbb{N}$.

Remark In the case of $k = 1$, the non-existence of the homomorphism is proved in [Boe-Collingwood 1990] for the regular integral case.

§ 5. Elementary homomorphisms

Here, we explain that we can construct homomorphisms in the setting of general parabolic subalgebras from the case of maximal parabolic subalgebras.

5.1 A comparison result

Here, we review some notion in [Matumoto 1993] §3. Hereafter, $\mathfrak{g}$ means a reductive Lie algebra over $\mathbb{C}$ and retain the notations in §1. We fix a subset Θ of Π. For $\alpha \in \Delta$, we put

$$\begin{aligned}\Delta(\alpha) &= \{\beta \in \Delta \mid \exists c \in \mathbb{R} \ \ \beta|_{\mathfrak{a}_\Theta} = c\alpha|_{\mathfrak{a}_\Theta}\},\\ \Delta^+(\alpha) &= \Delta(\alpha) \cap \Delta^+,\\ U_\alpha &= \mathbb{C}S + \mathbb{C}\alpha \subseteq \mathfrak{h}^*.\end{aligned}$$

Then $(U_\alpha, \Delta(\alpha), \langle\ ,\ \rangle)$ is a subroot system of $(\mathfrak{h}^*, \Delta, \langle\ ,\ \rangle)$. The set of simple roots for $\Delta^+(\alpha)$ is denoted by $\Pi(\alpha)$. If $\alpha|_{\mathfrak{a}_\Theta} \neq 0$, then there exists a unique $\tilde{\alpha} \in \Delta^+$ such that $\Pi(\alpha) = \Theta \cup \{\tilde{\alpha}\}$. If $\alpha \in \Delta$ satisfies $\alpha|_{\mathfrak{a}_\Theta} \neq 0$ and $\alpha = \tilde{\alpha}$, then we call α Θ-reduced. For $\alpha \in \Delta^+$, we denote by $W_\Theta(\alpha)$ the Weyl group of $(\mathfrak{h}^*, \Delta(\alpha))$. Clearly, $W_\Theta \subseteq W_\Theta(\alpha) \subseteq W$. We denote by w^α the longest element of $W_\Theta(\alpha)$. We call $\alpha \in \Delta$ Θ-acceptable if $w^\alpha w_\Theta = w_\Theta w^\alpha$. We denote by Δ_r^Θ the set of Θ-reduced Θ-acceptable roots. Put $(\Delta_r^\Theta)^+ = \Delta^+ \cap \Delta_r^\Theta$. For $\alpha \in \Delta_r^\Theta$, we define

$$\sigma_\alpha = w^\alpha w_\Theta = w_\Theta w^\alpha.$$

Clearly, ${\sigma_\alpha}^2 = 1$. For $\alpha \in \Delta$, we put

$$V_\alpha = \{\lambda \in \mathfrak{a}_\Theta^* \mid \langle\lambda, \alpha\rangle = 0\}.$$

We denote by $\omega_\alpha \in \mathfrak{a}_\Theta^* \subseteq \mathfrak{h}^*$ the fundamental weight for α with respect to the basis $\Pi(\alpha) = \Theta \cup \{\alpha\}$. Namely ω_α satisfies that $\langle\omega_\alpha, \beta\rangle = 0$ for $\beta \in \Theta$, $\langle\beta, \check{\alpha}\rangle = 1$, and $\omega_\alpha|_{\mathfrak{h}\cap\mathfrak{c}(\mathfrak{g}(\alpha))} = 0$. Here, $\mathfrak{c}(\mathfrak{g}(\alpha))$ is the center of $\mathfrak{g}(\alpha)$. We see that there is some positive real number a such that $\omega_\alpha = a\alpha|_{\mathfrak{a}_\Theta}$, since $\alpha|_{\mathfrak{h}\cap\mathfrak{c}(\mathfrak{g}(\alpha))} = 0$. Hence, we have $V_\alpha = \{\lambda \in \mathfrak{a}_\Theta^* \mid \langle\lambda, \omega_\alpha\rangle = 0\}$.

We can easily see:

Lemma 5.1.1. *Let $\alpha \in \Delta_r^\Theta$. Then, we have*

(1) σ_α *preserves* $\mathfrak{a}_\Theta^*$.

(2) $\sigma_\alpha \in W(\Theta)$. *In particular,* $\sigma_\alpha\rho_\Theta = \rho_\Theta$.

(3) $\sigma_\alpha\omega_\alpha = -\omega_\alpha$.

(4) $\sigma_\alpha|_{\mathfrak{a}_\Theta}$ *is the reflection with respect to* V_α.

For $\alpha \in (\Delta_r^\Theta)^+$, we define

$$\mathfrak{g}(\alpha) = \mathfrak{h} + \sum_{\beta\in\Delta(\alpha)} \mathfrak{g}_\beta, \qquad \mathfrak{p}_\Theta(\alpha) = \mathfrak{g}(\alpha) \cap \mathfrak{p}_\Theta.$$

Then, $\mathfrak{g}(\alpha)$ is a reductive Lie subalgebra of $\mathfrak{g}$ whose root system is $\Delta(\alpha)$ and $\mathfrak{p}_\Theta(\alpha)$ is a maximal parabolic subalgebra of $\mathfrak{g}(\alpha)$.

Put $\rho(\alpha) = \frac{1}{2}\sum_{\beta\in\Delta^+(\alpha)}\beta$, For $\nu \in \mathfrak{a}_\Theta^*$, we denote by $\mathbb{C}_\nu$ the one-dimensional $U(\mathfrak{p}_\Theta(\alpha))$-module corresponding to ν. For $\nu \in \mathfrak{a}_\Theta^*$ we define a generalized Verma module for $\mathfrak{g}(\alpha)$ as follows.

$$M_\Theta^{\mathfrak{g}(\alpha)}(\rho_\Theta + \nu) = U(\mathfrak{g}(\alpha)) \otimes_{U(\mathfrak{p}_\Theta(\alpha))} \mathbb{C}_{\nu-\rho(\alpha)}.$$

Then, we have:

Theorem 5.1.2. *Let ν be an arbitrary element in V_α and let c be either 1 or $\frac{1}{2}$. Assume that $M_\Theta^{\mathfrak{g}(\alpha)}(\rho_\Theta - nc\omega_\alpha) \subseteq M_\Theta^{\mathfrak{g}(\alpha)}(\rho_\Theta + nc\omega_\alpha)$ for all $n \in \mathbb{N}$. Then, we have $M_\Theta(\rho_\Theta + \nu - nc\omega_\alpha) \subseteq M_\Theta(\rho_\Theta + \nu + nc\omega_\alpha)$ for all $n \in \mathbb{N}$. (We call the above homomorphism of $M_\Theta(\rho_\Theta + \nu - nc\omega_\alpha)$ into $M_\Theta(\rho_\Theta + \nu + nc\omega_\alpha)$ an elementary homomorphism.)*

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