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Characters of wreath products of compact groups with the infinite symmetric group

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ABSTRACT. Characters of wreath products $G = \mathfrak{S}_\infty(T)$ of compact groups T with the infinite symmetric group $\mathfrak{S}_\infty$ are studied, and all the extremal continuous positive definite class functions on G are explicitly given (joint work with E. Hirai).

The case of the infinite symmetric group $\mathfrak{S}_\infty$ has been worked out early in 60's by E. Thoma in [Th2], and it is reexamined in [VK], [KO], [Bi] etc. from the point of view of approximation from $\mathfrak{S}_n$ $(n \to \infty)$, and recently in [Hi3]–[Hi4] from the standpoint of taking limits of centalizations of positive definite functions obtained as matrix elements of simple unitary representations.

The case where T is a finite abelian group, contains the cases of infinite Weyl groups and the limits $\mathfrak{S}_\infty(\boldsymbol{Z}_r) = \lim_{n\to\infty} G(r,1,n)$ of complex reflexion groups. For this abelian case a general explicit formula for characters is given, and so all the factor representations of finite type are classified in [HH1]. The case of $\mathfrak{S}_\infty(T)$ with T any finite group was treated in [HH2].

1 Wreath products of compact groups with the infinite symmetric group

For a set I, we denote by $\mathfrak{S}_I$ the group of all finite permutations on A. A permutation σ on I is called finite if its support $\mathrm{supp}(\sigma) := \{\, i \in I \,;\, \sigma(i) \neq i \,\}$ is finite. We call the *infinite symmetric group* the permutation group $\mathfrak{S}_{\boldsymbol{N}}$ on the set of natural numbers $\boldsymbol{N}$. The index $\boldsymbol{N}$ is frequently replaced by ∞. The symmetric group $\mathfrak{S}_n$ is naturally imbedded in $\mathfrak{S}_\infty$ as the permutation group of the set $I_n := \{\, 1, 2, \ldots, n \,\} \subset \boldsymbol{N}$.

Let T be a compact group. We consider a wreath product group $\mathfrak{S}_I(T)$ of T with a permutation group $\mathfrak{S}_I$ as follows:

$$\mathfrak{S}_I(T) = D_I(T) \rtimes \mathfrak{S}_I\,, \quad D_I(T) = \textstyle\prod'_{i\in I} T_i\,, \quad T_i = T \;(i \in I), \tag{1}$$

where the symbol $\prod'$ means the restricted direct product, and $\sigma \in \mathfrak{S}_I$ acts on $D_I(T)$ as

$$D_I(T) \ni d = (t_i)_{i\in I} \stackrel{\sigma}{\longmapsto} \sigma(d) = (t'_i)_{i\in I} \in D_I(T), \quad t'_i = t_{\sigma^{-1}(i)} \;(i \in I). \tag{2}$$

Identifying groups $D_I(T)$ and $\mathfrak{S}_I$ with their images in semidirect product $\mathfrak{S}_I(T)$, we have $\sigma\, d\, \sigma^{-1} = \sigma(d)$. The group $\mathfrak{S}_{I_n}(T)$ is denoted as $\mathfrak{S}_n(T)$, then $G := \mathfrak{S}_\infty(T)$ is an inductive limit of $G_n := \mathfrak{S}_n(T)$. Since T is compact, G_n is also compact, and we introduce in G its inductive limit topology τ_{ind}. Then G with τ_{ind} becomes a topological groups (cf. Theorem 2.7 in [TSH]). Recall that a subset $B \subset G$ is τ_{ind}-open if and only if $B \cap G_n$ is open in G_n for any $n \geq 1$. When T is a finite group, the topology τ_{ind} in G is again discrete.

A natural subgroup of $G := \mathfrak{S}_\infty(T)$ is given as a wreath product of T with the alternating group $\mathfrak{A}_\infty$ as $G' := \mathfrak{A}_\infty(T) = D_\infty(T) \rtimes \mathfrak{A}_\infty$.

In the case where T is abelian, we put

$$P_I(d) = \textstyle\prod_{i\in I} t_i \quad \text{for} \quad d = (t_i)_{i\in I} \in D_I(T), \tag{3}$$

and define a subgroup of $\mathfrak{S}_I(T)$ as

$$\mathfrak{S}_I^e(T) = D_I^e(T) \rtimes \mathfrak{S}_I \quad \text{with} \quad D_I^e(T) := \{\, d = (t_i)_{i \in I} \ ; \ P_I(d) = e_T \,\}, \tag{4}$$

where e_T denotes the identity element of T.

This kind of groups $\mathfrak{S}_\infty(T)$ and $\mathfrak{S}_\infty^e(T)$ with T abelian, contain the infinite Weyl groups of classical types, $W_{\mathbf{A}_\infty} = \mathfrak{S}_\infty$ of type $\mathbf{A}_\infty$, $W_{\mathbf{B}_\infty} = \mathfrak{S}_\infty(\boldsymbol{Z}_2)$ of type $\mathbf{B}_\infty/\mathbf{C}_\infty$, and $W_{\mathbf{D}_\infty} = \mathfrak{S}_\infty^e(\boldsymbol{Z}_2)$ of type $\mathbf{D}_\infty$, and moreover the inductive limits $\mathfrak{S}_\infty(\boldsymbol{Z}_r) = \lim_{n\to\infty} G(r,1,n)$ of complex reflexion groups $G(r,1,n) = \mathfrak{S}_n(\boldsymbol{Z}_r)$ (cf. [Ka], [Sh]).

In general, for a topological group G, let $E(G)$ be the set of all indecomposable (or extremal) continuous positive definite class functions on G. Then every $f \in E(G)$ gives canonically a character of a quasi-equivalent class of factor represetations of G of finite type, type $\mathrm{I}_n, n < \infty$, or II_1, and is called itself a character of such representations (see §2 below). When G is locally compact and in particular discrete, $E(G)$ covers all the characters of factor representations of finite type. This is the case of $G = \mathfrak{S}_\infty(T)$ with T finite. When T is not finite, this group G with the topology τ_{ind} is no longer locally compact and the problem of such completeness of $E(G)$ is left open (cf. Question 2002-7 in §2).

T compact
$\mathfrak{S}_\infty(T) \supset T$ open compact

The purpose of the present paper is to give explicitly all the elements f in $E(G)$ ~~for this non-locally-compact case~~. The case with a finite group T, has been treated in [Th2], [Hi3]–[Hi4] and [HH1]–[HH2]. Many of discussions in our previous papers in discrete case can be transferred to the case of a general compact group T.

2 Preliminaries for characters of locally compact groups

Here we give some preliminaries on characters of factor representations of topological groups and especially of inductive limits of locally compact groups. We refer [GR], [Di, §6, §13, §17] and [Th1].

2.1. Continuous positive definite functions. Firstly let G be a topological group. Denote by $\mathfrak{F}(G)$ a $*$-algebra of all functions on G zero outside a finite number of points, with operations

$$\varphi * \psi(g) = \sum_{h \in G} \varphi(gh^{-1})\,\psi(h), \qquad \varphi^*(g) = \overline{\varphi(g^{-1})} \qquad (\varphi, \psi \in \mathfrak{F}(G), g \in G).$$

A function f on G is called positive definite if the hermitian inner product

$$\begin{aligned} \langle \varphi, \psi \rangle_f &:= f(\varphi * \psi^*) = \sum_{g \in G} f(g)(\varphi * \psi^*)(g) \\ &= \sum_{g,h \in G} f(h^{-1}g)\varphi(g)\overline{\psi(h)} \qquad \text{for} \quad \varphi, \psi \in \mathfrak{F}(G) \end{aligned} \tag{5}$$

is positive definite. Let $\mathcal{P}(G)$ be the set of all *continuous* positive definite functions f on G, and $\mathcal{P}_1(G)$ be the set of all such f's that $f(e) = 1$, where e denotes the identity element of G. Let f and f' be positive definite functions on G, then f' is *majorized* by f if, for some $\lambda > 0$, $\lambda f - f'$ is again positive definite. *Let f' be positive definite and majorized by some $f \in \mathcal{P}(G)$, then f' is necessarily continuous, that is, $f' \in \mathcal{P}(G)$* [GR, p.3]. An element $f \in \mathcal{P}(G)$ is called *elementary* if any $f' \in \mathcal{P}(G)$ majorized by f is a scalar multiple of f. The set $E(G)$ of all extremal points in the convex set $\mathcal{P}_1(G)$ is equal to the set of all normalized elementary elements (cf. [GR]).

Let π be a unitary representation of G on a Hilbert space $V(\pi)$ for which the strong continuity of $G \ni g \mapsto \pi(g)$ is assumed by definition. Assume π be cyclic and take a normalized cyclic vector $v_0 \in V(\pi)$, $\|v_0\| = 1$. Then,

$$f(g) = \langle \pi(g)v_0, v_0 \rangle \qquad (g \in G) \tag{6}$$

is in $\mathcal{P}_1(G)$. Here f (resp. π) is said to be *associated* to π (resp. f).

Conversely, take an $f \in \mathcal{P}_1(G)$. Then, by GNS construction, we have a cyclic unitary representation $(\pi_f, \mathfrak{H}_f)$ associated to f through (6). Let us recall it. Introduce in $\mathfrak{F}(G)$ a hermitian positive definite inner product by (5). Then it is invariant under G through the left translation $(L(g_0)\varphi)(g) := \varphi(g_0{}^{-1}g)$ $(g_0, g \in G)$. Let J_f be the kernel of $\langle \cdot, \cdot \rangle_f$, that is, $J_f = \{ \varphi \in \mathfrak{F}(G) \; ; \; \langle \varphi, \psi \rangle_f = 0 \; (\psi \in \mathfrak{F}(G)) \}$. Then we get a strictly positive definite inner product on $\mathfrak{F}(G)/J_f$, denoted again by the same symbol. A Hilbert space $\mathfrak{H}_f$ is obtained as a completion of $\mathfrak{F}(G)/J_f$, and on it a unitary representation π_f is induced from $L(g), g \in G$. For $\varphi \in \mathfrak{F}(G)$, its natural image in $\mathfrak{F}(G)/J_f \subset \mathfrak{H}_f$ is denote by φ^f. Let $v_0 = \delta_e^f$ be the image of the delta function $\delta_e \in \mathfrak{F}(G)$ supported by the one point set $\{ e \}$, then it is a unit cyclic vector for π_f, and f is recovered by the formula (6). Furthermore the representation π_f is actually strongly continuous because $G \ni g \mapsto \langle \pi_f(g)v_0, v_0 \rangle = f(g)$ is continuous. The von Neumann algebra generated by $\pi_f(G) = \{ \pi_f(g) \; ; \; g \in G \}$ is $\mathfrak{U}_f := \pi_f(G)'' = \pi_f\big(\mathfrak{F}(G)\big)''$ (bicommutant).

We know in [GR] that *a cyclic representation π, associated to an $f \in \mathcal{P}_1(G)$, is irreducible if and only if f is extremal.*

2.2. Positive definite class functions. Now assume that an $f \in \mathcal{P}(G)$ is invariant under G or $f(g_0 g g_0{}^{-1}) = f(g)$ $(g, g_0 \in G)$. We call f also a class function. Note that $f \in \mathcal{P}(G)$ is a class function if and only if the inner product satisfies the condition

$$\langle \varphi, \psi \rangle_f = \langle \psi^*, \varphi^* \rangle_f \qquad (\varphi, \psi \in \mathfrak{F}(G)).$$

In this case, the kernal J_f is a two-sided $*$-ideal of $\mathfrak{F}(G)$, and the quotient $\mathfrak{F}(G)/J_f$ has naturally a structure of $*$-algebra. By right multiplication, we have a representation ρ_f of $\mathfrak{F}(G)$ and also that of the group G denoted by the same symbol:

$$\rho_f(\psi)\varphi^f = (\varphi * \psi^\vee)^f, \qquad \rho_f(g_0)\varphi^f = (R(g_0)\varphi)^f \qquad (\varphi, \psi \in \mathfrak{F}(G), g_0 \in G),$$

where $\psi^\vee(g) := \psi(g^{-1})$, $(R(g_0)\varphi)(g) := \varphi(gg_0)$ $(g \in G)$. Then, $\rho_f\big(\mathfrak{F}(G)\big)$ and $\rho_f(G)$ are contained in the commutant $\pi_f(G)' = (\mathfrak{U}_f)'$. They generate a von Neumann algebra $\mathfrak{V}_f = \rho_f\big(\mathfrak{F}(G)\big)'' = \rho_f(G)''$ and

$$\mathfrak{V}_f \subset (\mathfrak{U}_f)', \qquad \mathfrak{U}_f \subset (\mathfrak{V}_f)'.$$

Let us describe the common center $\mathfrak{Z}_f = \mathfrak{U}_f \cap \mathfrak{V}_f$. Let $L^+(G)$ be the set of all invariant $f \in \mathcal{P}(G)$. For an $f \in L^+(G)$, let $M^+(f)$ be the set of all $f' \in L^+(G)$ majorized by f. After [Th1, Lemma 2], we give a bijective map from the set $M^+(f)$ onto the set $\mathfrak{Z}_f^+$ of all positive hermitian operators in $\mathfrak{Z}_f$ as follows. Take an $f' \in M^+(f)$, then $\lambda f - f' \in L^+(G)$ for some $\lambda > 0$, and so $0 \le \langle \varphi, \varphi \rangle_{f'} \le \lambda \langle \varphi, \varphi \rangle_f$. Therefore there exists a unique positive hermitian operator $0 \le A \le \lambda I$ on $\mathfrak{H}_f$, with the identity operator I on $\mathfrak{H}_f$, such that $\langle \varphi, \psi \rangle_{f'} = \langle A\varphi, \psi \rangle_f$ $(\varphi, \psi \in \mathfrak{F}(G))$. Then we can prove that $A \in \mathfrak{Z}_f^+$. Conversely take an $A \in \mathfrak{Z}_f^+$. Put

$$f'(g) = \langle A\, \pi_f(g)v_0, v_0 \rangle_f = \langle \pi_f(g)\sqrt{A}v_0, \sqrt{A}v_0 \rangle_f \quad \text{with} \quad v_0 = \delta_e^f \qquad (g \in G).$$

Then f' is continuous and positive definite. Moreover, $||A||\,f-f'$ is positive definite because, $(||A||\,f-f')(g)=\langle\pi_f(g)Bv_0, Bv_0\rangle_f$ with $B=\sqrt{||A||\,I-A}$. Hence f' is majorized by f and so $f'\in M^+(f)$.

Thus, the common center $\mathfrak{Z}_f$ is reduced to $\boldsymbol{C}I$ if and only if any $f'\in L^+(G)$, majorized by f, is a scalar multiple of f. This gives us a criterion for that the representation π_f is a factor or that the von Neumann algebra $\mathfrak{U}_f$ is a factor.

We refer [Di, §6, §17] for the theory of characters and quote some essential parts in the next subsection **2.3**. Then we can say that, with the normalization $f(e)=1$, the map $\mathfrak{F}(G)\in\varphi\mapsto f(\varphi)=\sum_{g\in G}f(g)\varphi(g)$ gives a character $\boldsymbol{t}:\mathfrak{U}_f\ni\pi_f(\varphi)\mapsto f(\varphi)\in\boldsymbol{C}$ of the factor representation π_f of $\mathfrak{F}(G)$ or of G, normalized as $\boldsymbol{t}(I)=1$. In this case the factor is of finite type, I_n with $n<\infty$ or II_1 depending on whether $\dim\pi_f<\infty$ or $=\infty$.

Denote by $K(G)$ the set of all $f\in L^+(G)$ normalized as $f(e)=1$, and by $E(G)$ the set of all extremal points in the convex set $K(G)$.

Proposition 2.1. *Let G be a topological group and $E(G)$ the set of all continuous extremal positive definite class functions on G, normalized as $f(e)=1$. Then, each $f\in E(G)$ is a character of a factor representation of G of finite type.*

Question 2002-7: *For what kind of G, does $E(G)$ cover all the characters of factor representations of finite type ?*

2.3. Characters of C^*-algebras and characters of representations.

We refer [Di, §6] for the theory of traces and characters for representations of C^*-algebras. For a C^*-algebra A, a *character* $\boldsymbol{t}$ of A is, by definition, a trace on $A^+:=\{x\geq 0\,;\,x\in A\}$ which is semifinite, lower semicontinuous, and such that any such trace majorized by $\boldsymbol{t}$ is proportional to $\boldsymbol{t}$. In turn, a *trace* on A^+ is a function $\boldsymbol{t}:A^+\to[0,+\infty]$ satisfying

$$\begin{aligned}
&\text{(i)}\quad \boldsymbol{t}(x+y) &&= \boldsymbol{t}(x)+\boldsymbol{t}(y)\ \ (x,y\in A^+)\,;\quad \text{(ii)}\ \ \boldsymbol{t}(\lambda x)=\lambda\boldsymbol{t}(x)\ \ (\lambda\geq 0, x\in A^+)\,;\\
&\text{(iii)}\quad \boldsymbol{t}(zz^*) &&= \boldsymbol{t}(z^*z)\ \ (z\in A)\,.
\end{aligned}$$

A trace is called *finite* if $\boldsymbol{t}(x)<+\infty$ for any $x\in A^+$. A finite trace $\boldsymbol{t}$ can be uniquely extended to a positive and central linear form $\boldsymbol{f}$ on A, and vice versa. A linear form $\boldsymbol{f}$ is called *positive* if $\boldsymbol{f}(x)\geq 0$ $(x\in A^+)$, and $\boldsymbol{f}$ is called *central* if $\boldsymbol{f}(xy)=\boldsymbol{f}(yx)$ $(x,y\in A)$. A positive linear form on a C^*-algebra A is automatically continuous [Di, 2.1.8], and so *a finite trace on A^+ is automatically continuous too.*

A *representation with trace* of a C^*-algebra A is a couple (π,t) with the following properties:

(i) π is a non-degenerated representation of A on a Hilbert space,

(ii) t is a normal faithful trace on $\mathfrak{U}^+$ for the von Neumann algebra $\mathfrak{U}=\pi(A)''$,

(iii) $\pi(A)\cap\mathfrak{n}_t$ generates $\mathfrak{U}$, where $\mathfrak{n}_t=\{U\in\mathfrak{U}\,;\,t(UU^*)<\infty\}$.

In this case, t is semifinite and π is a sum of representations of type I and of type II. Put $\boldsymbol{t}=(t\circ\pi)|_{A^+}$, then it is a trace on A^+, semifinite and lower semicontinuous. A representation π of A is said to be *traceable* if there exists a trace t on $\mathfrak{U}^+$ which, together with π, satisfies the above conditions (i)–(iii).

Theorem 6.7.3 in [Di] says that *for a representation with trace (π,t), π is non-zero factorial if and only if $\boldsymbol{t}=(t\circ\pi)|_{A^+}$ is a character.*

(2.3.1) There exists a canonical bijective correspondence [Di, 6.7.4] between *a.* the set of quasi-equivalence classes of traceable non-zero factorial representations of A;

b. the set of characters of A modulo multiplicative factors > 0.

A von Neumann algebra $\mathfrak{U}$ is called *finite* if for any non-zero $U \in \mathfrak{U}^+$ there exists a normal finite trace t on $\mathfrak{U}^+$ such that $t(U) > 0$. A factorial representation π is called *of finite type* if its factor $\mathfrak{U} = \pi(A)''$ is finite.

(2.3.2) There exists a canonical bijective correspondence [Di, 6.8.6] between

a'. the set of quasi-equivalence classes of traceable non-zero factorial representations of finite type of A;

b'. the set of characters of A with norm 1.

2.4. The case of locally compact groups. Now suppose G be locally compact and unimodular. Let $\mathfrak{A}(G) = C_c(G)$ be the $*$-algebra of all compactly supported complex-valued continuous functions on G with operations

$$\varphi * \psi(g) = \int_G \varphi(gh^{-1})\, \psi(h)\, dh, \qquad \varphi^*(g) = \overline{\varphi(g^{-1})} \qquad (\varphi, \psi \in \mathfrak{A}, g \in G),$$

where dh denotes a Haar measure on G. This structure of $*$-algebra can be extended to a $*$-Banach algebra $L^1(G)$ of L^1-functions with respect to the Haar measure. The completion of $L^1(G)$ with respect to a so-called C^*-norm is the C^*-algebra $C^*(G)$ of G.

A unitary representation π of G corresponds bijectively to a non-degenerate representation of $L^1(G)$ and that of $C^*(G)$ through $\pi(\psi) := \int_G \pi(g)\, \psi(g)\, dg \;\; (\psi \in L^1(G))$.

A continuous linear form on $L^1(G)$ is canonically given by an element in $L^\infty(G)$, and an $f \in L^\infty(G)$ is called *integrally positive definite* if

$$f(\varphi * \varphi^*) = \int_G \int_G f(h^{-1}g)\varphi(g)\overline{\varphi(h)}\, dg\, dh \;\geq\; 0 \quad (\varphi \in L^1(G) \;\text{ or }\; \mathfrak{A}(G)).$$

We know in [GR, Theorem 4 and p.10] that such an f equals to a continuous one almost everywhere.

From the general theory in **2.3**, positive central linear forms on C^*-algebra $A = C^*(G)$ are automatically continuous and correspond bijectively to finite traces on A^+. Since $L^1(G)$ and $\mathfrak{A}(G)$ are both dense in A, a finite trace $\boldsymbol{t}$ on A^+ is determined by its restriction on $L^1(G)$ or on $\mathfrak{A}(G)$. As mentioned above for $L^1(G)$, the restriction of $\boldsymbol{t}$ is written with a continuous positive definite function f as

$$\boldsymbol{t} : \mathfrak{A}(G) \ni \psi \longmapsto f(\psi) := \int_G f(g)\psi(g) \;\in \boldsymbol{C}. \tag{7}$$

Since $\boldsymbol{t}$ is central, that is, $\boldsymbol{t}(\psi_1 * \psi_2) = \boldsymbol{t}(\psi_2 * \psi_1)$ $(\psi_1, \psi_2 \in \mathfrak{A}(G))$, f is invariant or a class function. A trace $\boldsymbol{t}$, associated to f, is a *character* if and only if f is indecomposable or extremal. Thus we have an affirmative answer to Question 2002-7 in this case.

(2.4.1) *In the case of a locally compact group G, the set $E(G)$ of extremal continuous positive definite class functions covers all the characters of factorial representations of G of finite type.*

(2.4.2) For a locally compact group G, GNS construction of a cyclic representation associated to an $f \in \mathcal{P}_1(G)$, is usually given by using integration with respect to a Haar measure. We remark here that this construction is equivalent to the one in **2.1** not using any integration.

Fix an $f \in \mathcal{P}_1(G)$. Introduce in $\mathfrak{A}(G)$ a positive semidefinite inner product as

$$(\varphi, \psi)_f := \iint_{G\times G} f(h^{-1}g)\varphi(g)\overline{\psi(h)}\,dg\,dh \qquad (\varphi, \psi \in \mathfrak{A}(G)).$$

Let J'_f be the kernel of $(\cdot\,,\cdot)_f$, and take a quotient $\mathfrak{A}(G)/J'_f$. Completing it with respect to the positive definite inner product, we get a Hilbert space $\mathfrak{H}'_f$. The left multiplication of $\mathfrak{A}(G)$ generates representation π'_f of $\mathfrak{A}(G)$ and also of G.

We define a liniear map Φ of $\mathfrak{A}(G)$ into $\mathfrak{H}_f$ as follows. As an operator-valued function on G, $G \ni g \mapsto \pi_f(g)$ is strongly continuous and the value $\pi_f(g)$ is unitary. So, for every $\varphi \in \mathfrak{A}(G)$ the operator-valued integration $\pi_f(\varphi) = \int_G \pi_f(g)\,\varphi(g)\,dg$ is strongly convergent and $\|\pi_f(\varphi)\| \le \|\varphi\|_{L^1}$. It defines a representation of $*$-algebra $\mathfrak{A}(G)$ and also of $L^1(G)$ on the space $\mathfrak{H}_f$. For $v_0 = \delta_e^f \in \mathfrak{H}_f$, put $\Phi(\varphi) := \pi_f(\varphi)v_0$. Then, for $\varphi, \psi \in \mathfrak{A}(G)$,

$$\begin{aligned}\langle \Phi(\varphi), \Phi(\psi)\rangle_f &= \int_G\int_G \varphi(g)\,\overline{\psi(h)}\,\langle \pi_f(g)v_0, \pi_f(h)v_0\rangle_f\,dg\,dh \\ &= \int_G\int_G \varphi(g)\,\overline{\psi(h)}\,f(h^{-1}g)\,dg\,dh \;=\; (\varphi, \psi)_f\,.\end{aligned}$$

This means that Φ induces a linear map Φ' from $\mathfrak{A}(G)/J'_f$ into $\mathfrak{H}_f$ which conserves the inner product. The map Φ' extends to an isomorphism of two Hilbert spaces $\mathfrak{H}'_f$ and $\mathfrak{H}_f$.

Proposition 2.2. *The extended linear map $\Phi' : \mathfrak{H}'_f \to \mathfrak{H}_f$ intertwines two unitary representations of G as $\Phi'\,\pi'_f(g) = \pi_f(g)\,\Phi'$ $(g \in G)$.*

(2.4.3) We give a remark about $\mathfrak{H}_f$ and $\mathfrak{H}'_f$. Let $\{V\}$ be the set of all relatively compact neighbourhoods of $e \in G$ with the order of inclusion. Take functions $\varphi_V \in \mathfrak{A}(G)$ such that

$$\varphi_V \ge 0, \quad \mathrm{supp}(\varphi_V) \subset V, \quad \int_G \varphi_V(g)\,dg = 1.$$

Then, as is easily proved, $\|\pi_f(\varphi_V)v_0 - v_0\| \to 0$ as $V \to \{e\}$, for $v_0 = \delta_e^f$. Corresponding to this strong convergence in $\mathfrak{H}_f$, through the isomorphism Φ', the image of φ_V in $\mathfrak{A}(G)/J'_f$ should converge strongly to an element $\xi_0 = {\Phi'}^{-1}(v_0) \in \mathfrak{H}'_f$. To prove this directly, inside $\mathfrak{H}'_f$, is not so simple as is seen in the proof of [GR, Theorem 4].

2.5. Case of inductive limits of locally compact groups.

Now let G be an inductive limit of a sequence of locally compact groups G_n. We assume that all G_n are unimodular and that for each n a continuous isomomorphism $\iota_n : G_n \hookrightarrow G_{n+1}$ is given. The limit group $G = \lim_{n\to\infty} G_n$ is equiped with the inductive limit τ_{ind} of topologies τ_{G_n} on G_n, which is proved to be a group topology (cf. [TSH, Theorem 2.7]). A complex valued function on G is τ_{ind}-continuous if its restriction on each G_n is τ_{G_n}-continuous.

Let $\mathfrak{A}(G_n) = C_c(G_n)$ be the $*$-algebra of compactly supported continuous functions on G_n. Fix a Haar measure $d_n g$ on G_n, and identify $\psi \in \mathfrak{A}(G_n)$ with a measure $\psi(g)\,d_n g$ $(g \in G_n)$. For a continuous function F on G, we define its integral against this measure as $\int_{G_n} F(g)\,\psi(g)\,d_n g$. The convolution of $d\mu_n(g) = \psi_n(g)\,d_n g$ on G_n and $d\mu_m(h) = \psi_m(h)\,d_m h$ on G_m with $n \ge m$ is defined by

$$\int_{G_n} F(g)d(\mu_n * \mu_m)(g) := \int_{G_n}\int_{G_m} F(gh)\,\psi_n(g)\,\psi_m(h)\,d_n g\,d_m h.$$

Hence we have $d(\mu_n * \mu_m)(g) = \psi(g)\, d_n g$ with $\psi(g) = \int_{G_m} \psi_n(gh^{-1})\psi_m(h)\, d_m h \in \mathfrak{A}(G_n)$. Define μ_n^* by $d\mu_n^*(g) = \overline{d\mu_n(g^{-1})}$, and let the norm $||\mu_n||$ be as usual, for example, if ψ_n is real valued, then $||\mu_n|| = \int_{G_n} |\psi_n(g)| d_n g$. We have $||\mu_n^*|| = ||\mu_n||$.

With these operations and norm, the union $\mathfrak{A}(G) := \cup_{n\geq 1}\mathfrak{A}(G_n)$ generates a $*$-Banach algebra, denoted by $\mathfrak{M}(G)$, which depends on the series $(G_n, d_n g)_{n\geq 1}$.

Now take a continuous positive definite function f, i.e., $f \in \mathcal{P}(G)$, then the cyclic representation π_f of G in **2.1**, associated to f, gives a representation of the algebra $\mathfrak{M}(G)$ through $\pi_f(\psi) = \int_{G_n} \pi_f(g)\, \psi(g)\, d_n g$ for $\psi \in \mathfrak{A}(G_n)$. It generates the same von Neumann algebra as that in **2.1**, that is, $\pi_f(\mathfrak{M}(G))'' = \pi_f(\mathfrak{F}(G))'' = \pi_f(G)'' = \mathfrak{U}_f$. Here the dependence on $(G_n, d_n g)_{n\geq 1}$ has disappeared.

If f is taken from $E(G)$, i.e., f is invariant, extremal and $f(e) = 1$, then, by **2.2**, π_f gives a factor representation of finite type, and also by **2.3**, its character is given by f as $\mathfrak{U}_f \ni \pi_f(\psi) \mapsto f(\psi) = \int_{G_n} f(g)\, \psi(g)\, d_n g\ (\psi \in \mathfrak{A}(G_n))$.

Conversely take a factor representation π of G of finite type and let t be its normalized character on $\mathfrak{U}_\pi = \pi(\mathfrak{M}(G))'' = \pi(G)''$. We should ask if (π, t) can be realized from an $f \in E(G)$ as above. Consider $\boldsymbol{f} = (t \circ \pi)$ on $\mathfrak{M}(G)$. Then, its restriction on each $\mathfrak{A}(G_n)$ is positive and central. Since G_n is locally compact, it is given, as in the case of **2.4**, by a continuous positive definite class function f_n on G_n as $\psi_n(g)d_n g \mapsto f_n(\psi_n) = \int_{G_n} f_n(g)\psi_n(g)d_n g$. According to the inclusion $G_n \hookrightarrow G_{n+1}$, if the the consistency condition

$$f_{n+1}|_{G_n} = f_n \qquad (n \geq 1) \tag{8}$$

holds, we get a function f on $G = \lim_{n\to\infty} G_n$. Clearly f is positive definite, invariant and continuous in τ_{ind} because $f|_{G_n} = f_n$ is continuous on G_n for each n. From the general theory, we see that $f \in E(G)$. Thus our question here is the following.

Question 2002-8: *Let G be an inductive limit of a sequence of locally compact groups $G_n \hookrightarrow G_{n+1}$ $(n \geq 1)$ with continuous isomorphisms. For a factor representations π of G of finite type, let a series of positive definite class functions f_n on G_n be as above. Then, does the consistency condition* (8) *hold ?*

The problem of *determining explicitly all the characters* of factorial representations of G of finite type is, for good categories of G, equivalent to determining all elements in $E(G)$. For discrete groups, this is the case and the problem has been studied in [Th1]–[Th2], [Sk], [Hi3]–[Hi4] and [HH1]–[HH2].

3 Structure of wreath product groups $\mathfrak{S}_\infty(T) = D_\infty(T) \rtimes \mathfrak{S}_\infty$

Fix a compact group T, and take the wreath product group $\mathfrak{S}_\infty(T)$ of T with the symmetric group $\mathfrak{S}_\infty$:

$$\mathfrak{S}_\infty(T) = D_\infty(T) \rtimes \mathfrak{S}_\infty, \quad D_\infty(T) := {\prod}'_{i\in \boldsymbol{N}} T_i\,, \quad T_i = T \quad (i \in \boldsymbol{N}). \tag{9}$$

Here $\sigma \in \mathfrak{S}_\infty$ acts on $d = (t_i)_{i\in\boldsymbol{N}} \in D_\infty(T)$ as $\sigma(d) = (t_{\sigma^{-1}(i)})_{i\in\boldsymbol{N}}$. We identify frequently d and σ with their images in $\mathfrak{S}_\infty(T)$ respectively, then $\sigma d \sigma^{-1} = \sigma(d)$ and

$$(d, \sigma)(d', \sigma') = \big(d(\sigma d' \sigma^{-1}), \sigma\sigma'\big) \qquad (d, d' \in D_\infty(T), \sigma, \sigma' \in \mathfrak{S}_\infty).$$

3.1. Standard decomposition of elements and conjugacy classes.

An element $g = (d, \sigma) \in G = \mathfrak{S}_\infty(T)$ is called *basic* in the following two cases:

CASE 1: σ is cyclic and $\mathrm{supp}(d) \subset \mathrm{supp}(\sigma)$;

CASE 2: $\sigma = \mathbf{1}$ and for $d = (t_i)_{i \in \boldsymbol{N}}$, $t_q \neq e_T$ only for one $q \in \boldsymbol{N}$.

Here $\mathbf{1} \in \mathfrak{S}_\infty$ denotes the trivial permutation, and the element $(d, \mathbf{1})$ in Case 2 is denoted by ξ_q, and put $\mathrm{supp}(\xi_q) := \mathrm{supp}(d) = \{q\}$.

For a cyclic permutation $\sigma = (i_1, i_2, \ldots, i_\ell)$ of ℓ integers, we define its *length* as $\ell(\sigma) = \ell$, and for the identity permutation $\mathbf{1}$, put $\ell(\mathbf{1}) = 1$ for convenience. In this connection, ξ_q is also denoted by $(t_q, (q))$ with a trivial cyclic permutation (q) of length 1. In Cases 1 and 2, put $\ell(g) = \ell(\sigma)$ for $g = (d, \sigma)$, and $\ell(\xi_q) = 1$.

An arbitrary element $g = (d, \sigma) \in G$, is expressed as a product of basic elements as

$$g = \xi_{q_1}\xi_{q_2}\cdots\xi_{q_r} g_1 g_2 \cdots g_m \tag{10}$$

with $g_j = (d_j, \sigma_j)$ in Case 1, in such a way that the supports of these components, $q_1, q_2, \ldots, q_r$, and $\mathrm{supp}(g_j) = \mathrm{supp}(\sigma_j)$ $(1 \le j \le m)$, are mutually disjoint. This expression of g is unique up to the orders of ξ_{q_k}'s and g_j's, and is called *standard decomposition* of g. Note that $\ell(\xi_{q_k}) = 1$ for $1 \le k \le r$ and $\ell(g_j) = \ell(\sigma_j) \ge 2$ for $1 \le j \le m$, and that, for $\mathfrak{S}_\infty$-components, $\sigma = \sigma_1\sigma_2\cdots\sigma_m$ gives the cycle decomposition of σ.

To write down conjugacy class of $g = (d, \sigma)$, there appear products of components t_i of $d = (t_i)$, where the orders of taking products are crucial when T is not abelian. So we should fix notations well.

We denotes by $[t]$ the conjugacy class of $t \in T$, and by $T/\!\sim$ the set of all conjugacy classes of T, and $t \sim t'$ denotes that $t, t' \in T$ are mutually conjugate in T. For a basic component $g_j = (d_j, \sigma_j)$ of g, let $\sigma_j = (i_{j,1}\ i_{j,2}\ \ldots\ i_{j,\ell_j})$ and put $K_j := \mathrm{supp}(\sigma_j) = \{i_{j,1}, i_{j,2}, \ldots, i_{j,\ell_j}\}$ with $\ell_j = \ell(\sigma_j)$. For $d_j = (t_i)_{i \in K_j}$, we put

$$P_{\sigma_j}(d_j) := [t'_{\ell_j} t'_{\ell_j - 1} \cdots t'_2 t'_1] \in T/\!\sim \qquad \text{with} \quad t'_k = t_{i_{j,k}} \ (1 \le k \le \ell_j). \tag{11}$$

Note that the product $P_{\sigma_j}(d_j)$ is well-defined, because, for $t_1, t_2, \ldots, t_\ell \in T$, we have $t_1 t_2 \cdots t_\ell \sim t_k t_{k+1} \cdots t_\ell t_1 \cdots t_{k-1}$ for any k, that is, the conjugacy class does not depend on any cyclic permutation of $(t_1, t_2, \ldots, t_\ell)$.

Lemma 3.1. (i) *Let $\sigma \in \mathfrak{S}_\infty$ be a cycle, and put $K = \mathrm{supp}(\sigma)$. Then, an element $g = (d, \sigma) \in \mathfrak{S}_K(T)(=: G_K$ (put)) is conjugate in it to $g' = (d', \sigma) \in G_K$ with $d' = (t'_i)_{i \in K}, t'_i = e_T$ $(i \neq i_0)$, $[t'_{i_0}] = P_\sigma(d)$ for some $i_0 \in K$.*

(ii) *Identify $\tau \in \mathfrak{S}_\infty$ with its image in $G = \mathfrak{S}_\infty(T)$. Then we have, for $g = (d, \sigma)$,*

$$\tau g \tau^{-1} = (\tau(d), \tau\sigma\tau^{-1}) \ \ (=: (d', \sigma') \ (put)\ ),$$

and $P_{\sigma'}(d') = P_\sigma(d)$.

Theorem 1. *Let T be a compact group. Take an element $g \in G = \mathfrak{S}_\infty(T)$ and let its standard decomposition into basic elements be $g = \xi_{q_1}\xi_{q_2}\cdots\xi_{q_r} g_1 g_2 \cdots g_m$ in (10), with $\xi_{q_k} = (t_{q_k}, (q_k))$, and $g_j = (d_j, \sigma_j)$, σ_j cyclic, $\mathrm{supp}(d_j) \subset \mathrm{supp}(\sigma_j)$. Then the conjugacy class of g is determined by*

$$[t_{q_k}] \in T/\!\sim \quad (1 \le k \le r) \quad \textit{and} \quad (P_{\sigma_j}(d_j), \ell(\sigma_j)) \quad (1 \le j \le m), \tag{12}$$

where $P_{\sigma_j}(d_j) \in T/\!\sim$ *and* $\ell(\sigma_j) \geq 2$. (*Note that we put* $\ell(\xi_{q_k}) = 1$, $\ell(g_j) = \ell(\sigma_j) \geq 2$.)

3.2. The case where T is abelian.

In the case where T is abelian, the set $T/\!\sim$ of conjugacy classes is equal to T itself. Take $g \in G$, and take its standard decompositon (10). For $g_j = (d_j, \sigma_j)$, put $g'_j := (d'_j, \sigma_j)$, where $d'_j = (t'_i)_{i\in \boldsymbol{N}}$ with $t'_{i_0} = P(d_j) = \prod_{i\in K_j} t_i$ for some $i_0 \in K_j := \mathrm{supp}(\sigma_j)$, and $t'_i = e_T$ elsewhere.

Lemma 3.2. *Let T be abelian. For a $g = (d,\sigma) \in \mathfrak{S}_\infty(T)$, let its standard decomposition be $g = \xi_{q_1}\xi_{q_2}\cdots\xi_{q_r}g_1g_2\cdots g_m$ in (10). Define g'_j $(1 \leq j \leq m)$ as above and put $g' = \xi_{q_1}\xi_{q_2}\cdots\xi_{q_r}g'_1g'_2\cdots g'_m$. Then, g and g' are mutually conjugate in $\mathfrak{S}_\infty(T)$.*

Corollary. *A complete set of parameters of the conjugacy classes of non-trivial elements $g \in \mathfrak{S}_\infty(T)$ is given by*

$$\{\, t'_1, t'_2, \ldots, t'_r \,\} \quad \text{and} \quad \{\, (u_j, \ell_j)\ ;\ 1 \leq j \leq m \,\}, \tag{13}$$

where $t'_k = t_{q_k} \in T^* := T \setminus \{\, e_T \,\}$, $u_j = P(d_j) \in T$, $\ell_j \geq 2$, and $r + m > 0$.

3.3. Finite-dimensional irreducible representations.

Let us study finite-dimensional continuous irreducible representations (= IRs) of $G = \mathfrak{S}_\infty(T)$. Similarly as in the case a finite group T in [HH1], we can prove the following facts.

Lemma 3.3. *A finite-dimensional continuous irreducible representation π of $\mathfrak{S}_\infty(T)$ is a one-dimensional character, and is given in the form $\pi = \pi_{\zeta,\varepsilon}$ with*

$$\pi_{\zeta,\varepsilon}(g) = \zeta(P(d))\,(\mathrm{sgn}_{\mathfrak{S}})^{\varepsilon}(\sigma) \quad \text{for } g = (d,\sigma) \in \mathfrak{S}_\infty(T) = D_\infty(T) \rtimes \mathfrak{S}_\infty,$$

where ζ is a one-dimensional character of T, $P(d)$ is a product of components t_i of $d = (t_i)$, and $\mathrm{sgn}_{\mathfrak{S}}(\sigma)$ denotes the usual sign of σ and $\varepsilon = 0, 1$. (Since $\zeta(P(d)) = \prod_{i\in \boldsymbol{N}} \zeta(t_i)$, the order of taking product for $P(d)$ has no meaning even if T is not abelian.)

Lemma 3.4. *Assume that T is abelian. Then, a finite-dimensional continuous irreducible representation π of $\mathfrak{S}^e_\infty(T)$ is a one-dimensional character, and is given in the form*

$$\pi(g) = (\mathrm{sgn}_{\mathfrak{S}})^{\varepsilon}(\sigma) \quad \text{for } g = (d,\sigma) \in \mathfrak{S}^e_\infty(T) = D^e_\infty(T) \rtimes \mathfrak{S}_\infty.$$

4 Characters of $\mathfrak{S}_\infty(T)$ with T any compact group

4.1. Character formula for factor representations of G of finite type.

Let $\widehat{T}$ be the dual of T consisting of all equivalence classes of continuous irreducible unitary representations. We identify every equivalence class with one of its representative. Thus $\zeta \in \widehat{T}$ is an IR and denote by χ_ζ its character: $\chi_\zeta(t) = \mathrm{tr}(\zeta(t))$ $(t \in T)$, then $\dim \zeta = \chi_\zeta(e_T)$.

For a $g \in G = \mathfrak{S}_\infty(T)$, let its standard decomposition into basic components be

$$g = \xi_{q_1}\xi_{q_2}\cdots\xi_{q_r}g_1g_2\cdots g_m\,, \tag{14}$$

where the supports of components, $q_1, q_2, \ldots, q_r$, and $\mathrm{supp}(g_j) := \mathrm{supp}(\sigma_j)$ $(1 \leq j \leq m)$, are mutually disjoint. Furthermore, $\xi_{q_k} = (t_{q_k}, (q_k)), t_{q_k} \neq e_T$, with $\ell(\xi_{q_k}) = 1$ for $1 \leq k \leq r$, and σ_j is a cycle of length $\ell(\sigma_j) \geq 2$ and $\mathrm{supp}(d_j) \subset K_j = \mathrm{supp}(\sigma_j)$. For $\mathfrak{S}_\infty$-components, $\sigma = \sigma_1\sigma_2\cdots\sigma_m$ gives the cycle decomposition of σ. For $d_j = (t_i)_{i\in K_j} \in D_{K_j}(T) \hookrightarrow D_\infty(T)$, put $P_{\sigma_j}(d_j)$ as in (11).

For one-dimensional charcters of $\mathfrak{S}_\infty$, we introduce simple notation as

$$\chi_\varepsilon(\sigma) := \mathrm{sgn}_{\mathfrak{S}}(\sigma)^\varepsilon \quad (\sigma \in \mathfrak{S}_\infty\,;\ \varepsilon = 0,1). \tag{15}$$

As a parameter for characters of $G = \mathfrak{S}_\infty(T)$, we prepare a set

$$\alpha_{\zeta,\varepsilon}\ (\zeta \in \widehat{T}, \varepsilon \in \{0,1\}) \quad \text{and} \quad \mu = (\mu_\zeta)_{\zeta\in\widehat{T}}, \tag{16}$$

of decreasing sequences of non-negative numbers

$$\alpha_{\zeta,\varepsilon} = (\alpha_{\zeta,\varepsilon,i})_{i\in \boldsymbol{N}}, \quad \alpha_{\zeta,\varepsilon,1} \geq \alpha_{\zeta,\varepsilon,2} \geq \alpha_{\zeta,\varepsilon,3} \geq \cdots \geq 0\,;$$

and a set of non-negative $\mu_\zeta \geq 0$ $(\zeta \in \widehat{T})$, which altogether satisfies the condition

$$\sum_{\zeta\in\widehat{T}} \sum_{\varepsilon\in\{0,1\}} \|\alpha_{\zeta,\varepsilon}\| + \|\mu\| \leq 1, \tag{17}$$

$$\text{with} \qquad \|\alpha_{\zeta,\varepsilon}\| = \textstyle\sum_{i\in\boldsymbol{N}} \alpha_{\zeta,\varepsilon,i}, \quad \|\mu\| = \sum_{\zeta\in\widehat{T}} \mu_\zeta\,.$$

Theorem 2. *Let $G = \mathfrak{S}_\infty(T)$ be a wreath product group of a compact group T with $\mathfrak{S}_\infty$. Then, for a parameter*

$$A := \Big((\alpha_{\zeta,\varepsilon})_{(\zeta,\varepsilon)\in\widehat{T}\times\{0,1\}}\,;\,\mu\Big), \tag{18}$$

in (16)–(17), *the following formula determines a character f_A of G: for an element $g \in G$, let* (14) *be its standard decomposition, then*

$$\begin{aligned} f_A(g) \;&=\; \prod_{1\leq k\leq r} \left\{ \sum_{\zeta\in\widehat{T}} \left(\sum_{\varepsilon\in\{0,1\}} \sum_{i\in\boldsymbol{N}} \frac{\alpha_{\zeta,\varepsilon,i}}{\dim\zeta} + \frac{\mu_\zeta}{\dim\zeta} \right) \chi_\zeta(t_{q_k}) \right\} \\ &\times \prod_{1\leq j\leq m} \left\{ \sum_{\zeta\in\widehat{T}} \left(\sum_{\varepsilon\in\{0,1\}} \sum_{i\in\boldsymbol{N}} \left(\frac{\alpha_{\zeta,\varepsilon,i}}{\dim\zeta}\right)^{\ell(\sigma_j)} \chi_\varepsilon(\sigma_j) \right) \chi_\zeta(P_{\sigma_j}(d_j)) \right\}, \end{aligned} \tag{19}$$

where $\chi_\varepsilon(\sigma_j) = \mathrm{sgn}_{\mathfrak{S}}(\sigma_j)^\varepsilon = (-1)^{\varepsilon(\ell(\sigma_j)-1)}$.

Conversely any character of G is given in the form of f_A.

Example 4.1. (T : abelian) The ~~case where~~ $\alpha_{\zeta,\varepsilon,1} = 1$ for a fixed $(\zeta,\varepsilon) \in \widehat{T}\times\{0,1\}$ and all other parameters in A are zero, whence $\alpha_{\zeta,\varepsilon} = (1,0,0,\ldots)$, corresponds to one-dimensional character $\pi_{\zeta,\varepsilon}$ of G in Lemma 3.3. Except these cases of one-dimensional representations of G, a character f_A given above corresponds to a factor representation of G of type II_1.

The case "$\alpha_{\zeta,\varepsilon} = (\alpha_{\zeta,\varepsilon,i})_{i\in\boldsymbol{N}} = \boldsymbol{0}$ for all $(\zeta,\varepsilon) \in \widehat{T}\times\{0,1\}$ and $\mu = (\mu_\zeta)_{\zeta\in\widehat{T}} = \boldsymbol{0}$" corresponds to the regular representation λ_G of G.

Consider the case where $\|\alpha_{\zeta,0}\| + \|\alpha_{\zeta,1}\| + \mu_\zeta = 1$ for a fixed $(\zeta,\varepsilon) \in \widehat{T}\times\{0,1\}$ and all other parameters in A are zero. Put $\alpha = \alpha_{\zeta,0}, \beta = \alpha_{\zeta,1}$, and let $f_{\alpha,\beta}$ be Thoma's character for $\mathfrak{S}_\infty$. Denote by Ψ the natural homomorphism from G onto $\mathfrak{S}_\infty \cong G/D$ with normal subgroup $D = D_\infty(T)$, and put $f^\#_{\alpha,\beta} := f_{\alpha,\beta}\circ\Psi$. Then the character $f_A(g)$ in this case is equal to $f^\#_{\alpha,\beta}(g)\cdot\pi_{\zeta,0}(g)$ with a one-dimensional character $\pi_{\zeta,0}$ of G with $\varepsilon = 0$. In particular, the case where $\mu_\zeta = 1$ for a fixed $\zeta \in \widehat{T}$, corresponds to the induced representation $\mathrm{Ind}_D^G \zeta_D$, where $\zeta_D(d) := \zeta(P(d)), d \in D$, is a one-dimensional character of $D = D_\infty(T)$. The character f_A is equal to ζ_D on $D \hookrightarrow G$, and zero outside of D. In the case $\zeta = \mathbf{1}_T$, this induced representation is nothing but the regular representation of $G/D \cong \mathfrak{S}_\infty$.

4.2. Remarks on the case where T is a finite group.

Denote by $\mathbf{1}_T$ the identity representation of T, and put $\widehat{T}^* := \widehat{T} \setminus \{ \mathbf{1}_T \}$. Then,

$$|T|\delta_{e_T} = \sum_{\zeta \in \widehat{T}} (\dim \zeta)\, \chi_\zeta\,, \quad \text{as functions on } T, \tag{20}$$

$$0 = \sum_{\zeta \in \widehat{T}} (\dim \zeta)\, \chi_\zeta\,, \quad 1 = \chi_{\mathbf{1}_T} = - \sum_{\zeta \in \widehat{T}^*} (\dim \zeta)\, \chi_\zeta\,, \quad \text{on } T^*. \tag{21}$$

The parameter A of character is not necessarily unique because of the linear dependence (21) on T^* of functions χ_ζ, $\zeta \in \widehat{T}$. To establish uniqueness of parameter, we transfer from the parameter A, to another parameter $B = \phi(A)$ given by

$$\begin{aligned} B = \phi(A) &:= \left((\alpha_{\zeta,\varepsilon})_{(\zeta,\varepsilon) \in \widehat{T} \times \{0,1\}}\ ;\ \kappa \right), \\ \text{with} \qquad \kappa &= (\kappa_\zeta)_{\zeta \in \widehat{T}^*}\,, \quad \kappa_\zeta = \mu_\zeta - (\dim \zeta)^2 \mu_{\mathbf{1}_T} \quad (\zeta \in \widehat{T}^*). \end{aligned} \tag{22}$$

Then, the uniqueness of parameter is established. However the inequality (17) for the range of parameter A containing μ cannot be translated in a compact form in another parameter $\phi(A)$ containing κ in place of μ.

We can propose another normalization of the parameter $\mu = (\mu_\zeta)_{\zeta \in \widehat{T}}$, $\mu_\zeta \geq 0$, in the case where T is non-trivial. It is the following maximal condition on A, whose merit is that the character formula (19) is valid even for $t_{q_k} = e_T$ (not necessarily $t_{q_k} \in T^*$):

$$\text{(MAX)} \qquad \sum_{\zeta \in \widehat{T}} \sum_{\varepsilon \in \{0,1\}} \|\alpha_{\zeta,\varepsilon}\| + \|\mu\| = 1. \tag{23}$$

5 Characters of wreath product group $\mathfrak{S}_\infty(T)$ with T abelian

When T is abelian, the general character formula (19) for $\mathfrak{S}_\infty(T) = D_\infty(T) \rtimes \mathfrak{S}_\infty$ with a compact group T has a simplified form.

In this abelian case, $\widehat{T}$ is nothing but the dual group consisting of all one-dimensional characters of T, and for each $\zeta \in \widehat{T}$, its character χ_ζ is identified with ζ itself.

For a $g \in G = \mathfrak{S}_\infty(T)$, let its standard decomposition be as in (14), $g = \xi_{q_1}\xi_{q_2}\cdots\xi_{q_r} g_1 g_2 \cdots g_m$, with $\xi_{q_k} = (t_{q_k}, (q_k)), t_{q_k} \neq e_T$, for $1 \leq k \leq r$, and $g_j = (d_j, \sigma_j)$ for $1 \leq j \leq m$. Put $K_j = \mathrm{supp}(\sigma_j)$, and for $d_j = (t_i)_{i \in K_j} \in D_{K_j}(T) \hookrightarrow D_\infty(T)$, put

$$P_{K_j}(d_j) = \textstyle\prod_{i \in K_j} t_i\,, \quad \zeta(d_j) := \zeta(P_{K_j}(d_j)) = \prod_{i \in K_j} \zeta(t_i)\,. \tag{24}$$

As a parameter for characters of $G = \mathfrak{S}_\infty(T)$, we prepare a set

$$\alpha_{\zeta,\varepsilon} \ (\zeta \in \widehat{T}, \varepsilon \in \{0,1\}), \quad \text{and} \quad \mu = (\mu_\zeta)_{\zeta \in \widehat{T}}\,, \tag{25}$$

of decreasing sequences of non-negative numbers $\alpha_{\zeta,\varepsilon} = (\alpha_{\zeta,\varepsilon,i})_{i\in\boldsymbol{N}}$, and a set of non-negative $\mu_\zeta \geq 0$ $(\zeta \in \widehat{T})$, which satisfies the condition

$$\sum_{\zeta\in\widehat{T}} \sum_{\varepsilon\in\{0,1\}} \|\alpha_{\zeta,\varepsilon}\| + \|\mu\| \leq 1. \tag{26}$$

Theorem 3. *Let $G = \mathfrak{S}_\infty(T)$ be a wreath product group of a compact abelian group T with $\mathfrak{S}_\infty$. Then, for a parameter $A := \left((\alpha_{\zeta,\varepsilon})_{(\zeta,\varepsilon)\in\widehat{T}\times\{0,1\}}\,;\,\mu\right)$, in (25)–(26), the following formula determines a character f_A of G: for an element $g \in G$, let its standard decomposition be as above, then*

$$\begin{aligned} f_A(g) \;=\; & \prod_{1\leq k\leq r} \left\{ \sum_{\zeta\in\widehat{T}} \left(\sum_{\varepsilon\in\{0,1\}} \sum_{i\in\boldsymbol{N}} \alpha_{\zeta,\varepsilon,i} + \mu_\zeta \right) \zeta(t_{q_k}) \right\} \\ & \times \prod_{1\leq j\leq m} \left\{ \sum_{\zeta\in\widehat{T}} \left(\sum_{\varepsilon\in\{0,1\}} \sum_{i\in\boldsymbol{N}} (\alpha_{\zeta,\varepsilon,i})^{\ell(\sigma_j)} \cdot \chi_\varepsilon(\sigma_j) \right) \zeta(d_j) \right\}, \end{aligned} \tag{27}$$

where $\chi_\varepsilon(\sigma_j) = \mathrm{sgn}_{\mathfrak{S}}(\sigma_j)^\varepsilon = (-1)^{\varepsilon(\ell(\sigma_j)-1)}$, and $\zeta(d_j)$ as in (24).

Conversely any character of G is given in the form of f_A.

6 Characters of the subgroup $\mathfrak{S}^e_\infty(T) \subset \mathfrak{S}_\infty(T)$, T abelian

For the natural subgroup $G^e := \mathfrak{S}^e_\infty(T) = D^e_\infty(T) \rtimes \mathfrak{S}_\infty$ with

$$D^e_\infty(T) := \{\, d = (t_i)_{i\in\boldsymbol{N}}\,;\, P(d) = e_T \,\}, \quad P(d) := \textstyle\prod_{i\in\boldsymbol{N}} t_i, \tag{28}$$

we deduce a general character formula from the one for $G := \mathfrak{S}_\infty(T)$.

Take an element $g \in G^e = \mathfrak{S}^e_\infty(T)$ and let its standard decomposition be $g = \xi_{q_1}\xi_{q_2}\cdots\xi_{q_r}$ $g_1 g_2 \cdots g_m$ with $\xi_{q_k} = (t_{q_k}, (q_k))$ and $g_j = (d_j, \sigma_j), d_j = (t_i)_{i\in K_j}, K_j = \mathrm{supp}(\sigma_j)$. Note that each component ξ_{q_k} does not belong to G^e, and that the component $g_j = (d_j, \sigma_j)$ belongs to G^e if and only if $P(d_j) = \prod_{i\in K_j} t_i = e_T$. However, after careful discussions on the relation between G^e and G, we obtain the following result for the subgroup G^e from the result for G.

Theorem 4. (i) *Let $G^e = \mathfrak{S}^e_\infty(T)$ be the subgroup of $G = \mathfrak{S}_\infty(T)$ given by (28). For a parameter*

$$A := \left((\alpha_{\zeta,\varepsilon})_{(\zeta,\varepsilon)\in\widehat{T}\times\{0,1\}}\,;\,\mu\right), \tag{29}$$

in (25)–(26), the following formula determines a character f^e_A of G^e: for an element $g \in G^e$, let its standard decomposition be as above, then

$$\begin{aligned} f^e_A(g) \;=\; & \prod_{1\leq k\leq r} \left\{ \sum_{\zeta\in\widehat{T}} \left(\sum_{\varepsilon\in\{0,1\}} \sum_{i\in\boldsymbol{N}} \alpha_{\zeta,\varepsilon,i} + \mu_\zeta \right) \zeta(t_{q_k}) \right\} \\ & \times \prod_{1\leq j\leq m} \left\{ \sum_{\zeta\in\widehat{T}} \left(\sum_{\varepsilon\in\{0,1\}} \sum_{i\in\boldsymbol{N}} (\alpha_{\zeta,\varepsilon,i})^{\ell(\sigma_j)} \cdot \chi_\varepsilon(\sigma_j) \right) \zeta(d_j) \right\}, \end{aligned} \tag{30}$$

where $\chi_\varepsilon(\sigma_j) = \mathrm{sgn}_{\mathfrak{S}}(\sigma_j)^\varepsilon = (-1)^{\varepsilon(\ell(\sigma_j)-1)}$, and $\zeta(d_j)$ as in (24).

Conversely any character of G^e is given in the form of f_A^e.

(ii) *Assume that two parameters for characters*

$$A = \Big((\alpha_{\zeta,\varepsilon})_{(\zeta,\varepsilon)\in\widehat{T}\times\{0,1\}}\ ;\ \mu\Big) \quad \textit{and} \quad A' = \Big((\alpha'_{\zeta,\varepsilon})_{(\zeta,\varepsilon)\in\widehat{T}\times\{0,1\}}\ ;\ \mu'\Big)$$

satisfy the normalization condition (MAX) *for μ and μ' respectively. Then, they determine the same character, that is, $f_A^e = f_{A'}^e$, if and only if $A' = R(\zeta_0)A$ for some $\zeta_0 \in \widehat{T}$, where*

$$R(\zeta_0)A := \Big((\alpha'_{\zeta,\varepsilon})_{(\zeta,\varepsilon)\in\widehat{T}\times\{0,1\}}\ ;\ R(\zeta_0)\mu\Big) \tag{31}$$

$$\textit{with} \quad \alpha'_{\zeta,\varepsilon} = \alpha_{\zeta\zeta_0^{-1},\varepsilon}\ \big((\zeta,\varepsilon)\in\widehat{T}\times\{0,1\}\big); \quad R(\zeta_0)\mu = (\mu'_\zeta)_{\zeta\in\widehat{T}},\ \mu'_\zeta = \mu_{\zeta\zeta_0^{-1}}.$$

In this case, as characters on $G \supset G^e$, we have $f_{A'}(g) = \pi_{\zeta_0,0}(g)\cdot f_A(g)\ \ (g \in G)$.

7 Method of proving Theorem 2 : The first part

Our proof of Theorem 2 can be carried out just as in the case of $\mathfrak{S}_\infty(T)$ with finite groups T in [HH2]. It consists of two parts. The first part is to prepare seemingly sufficiently big family of factorizable (hence extremal by the criterion in Theorem 6 below) continuous positive definite class functions on $G = \mathfrak{S}_\infty(T)$. The second part is to guarantee that actually all extremal continuous positive definite class functions or characters have been already obtained in the first part.

Here in this section we explain the first part of the proof. It has two important ingredients.

7.1. Limits of centralizations of positive definite functions.

The first ingredient is a method of *taking limits of centralizations* of positive definite functions. For a continuous positive definite function f on a topological group G and a compact subgroup $G' \subset G$, we define a *centralization* of f with respect to G' as

$$f^{G'}(g) := \int_{g'\in G'} f(g'gg'^{-1})\, d\mu_{G'}(g'), \tag{32}$$

where $d\mu_{G'}$ denotes the normalized Haar measure on G'.

Assume that we have an increasing sequence of compact subgroups $G_N \nearrow G$. Consider a series f^{G_N} of centralizations of f with respect to G_N and study its pointwise convergence limit, $\lim_{N\to\infty} f^{G_N}$, which depends heavily on the choice of the series $G_N \nearrow G$. We study the case where the limit function is again continuous.

In the case of discrete groups, we have studied in [Hi3]–[Hi4] limits of centralizations of positive definite functions on the infinite symmetric group $G = \mathfrak{S}_\infty$, and in particular recovered all the characters of G given in [Th2]. We have also calculated in [HH1] for $G = \mathfrak{S}_\infty(T)$ with T any finite abelian group T, and in [HH2] for $G = \mathfrak{S}_\infty(T)$ with T any finite group T, various limits of centaralizations of positive definite matrix elements of irreducible or non-irreducible representations induced from subgroups of wreath product type.

Observation. *For a certain choice of a subgroup H and one of its unitary representation π, the family of limits of centralizations of matrix elements of the induced representation $\rho = \mathrm{Ind}_H^G\pi$ covers all the characters of the group $G = \mathfrak{S}_\infty(T)$ with T any finite group.*

7.2. Inducing up of positive definite functions and their centralizations.

The second ingredient is *inducing up positive definite functions* from subgroups. After choosing appropriate subgroups H and their representations π, we use their matrix elements f_π as positive definite functions on H to be induced up to G, and then to be centralized. We have constructed in [Hi1] a huge family of irreducible unitary representations (= IURs) of a wreath product group $G = \mathfrak{S}_\infty(T) = D_\infty(T) \rtimes \mathfrak{S}_\infty$ with any finite group T, by taking so-called wreath product type subgroups H in a 'saturated fashion', and their IURs π of a certain form to get IURs of G as induced representations $\rho = \mathrm{Ind}_H^G \pi$.

For our present purpose of getting (seemingly) all possible extremal continuous positive definite class functions on G, we choose simpler subgroups of degenerate wreath product type and their IURs. In this case, we get unitary representations $\rho = \mathrm{Ind}_H^G \pi$ which are very far from to be irreducible, but sufficient for our purpose. This method can be applied to the case with T any compact group.

7.2.1. Inducing up of positive definite functions. In a general setting, let G be a group and H its subgroup. For a positive definite function f on H, extend it onto G by putting 0 outside H and denote it by $\mathrm{Ind}_H^G f$ or again by the same symbol f in case of no danger of misunderstanding. Then, *the inducing up* $\mathrm{Ind}_H^G f$ *of* f *onto* G *is again positive definite.*

In the case where G is a topological group, the function $\mathrm{Ind}_H^G f$ is usually not continuous even if f is continuous on H with respect to the relative topology. Let G' be a compact subgroup of G and take a centralization $F^{G'}$ of $F = \mathrm{Ind}_H^G f$. Since F is zero outside of H, the value of centralization $F^{G'}(g)$ is $\neq 0$ only for elements g which are conjugate under G_N to some $h \in H$, and moreover, for $h \in H$, $F^{G'}(h) = \int_{G'} f(g'hg'^{\,-1})\, d\mu_{G'}(g')$, where $f(g'hg'^{\,-1}) = 0$ if $g'hg'^{\,-1} \notin H$. We can hope that $F^{G'}$ is continuous on G'.

We study limits of F^{G_N} for increasing sequences $G_N \nearrow G$ of compact subgroups of G, hoping that the limits give us positive definite functions which are continuous with respect to the inductive limit topology $\tau_{ind} = \lim_{N\to\infty} \tau_{G_N}$.

Let us give a remark on the case where G is discrete. Take a unitary representation π of a subgroup H, and consider an induced representation $\rho = \mathrm{Ind}_H^G \pi$. For a non-zero vector v in the representation space of π, consider a positive definite function on H associated to π as $f_\pi(h) = \langle \pi(h)v, v\rangle \ (h \in H)$.

Proposition 5. *Assume G be discrete. The inducing up $F = \mathrm{Ind}_H^G f_\pi$ of a positive definite function f_π on H associated to π is again a positive definite function on G associated to $\rho = \mathrm{Ind}_H^G \pi$.*

7.2.2. Centralizations of $F = \mathrm{Ind}_H^G f_\pi$ and combinatorial calculations.

Let $G_N \nearrow G$ be an increasing sequence of compact subgroups going up to G, then

$$F^{G_N}(h) = \int_{G_N} f_\pi(g'hg'^{-1})\, d\mu_{G_N}(g'), \tag{33}$$

where $f_\pi(g'hg'^{-1}) = 0$ if $g'hg'^{-1} \notin H$. The condition $g'hg'^{-1} \in H$ for $g' \in G_N$, is translated into certain combinatorial conditions, and to get the limit as $N \to \infty$, we have to calculate asymtotic behavior of several ratios of combinatorial numbers.

The details in the case of $G = \mathfrak{S}_\infty$ and $\mathfrak{S}_\infty(T)$ with T finite are given in the papers cited above.

8 The second part of the proof

The second part of our proof contains also two ingredients.

8.1. The first one is to generalize Thoma's criterion, Satz 1 in [Th2], for that a positive definite class function is extremal or indecomposable.

Theorem 6. *Let T be a compact group, and f a continuous positive definite class function on $G = \mathfrak{S}_\infty(T)$ normalized as $f(e) = 1$. Then f is extremal (or indecomposable), if and only if it has the following properties which are mutually equivalent:*

(FTP) [Factorizability Property] *For any $g = (d, \sigma) \in G$, let $g = \xi_{q_1}\xi_{q_2}\cdots\xi_{q_r}g_1g_2\cdots g_m$, $\xi_q = (t_q, (q))$, $g_j = (d_j, \sigma_j)$, be a standard decomposition. Then, $f(g) = \prod_{1\le k\le r} f(\xi_{q_k}) \times \prod_{1\le j\le m} f(g_j)$.*

(FTP$'$) *For any two elements g, g' with disjoint supports, $f(gg') = f(g)f(g')$.*

8.2. The second ingredient is to determine the range of parameters for extremal continuous positive definite class functions f. Since f is factorizable, $f(g)$ is written as $f(g) = \prod_{1\le k\le r} f(\xi_{q_k}) \prod_{1\le j\le m} f(g_j)$ for $g = \xi_{q_1}\cdots\xi_{q_m} g_1 \cdots g_m$. Then, we take a kind of *Fourier transform* of f on $G = D_\infty(T) \rtimes \mathfrak{S}_\infty$ with respect to the subgroup $D_\infty(T)$, and get a positive definite class function on $\mathfrak{S}_\infty$. Then for this we appeal to Korollar 1 to Satz 2 in [Th2]. Thus actually we find that the parameter for f is given as $A = \big((\alpha_{\zeta,\varepsilon})_{(\zeta,\varepsilon)\in\widehat{T}\times\{0,1\}}, \mu\big)$ in Theorem 2, and its range is given by (17) as asserted in Theorem 2.

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