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Corwin-Greenleaf multiplicity function for Hermitian Lie groups — restriction to a maximal compact subgroup*

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1 Introduction

Suppose G is a Lie group. A unitary representation of G is a Hilbert space endowed with a continuous linear action of G by unitary operators. A unitary representation is called irreducible if there is no closed G-invariant subspace except for $\{0\}$ and itself. We denote by $\widehat{G}$ the unitary dual of G, namely, the set of equivalence classes of irreducible unitary representations of G.

Suppose H is a closed subgroup of G. Following a programme in [8], we can consider two natural settings for the irreducible decomposition of a unitary representation: one is an induced representation from a smaller group (denoted by $\mathrm{Ind}_H^G \nu$ for $\nu \in \widehat{H}$), and the other is the restriction from a larger group (denoted by $\pi|_H$ for $\pi \in \widehat{G}$). The corresponding decomposition formulas are called *Plancherel formula* and *branching law* respectively. Then one of the fundamental problems in representation theory is to describe the *branching rule*

$$\pi|_H \simeq \int_{\widehat{H}}^{\oplus} m_\pi(\nu)\nu d\mu(\nu) \quad \text{(a direct integral).} \tag{1}$$

Such a decomposition is unique, for example, if H is a reductive Lie group, and the **multiplicity** $m_\pi : \widehat{H} \to \mathbb{N} \cup \{\infty\}$ makes sense as a measurable function on the unitary dual $\widehat{H}$. In general, branching laws face difficulties for the cause of infinite multiplicity (cf. [5]). So, among various settings of branching problems, one could expect a simple and detailed study if $\pi|_H$ would be multiplicity-free as a representation of H. Here is a basic question on the **multiplicity**:

Question. When is the multiplicity free in the branching law?

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As for the above question, Prof. KOBAYASHI recently proved a multiplicity free theorem for branching laws in great generality for both finite and infinite dimensional representations ([6], [7], [8], [9]). His theorem gives a uniform explanation for many known cases of multiplicity free results (e.g. Clebsch-Gordan formula, Pieri's rule, Plancherel formula for Riemannian symmetric spaces, GL_m-GL_n duality, the Hua-Kostant-Schmid-Kobayashi formula, ...) and also presents many new cases of multiplicity free branching laws. This article is concerned with the so-called *branching problems* from the view point of the orbit method.

Now we turn our attention to the title of the article from the orbit method due to Kirillov-Kostant, namely, the **Corwin-Greenleaf multiplicity function**. For this purpose, first we recall the **orbit method** in the unitary representation theory of Lie groups ([1, 2, 3, 4, 8, 10, 13, 16]).

Let $\mathfrak{g}$ be the Lie algebra of G and $\mathfrak{g}^*$ be the dual of $\mathfrak{g}$. Let us consider the contragradient representation $\mathrm{Ad}^* : G \to \mathrm{GL}(\mathfrak{g}^*)$ of the adjoint representation of G, $\mathrm{Ad} : G \to \mathrm{GL}(\mathfrak{g})$. This non-unitary finite dimensional representation often has a surprising intimate relation with the unitary dual $\widehat{G}$ (which are infinite dimensional in general). We define an equvalence relation in $\mathfrak{g}^*$ by

$$\lambda \sim \mu \Leftrightarrow \mathrm{Ad}^*(g)\lambda = \mu \quad (\text{for } g \in G).$$

The set of equivalence classes is denoted by $\mathfrak{g}^*/G$, the set of coadjoint orbits.

For example, if $G = \mathbb{R}^n$, then $\mathfrak{g}^*/G \simeq \mathbb{R}^n$, and we have a natural bijection between $\widehat{G}$ and $\sqrt{-1}\mathfrak{g}^*/G \simeq \sqrt{-1}\mathbb{R}^n$. This abelian example is generalized to the case of nilpotent Lie groups by Kirillov [4]. Namely, if G is a connected and simply connected nilpotent Lie group then we have the Kirillov correspondence

$$\widehat{G} \approx \mathfrak{g}^*/\,\mathrm{Ad}^*(G).$$

Taking Kirillov's theory as the governing principle, CORWIN and GREENLEAF have worked in the context of simply connected nilpotent Lie groups. Fujiwara and Lipsman have extended their work to certain solvable Lie groups. We refer the survey paper [2], [3] and [13] for further details concerning this.

In the context of nilpotent Lie groups, Kirillov's theory of coadjoint orbits allows one to determine the spectrum and multiplicities in terms of G and H-orbits.

Let $\pi \in \widehat{G}$, $\nu \in \widehat{H}$. Due to Kirillov's orbit method, suppose $\mathcal{O}_\pi^G$ and $\mathcal{O}_\nu^H$ are the corresponding coadjoint orbits in $\mathfrak{g}^*$ and $\mathfrak{h}^*$ respectively. We define a "mod H" intersection number

$$n_\pi(\nu) = \sharp\big(\mathrm{Ad}^*(H)\text{-orbit in } \mathcal{O}_\pi^G \cap \mathrm{pr}^{-1}(\mathcal{O}_\nu^H)\big), \tag{2}$$

where $\mathrm{pr} : \mathfrak{g}^* \to \mathfrak{h}^*$ is the canonical projection. The number defined in (2) is known as **Corwin-Greenleaf multiplicity function** in the *Corwin-Greenleaf orbital integral formula* (for the description of both induced and restricted represeentations). What the orbit method expects is the coincidence of two equations: one comes from the branching law of the unitary representation (1), and the other from the geometry of coadjoint orbits (2), that is,

$$m_\pi(\nu) = n_\pi(\nu).$$

This was proved for nilpotent case by Corwin and Greenleaf [1]. It is the **multiplicity function** that is the main focus of my article.

The orbit method works perfectly for nilpotent Lie groups. Unlike the nilpotent case, it has been observed by experts that the orbit method works only partially for reductive groups. However, the *multiplicity free* theorems ([6], [7], [9]) for reductive groups and other considerations motivated by the **Corwin-Greenleaf multiplicity function** lead Prof. KOBAYASHI to formulate the following general conjecture for reductive symmetric spaces.

Conjecture 1. Let (G, H) be a reductive symmetric pair. If a G-coadjoint orbit $\mathcal{O}_z^G$ in $\mathfrak{g}^*$ meets $\mathfrak{c}(\mathfrak{k})$ (see the notation in the next section), then

$$\sharp\big(\mathrm{Ad}^*(H)\text{-orbit in } \mathcal{O}_z^G \cap \mathrm{pr}^{-1}(\mathcal{O}_\nu^H)\big) \leq 1,$$

for any H-coadjoint orbit $\mathcal{O}_\nu^H$ in $\mathfrak{h}^*$.

2 Statement of the main result

Let G be a simple non-compact Lie group with finite center, Cartan involution θ, maximal compact subgroup $K := \{g \in G | \theta g = g\}$ and Lie algebra $\mathfrak{g}$. We write $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ for the Cartan decomposition, corresponding to the Cartan involution θ. We assume that G is of Hermitian type. This means that the corresponding Riemannian symmetric space G/K is a complex bounded symmetric domain. In terms of group theory, this is equivalent to saying that the center $\mathfrak{c}(\mathfrak{k})$ of $\mathfrak{k}$ is non-trivial. Moreover, it is well-known that $\dim \mathfrak{c}(\mathfrak{k}) = 1$ and every $\mathrm{ad}\, z|_{\mathfrak{p}}$, $z \in \mathfrak{c}(\mathfrak{k})\backslash\{0\}$ is regular.

Example. The following Hermitian Lie algebras are the classification of irreducible cases:

$$\mathfrak{g} = \mathfrak{sp}(n, \mathbb{R}),\ \mathfrak{su}(p, q),\ \mathfrak{so}^*(2n),\ \mathfrak{so}(2, n),\ \mathfrak{e}_{6(-14)},\ \mathfrak{e}_{7(-25)}.$$

Now, since G is reductive, the adjoint representation $\mathrm{Ad} : G \to \mathrm{GL}(\mathfrak{g})$ is isomorphic to the the contragradient representation $\mathrm{Ad}^* : G \to \mathrm{GL}(\mathfrak{g}^*)$ through the bijection $\mathfrak{g} \simeq \mathfrak{g}^*$ given by an $\mathrm{Ad}(G)$-invariant non-degenerate bilinear form on $\mathfrak{g}$ (e.g. the Killing form if $\mathfrak{g}$ is semisimple). Therefore we can identify the adjoint orbit with the coadjoint orbit provided G is real reductive. We consider the projection $\mathrm{pr} : \mathfrak{g} \to \mathfrak{h}$, which is defined by means of invariant bilinear form on $\mathfrak{g}$. Then the pullback $\mathrm{pr}^{-1}(\mathcal{O}_y^H)$ $(y \in \mathfrak{h})$ is $\mathrm{Ad}(H)$-stable.

For a non-zero $z \in \mathfrak{c}(\mathfrak{k})$ we consider the orbit

$$\mathcal{O}_z^G := \mathrm{Ad}(G)\, z \subset \mathfrak{g}.$$

Then, we rewrite Conjecture 1 in the following equivalent (simpler) form:

Conjecture 2. Let G be Hermitian, (G, H) a symmetric pair, and z the central element of $\mathfrak{k}$ as above. Then for any $y \in \mathfrak{h}$,

$$\sharp\big(\mathcal{O}_z^G \cap \mathrm{pr}^{-1}(\mathcal{O}_y^H)/\,\mathrm{Ad}(H)\big) \leq 1.$$

That is, $\mathcal{O}_z^G \cap \mathrm{pr}^{-1}(\mathcal{O}_y^H)$ is a single H-orbit, provided it is not empty.

Example. Suppose $G = \mathrm{SL}(2,\mathbb{R})$, $\mathrm{SU}(2)$ respectively and $H = K = \mathrm{SO}(2)$. Let X be a central element in $\mathfrak{k}$. Then in each case, for any $y \in \mathfrak{h}$ we have

$$O_X^G \cap \mathrm{pr}^{-1}(\mathcal{O}_y^H) = \text{ circle } = \text{single } \mathrm{SO}(2)\text{-orbit.}$$

In my Ph.D thesis [14] I have proved Conjecture 2 (and thus also Conjecture 1) in the Riemannian symmetric pair. That is, my main result can be stated as
Theorem. Kobayashi's Conjecture is true if $H = K$.

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