表現論シンポジウム講演集, 2002
pp.179-185

On the decomposition of the tensor K-modules

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§ 1. Decomposition of the K-module $\mathfrak{p}_{\mathbf{C}} \otimes V_\mu$.

Let $\mathbf{C}$(resp.$\mathbf{R}$) denote the complex (resp. real)number field. We consider a connected simply connected complex simple Lie group $G_{\mathbf{C}}$ and a connected noncompact inner type simple real form G of $G_{\mathbf{C}}$. Let K be a maximal compact subgroup of G. We denote the Lie algebras of G and K respectively by $\mathfrak{g}$ and $\mathfrak{k}$. Let θ be the Cartan involution of $\mathfrak{g}$ corresponding to $\mathfrak{k}$. Let's denote the eigensubspace of θ of $\mathfrak{g}$ with the eigenvalue -1 by $\mathfrak{p}$. Then we have a Cartan decomposition: $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$. Consequently the Lie algebra $\mathfrak{g}_{\mathbf{C}}$ of $G_{\mathbf{C}}$ is also decomposed by $\mathfrak{g}_{\mathbf{C}} = \mathfrak{k}_{\mathbf{C}} \oplus \mathfrak{p}_{\mathbf{C}}$, where $\mathfrak{k}_{\mathbf{C}}$ (resp. $\mathfrak{p}_{\mathbf{C}}$) is the complexification of $\mathfrak{k}$ (resp. $\mathfrak{p}$) in $\mathfrak{g}_{\mathbf{C}}$. Canonically K acts on the space $\mathfrak{p}_{\mathbf{C}}$. Let B be a maximal abelian subgroup of K. Since K is connected and G is an inner type simple Lie group, B is also a maximal abelian subgruop of G. Therefore B is a Cartan subgroup of G and K. Let $\mathfrak{b}_{\mathbf{C}}$ be the complexification of the Lie algebra $\mathfrak{b}$ of B. Let Σ be the root system of the pair $(\mathfrak{g}_{\mathbf{C}}, \mathfrak{b}_{\mathbf{C}})$. Then we have $\Sigma = \Sigma_K \cup \Sigma_n$, where Σ_K (resp. Σ_n) is the set of all compact (resp. noncompact) roots of Σ . We shall fix a positive root system P_K of Σ_K.

Let (π_μ, V_μ) be a simple K-module with the highest weight μ. In this article we shall state our results for the decompositins of the tensor K-modules $\mathfrak{p}_{\mathbf{C}} \otimes V_\mu$ and $\mathfrak{p}_{\mathbf{C}} \otimes \mathfrak{p}_{\mathbf{C}} \otimes V_\mu$. These K-modules are closely related with the classification of infinitesimal irreducible unitary representations of G. For example, by using the Clebsch-Gordan coefficients of the tensor K-moduls, the complete classification are obtained for the cases: $SL(2, \mathbf{R})$ in [1], De Sitter group in [2] and [9], SO(2n,1) in [5], [6], SU(n,1) in [7] and etc.

Let ν be a P_K-dominant integral form on $\mathfrak{b}_{\mathbf{C}}$ and (π_ν, V_ν) a simple K-module corresponding to ν. We define a projection operator P_ν on the K-module $\mathfrak{p}_{\mathbf{C}} \otimes V_\mu$ by

$$P_\nu(Z) = deg\pi_\nu \int_K kZ\overline{trace\pi_\nu(k)}dk \text{ for } Z \text{ in } \mathfrak{p}_{\mathbf{C}} \otimes V_\mu,$$

where dk is the Haar measure on K normalized as $\int_K dk = 1$. Let Γ_K be the set of all P_K-dominant integral form on $\mathfrak{b}_{\mathbf{C}}$. Then we have the following decomposition :

$$\mathfrak{p}_{\mathbf{C}} \otimes V_\mu = \oplus_{\omega \in \Sigma_n, \mu+\omega \in \Gamma_K} P_{\mu+\omega}(\mathfrak{p}_{\mathbf{C}} \otimes V_\mu), \tag{1.1}$$

where $P_{\mu+\omega}(\mathfrak{p}_{\mathbf{C}} \otimes V_\mu) = \{0\}$ or is a simple K-module. We shall give a characterization for this decomposion by using a rational function.

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Let $(\sqrt{-1}\mathfrak{b})^*$ be the dual space of the real vector space $\sqrt{-1}\mathfrak{b}$ and $\mathbf{R}[\eta]$ the polynomial ring in $\eta \in (\sqrt{-1}\mathfrak{b})^*$ over the real number field $\mathbf{R}$. We denote by $\mathbf{R}(\eta)$ the quotient field of $\mathbf{R}[\eta]$. Let Σ be the set of all roots on $\mathfrak{b}_{\mathbf{C}}$. Then we have

$$\mathfrak{g}_{\mathbf{C}} = \mathfrak{b}_{\mathbf{C}} \oplus \sum_{\alpha \in \Sigma} \mathfrak{g}_{\alpha},$$

where $\mathfrak{g}_\alpha$ is a one dimensional eigenspace corresponding to α. The real subalgebra $\mathfrak{g}_u = \mathfrak{k} \oplus \sqrt{-1}\mathfrak{p}$ of $\mathfrak{g}_{\mathbf{C}}$ is said to be a compact real form of $\mathfrak{g}_{\mathbf{C}}$. We choose a Weyl basis $X_\alpha \in \mathfrak{g}_\alpha, \alpha \in \Sigma$, satisfying the followings (cf. the proof of Theorem 6.3 in [4]).

$$X_\alpha - X_{-\alpha},\ \sqrt{-1}(X_\alpha + X_{-\alpha}) \in \mathfrak{g}_u \text{ and } \phi(X_\alpha, X_{-\alpha}) = 1,$$

where ϕ is the Killing form on $\mathfrak{g}_{\mathbf{C}}$. For the element $H_\alpha = ad(X_\alpha)X_{-\alpha}$ in $\sqrt{-1}\mathfrak{b}$, we have $\phi(H_\alpha, H) = \alpha(H)$ for all H in $\mathfrak{b}_{\mathbf{C}}$. Let μ be a linear form on $\sqrt{-1}\mathfrak{b}$. Then there exists a unique H_μ in $\sqrt{-1}\mathfrak{b}$ such that $\phi(H_\mu, H) = \mu(H)$ for all H in $\sqrt{-1}\mathfrak{b}$. Let $(\sqrt{-1}\mathfrak{b})^*$ be the dual space of $\sqrt{-1}\mathfrak{b}$. We define a positive definite bilinear form (λ, μ) by $(\lambda, \mu) = \phi(H_\mu, H_\lambda)$ for $\lambda, \mu \in (\sqrt{-1}\mathfrak{b})^*$. We put for each pair of α and β in Σ, a complex number $< \alpha, \beta >$ by

$$< \alpha, \beta > = \begin{cases} \phi(ad(X_\alpha)X_\beta, X_{-\alpha-\beta}) & \text{if } \alpha + \beta \in \Sigma \\ 0 & \textit{otherwise.} \end{cases}$$

Then $< \alpha, \beta >$ is a pure imaginary number. Let τ be the conjugation of $\mathfrak{g}_{\mathbf{C}}$ with respect to the real form $\mathfrak{g}_u$. By our choice for the Weyl basis of $\mathfrak{g}_{\mathbf{C}}$ we have

$$\tau(X_\alpha) = -X_{-\alpha} \text{ for } \alpha \in \Sigma.$$

We define a hermitian structure (X, Y) on $\mathfrak{p}_{\mathbf{C}}$ by

$$(X, Y) = -\phi(X, \tau(Y)) \text{ for } X, Y \in \mathfrak{p}_{\mathbf{C}}. \tag{3.1}$$

Thereby $\mathfrak{p}_{\mathbf{C}}$ is a unitary K-module. We can prove that $P_{\mu+\omega}(\mathfrak{p}_{\mathbf{C}} \otimes V_\mu) \neq \{0\}$ if and only if $P_{\mu+\omega}(X_\omega \otimes v(\mu)) \neq 0$, where $v(\mu)$ is the highest weight vector in V_μ normalized as $|v(\mu)| = 1$.

THEOREM I. *Let $\mu \in \Gamma_K$ and (π_μ, V_μ) a simple K-module with the highest weight μ. Consider a noncompact root ω in Σ satisfying $\mu + \omega \in \Gamma_K$ and $P_{\mu+\omega}(\mathfrak{p}_{\mathbf{C}} \otimes V_\mu) \neq \{0\}$. Then there exists a rational function $f(\eta; \omega) \in \mathbf{R}(\eta)$ such that $|P_{\mu+\omega}(X_\omega \otimes v(\mu))|^2 = f(\lambda + \omega; \omega)$, where $\lambda = \mu + \rho_K$, ρ_K is one half the sum of all roots in P_K.*

Outline of the proof. Fiest we shall define a rational function $f(\eta; \omega)$. Let p be a nonnegative integer. We define a set Π_p by

$$\Pi_0 = \{\tilde{\phi}\},\ \Pi_p = \{(\alpha_1, \alpha_2, \ldots, \alpha_p) : \alpha_i \in P_K\} \text{ for } p > 1,$$
$$\text{and put } \Pi = \cup_{p=0}^{\infty} \Pi_p.$$

Let $I = (\alpha_1, \alpha_2, ..., , \alpha_p)$ and $J = (\beta_1, \beta_2, ..., \beta_q)$ be two elements in Π. We define a multiplicative operation $\star$ in Π by

$$I \star J = (\alpha_1, \alpha_2, ..., \alpha_p, \beta_1, \beta_2, ..., \beta_q).$$

Then Π is a semigroup with the identity $\tilde{\phi}$. Let $U(\mathfrak{k}_{\mathbf{C}})$ be the universal enveloping algebra of $\mathfrak{k}_{\mathbf{C}}$. For each I in Π we define an element $Q(I)$ in $U(\mathfrak{k}_{\mathbf{C}})$ by

$$Q(I) = 1 \text{ for } I = \tilde{\phi} \text{ and } Q(I) = X_{-\alpha_1} X_{-\alpha_2} ... X_{-\alpha_p} \text{ for } I = (\alpha_1, \alpha_2, ..., \alpha_p).$$

Then Q is a semigroup homomorphism of Π to $U(\mathfrak{k}_{\mathbf{C}})$. Furthermore, $Q(I)$ acts on $\mathfrak{p}_{\mathbf{C}}$ by $Q(I)X = ad(Q(I))X$ for X in $\mathfrak{p}_{\mathbf{C}}$. We also define the adioint operator $Q(I)^*$ of $Q(I)$ by $(Q(I)X, Y) = (X, Q(I)^* Y)$ for $X, Y \in \mathfrak{p}_{\mathbf{C}}$.

Definition 1.1. *For a generic point* $\eta \in (\sqrt{-1}\mathfrak{b})^*$, $\omega \in \Sigma_n$ *and* $I \in \Pi$, *we define* $R(\eta; I), S(\eta; I), T(\eta; I)$, *and* $f(\eta; I)$ *as follows:* $R(\eta; \widehat{\phi}) = S(\eta; \widehat{(\phi)}) = T(\eta; \widehat{\phi}) = a_\omega(\widehat{\phi}) = 1$ *and for* $I = (\alpha_1, \alpha_2, ..., \alpha_p) \in \Pi$

$$\begin{aligned}
R(\eta; I) &= (|\eta + < I >|^2 - |\eta|^2)^{-1}, \\
S(\eta; I) &= \prod_{J, L \in \Pi, J \star L = I, J \neq \widehat{\phi}} R(\eta; J), \\
T(\eta; I) &= \prod_{J, L \in \Pi, J * L = I} R(\eta + < J >; L), \\
a_\omega(I) &= 2^{\sharp I} |\phi(Q(I)^* X_\omega, X_{-\omega - <I>})|^2, \\
f(\eta; \omega) &= \sum_{I \in \Pi} (-1)^{\sharp I} a_\omega(I) S(\eta; I),
\end{aligned}$$

where $\sharp I = p$ *and* $< I > = \sum_{i=1}^{p} \alpha_i$.

Then Theorem I can be proved by using three lemmas below.

Lemma 1.2. *Let* $\mu \in \Gamma_K$, *and assume that* $\mu + \gamma \in \Gamma_K$ *for all* $\gamma \in \Sigma_n$. *Then, for* $\omega \in \Sigma_n$, *we have* $|P_{\mu+\omega}(X_\omega \otimes v(\mu))|^2 = f(\lambda + \omega; \omega)$.

Lemma 1.3. *Let* $\omega \in \Sigma_n$, *and assume* $P_{\mu+\omega}(\mathfrak{p}_{\mathbf{C}} \otimes V_\mu) \neq \{0\}$. *Then we have*

$$f(-\lambda - \omega; -\omega) |P_{\mu+\omega}(X_\omega \otimes v(\mu))|^2 = \prod_{\alpha \in P_K} (\lambda + \omega, \alpha)(\lambda, \alpha)^{-1}.$$

Lemma 1.4. *Let* ω *be an element in* Σ_n. *Then we have the following functional equations in* $\mathbf{R}(\eta)$.

$$\prod_{\alpha \in P_K} (\eta, \alpha) f(\eta + \omega; \omega) = \prod_{\alpha \in P_K} (\eta + \omega, \alpha) f(\eta; -\omega), \tag{1.2}$$

$$f(\eta + \omega; \omega) f(-\eta - \omega; -\omega) = \prod_{\alpha \in P_K} (\eta + \omega, \alpha)(\eta, \alpha)^{-1}. \tag{1.3}$$

REMARK. *(1.2) is a modefied version of a result due to N. Tatsuuma (cf. [8]).*

We shall give a product formula for $f(\eta;\omega)$. Let ω be a noncompact root in Σ. We define the subsets $\Delta(\omega), \Delta_-(\omega), \Delta_m(\omega)$ and $\Delta_m(\omega)^*$ of P_K, where m is an integer, by

$$\begin{aligned}
\Delta(\omega) &= \{\alpha \in P_K : \omega + \alpha \in \Sigma\},\\
\Delta_-(\omega) &= \{\alpha \in P_K : (\omega, \alpha) > 0\},\\
\Delta_m(\omega) &= \{\alpha \in P_K : 2(\omega, \alpha)|\alpha|^{-2} = m, \omega + \alpha \in \Sigma\},\\
\Delta_m(\omega)^* &= \{\alpha \in \Delta_m(\omega) : \omega - \alpha \in \Sigma\}.
\end{aligned}$$

Then by using the classification of the inner type noncompact real simple Lie groups we have $\Delta(\omega) = \Delta_-(\omega) \cup \Delta_0(\omega)^* \cup \Delta_1(\omega)^*$.

THEOREM II. *Let ω a noncompact root. We define $f(\eta;\omega)$ by Definition 1.1. Then $f(\eta;\omega)$ has one of the following product formulae.*

(1) *If $\Delta_0(\omega)^* \cup \Delta_{-1}(\omega)^* \cup \Delta_1(\omega)^* = \phi$, then*

$$f(\eta + \omega; \omega) = \prod_{\alpha \in \Delta_-(\omega)} (\eta + \omega, \alpha)(\eta, \alpha)^{-1}.$$

(2) *If $\Delta_0(\omega)^* \neq \phi$, then $\Delta_1(\omega)^* \cup \Delta_{-1}(\omega)^* = \phi$ and*

$$f(\eta + \omega; \omega) = \prod_{\alpha \in \Delta_0(\omega)^*} (2(\eta, \alpha) - |\alpha|^2)(2(\eta, \alpha) + |\alpha|^2)^{-1} \times \prod_{\alpha \in \Delta_{-1}(\omega)} (\eta + \omega, \alpha)(\eta, \alpha)^{-1}.$$

(3) *If $\Delta_1(\omega)^* \cup \Delta_{-1}(\omega)^* \neq \phi$, then $\Delta_0(\omega)^* = \phi$ and*

$$f(\eta + \omega; \omega) = \prod_{\alpha \in \Delta_-(\omega)} (\eta + \omega, \alpha)(\eta, \alpha)^{-1} \prod_{\alpha \in \Delta_1(\omega)^*} (2(\eta, \alpha) - |\alpha|^2)(2\{(\eta, \alpha) + |\alpha|^2\})^{-1} \times \prod_{\alpha \in \Delta_{-1}(\omega)^*} 2\{(\eta, \alpha) - |\alpha|^2\}(2(\eta.\alpha) + |\alpha|^2)^{-1}.$$

The formulae in Theorem II is proved mainly by using the identities in Lemma 1.4.

REMARK. *Theorem I and Theorem II are reported in " Clebsch-Gordan coefficients for a tensor product representation $Ad \otimes \pi$ of a maximal compact subgroup of real semisimple Lie group", Lect. in Math., Kyoto univ. No. 14 pp.149-175.*

DEFINITION 1.5. *Let $\mu \in \Gamma_K$, and define the following six sets for $\lambda = \mu + \rho_K$.*

$$\begin{aligned}
w(\lambda) &= \{\lambda + \omega : \omega \in \Sigma_n\},\\
sw(\lambda) &= \{\xi \in w(\lambda) : \prod_{\alpha \in P_K} (\xi, \alpha) = 0\},\\
rw(\lambda) &= \{\xi \in w(\lambda) : \prod_{\alpha \in P_K} (\xi, \alpha) \neq 0\},\\
rw_0(\lambda) &= \{\lambda + \omega \in rw(\lambda) : f(\lambda + \omega; \omega) = 0\},\\
rw_+(\lambda) &= \{\lambda + \omega : \mu + \omega \in \Gamma_K, f(\lambda + \omega; \omega) > 0\},\\
rw_-(\lambda) &= rw(\lambda) \backslash (rw_0(\lambda) \cup rw_+(\lambda)).
\end{aligned}$$

THEOREM III. *Let ω be a noncompact root in Σ_n satisfying $\mu + \omega \in \Gamma_K$ and $f(\lambda + \omega; \omega) > 0$. Then we have $P_{\mu+\omega}(\mathfrak{p}_{\mathbf{C}} \otimes V_\mu) \neq \{0\}$.*

Outlinee of a proof. Choosing a suitable covering group K^* of K, we can define the character ξ_{ρ_K} of the analytic subgroup B^* of K^* corresponding to $\mathfrak{b}$. By Weyl's character formula we have

$$(\Delta_K trace(Ad \otimes \pi_\mu))(expH) = \sum_{\lambda+\omega \in w(\lambda)} \sum_{t \in W_K} \epsilon(t) e^{t(\lambda+\omega)(H)}$$

for all $expH \in B^*$. We shall prove that

$$(\Delta_K trace(Ad \otimes \pi_\mu))(expH) = \sum_{\lambda+\omega \in rw_+(\lambda)} \sum_{t \in W_K} \epsilon(t) e^{t(\lambda+\omega)(H)}. \tag{1.4}$$

If $\lambda + \omega \in w(\lambda)$ is P_K-singular, then

$$\sum_{t \in W_K} \epsilon(t) e^{t(\lambda+\omega)(H)} = 0.$$

By using Theorem II we can prove

$$\sum_{\lambda+\omega \in rw_0(\lambda) \cup rw_-(\lambda)} \sum_{t \in W_K} \epsilon(t) e^{t(\lambda+\omega)(H)} = 0. \tag{1.5}$$

Since $w(\lambda) = sw(\lambda) \cup rw_0(\lambda) \cup rw_-(\lambda) \cup rw_+(\lambda)$,

$$trace(Ad \otimes \pi_\mu)(k) = \sum_{\mu+\omega \in \Gamma_K, f(\lambda+\omega;\omega)>0} trace \pi_{\mu+\omega}(k).$$

Thus if $\mu + \omega \in \Gamma_K$ and $f(\lambda + \omega; \omega) > 0$ then $P_{\mu+\omega}(\mathfrak{p}_{\mathbf{C}} \otimes V_\mu) \neq \{0\}$ as claimed.

§ 2. The multiplicity of V_μ in $P_\mu(\mathfrak{p}_{\mathbf{C}} \otimes \mathfrak{p}_{\mathbf{C}} \otimes V_\mu)$.

We consider the tensor K-module $\mathfrak{p}_{\mathbf{C}} \otimes \mathfrak{p}_{\mathbf{C}} \otimes V_\mu$. Then each simple submodule in $P_\mu(\mathfrak{p}_{\mathbf{C}} \otimes \mathfrak{p}_{\mathbf{C}} \otimes V_\mu)$ is K-isomophic to V_μ. Therefore

$$P_\mu(\mathfrak{p}_{\mathbf{C}} \otimes \mathfrak{p}_{\mathbf{C}} \otimes V_\mu) \cong m(\mu)V_\mu,$$

where $m(\mu)$ is the multiplicity of V_μ. One of our purposes of this section is to determine the number $m(\mu)$. Let us state our results more precisely after the followings. Let H_μ be the element in $\mathfrak{b}_{\mathbf{C}}$ satisfying $\phi(H_\mu, H) = \mu(H)$ for all $H \in \mathfrak{b}_{\mathbf{C}}$. Then the centralizer $K(\mu)$ of H_μ in K is reductive, and contains B. Let $\Sigma_{K(\mu)}$ be the root system of the pair $(\mathfrak{k}(\mu)_{\mathbf{C}}, \mathfrak{b}_{\mathbf{C}})$, where $\mathfrak{k}(\mu)$ is the Lie algebra of $K(\mu)$. We put $P_{K(\mu)} = P_K \cap \Sigma_{K(\mu)}$. Then $P_{K(\mu)}$ is a positive root system of $\Sigma_{K(\mu)}$. A noncompact root $\omega \in \Sigma_n$ is said to be $P_{K(\mu)}$-highest if $\omega + \alpha \notin \Sigma$ for all α in $P_{K(\mu)}$.

DEFINITION 2.1. *An element μ in Γ_K is admissible if μ has the following properties: (1) For one of the groups $S_p(n,\mathbf{R})andSO(2m,2n+1)$, $2(\mu,\alpha)|\alpha|^{-2} \geq 2$ for all short roots α in $P_K \backslash P_{K(\mu)}$. (2) For the group G_2, $2(\mu,\alpha)|\alpha|^{-2} \geq 3$ for a short root α in $P_K \backslash P_{K(\mu)}$.*

REMARK. *If G satisfies that all noncompact roots in Σ have the same length then we have no assumptions for the admissibilty of μ.*

The following Theorem IV and Theorem VI are proved by using three theorems in §1.

THEOREM IV. *Let $\mu \in \Gamma_K$, and assume that μ is admissible. Then the multiplicity $m(\mu)$ of V_μ in $P_\mu(\mathfrak{p}_{\mathbf{C}} \otimes \mathfrak{p}_{\mathbf{C}} \otimes V_\mu)$ is given by*

$$m(\mu) = \sharp\{\omega \in \Sigma_n : \omega \text{ is } K(\mu)\text{-highest } \},$$

where $\sharp A$ is the number of the elements in a set A.

Let P be a positive root system containing P_K. For a subset Θ in the simple root system Ψ of P, we denote by $P(\Theta)$ the set of all positive roots in P generated by Θ over the ring of integers. Let C be the positive Weyl chamber of $\sqrt{-1}\mathfrak{b}$ corresponding to P. We define a subset $C(\Theta)$ contained in the topological closure $cl(C)$ of C by

$$C(\Theta) = \{H \in \sqrt{-1}\mathfrak{b} : \alpha(H) = 0 \text{ for all } \alpha \in P(\Theta) \text{ and } \alpha(H) > 0 \text{ for all } \alpha \in P\backslash P(\Theta)\}.$$

Let H_0 be an element in $C(\Theta)$. Then the centralizer $M(\Theta)$ of H_0 in G is a reductive subgroup of G with a Cartan subgroup B. $M(\Theta)$ is uniquely determined by $C(\Theta)$. Let $\mathfrak{p}^+$ be the subspace of $\mathfrak{p}_{\mathbf{C}}$ generated by the set of all root vectors cprresponding to $P \cap \Sigma_n$. Let τ be the conjugation of $\mathfrak{g}_{\mathbf{C}}$ with respect to the compact real form $\mathfrak{g}_u$. A simple $K(\Theta)$-submodule $\mathfrak{q}$ of $\mathfrak{p}_{\mathbf{C}}$ is said to be the first (resp. the second) kind if $\tau(\mathfrak{q}) = \mathfrak{q}$ (resp. $\mathfrak{q} \subset \mathfrak{p}^+$ or $\tau(\mathfrak{q}) \subset \mathfrak{p}^+$). A noncompact root ω in P is said to be the first (resp. second) kind if ω is a weight of a simple $K(\Theta)$-submodule of $\mathfrak{p}_{\mathbf{C}}$ of the first (resp. the second) kind.

Definition 2.2. *The triple* $(P_K, P(\Theta), P)$ *is standard if each simple* $K(\Theta)$*-submodule* $\mathfrak{q}$ *of* $\mathfrak{p}_{\mathbf{C}}$ *is the first kind or the second kind.*

Theorem V. *For* $\mu \in \Gamma_K$ *there exists a standard triple* $(P_K, P(\Theta), P)$ *such that* $\mu \in C(\Theta)$.

Let $(P_K, P(\Theta), P)$ be a standard triple. We consider an element μ in $\Gamma_K \cap C(\Theta)$ and a noncompact root ω satisfying $\mu + \omega \in \Gamma_K$. We now define a projection operator $P_{\mu+\omega}$ on $\mathfrak{p}_{\mathbf{C}} \otimes V_\mu$ by the same as in (1.1). We put

$$P_\mu^+ = \sum_{\omega \in \Sigma_n \cap P, \mu+\omega \in \Gamma_K} P_{\mu+\omega}.$$

Let us define a K-submodule $N(\mu)$ of $P_\mu(\mathfrak{p}_{\mathbf{C}} \otimes \mathfrak{p}_{\mathbf{C}} \otimes V_\mu)$ by

$$N(\mu) = \text{ the } K\text{-module generated by the set}$$
$$\{P_\mu(X \otimes P_\mu^+(Y \otimes v) - Y \otimes P_\mu^+(X \otimes v)) : X, Y \in \mathfrak{p}_{\mathbf{C}}, v \in V_\mu\}.$$

Theorem VI. *Let* $(P_K, P(\Theta), P)$ *be a standard triple and* $\mu \in \Gamma_K \cap C(\Theta)$. *Suppose that* μ *is sufficiently* $P_K \backslash P_{K(\Theta)}$*-regular. Then* μ *is admissible. Furthermore, we have*

$$n(\mu) = \sharp\{\omega \in P \cap \Sigma_n : \omega \text{ is } P_{K(\Theta)}\text{-highest and the second kind }\},$$

where $n(\mu)$ is the multiplicity of V_μ in $N(\mu)$.

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