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The Euler-Selberg constants for nonuniform lattices of rank one symmetric spaces

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Abstract

It is well-known that the Euler constant γ is taken for the constant term of the Laurent expansion at $s = 1$ of the Riemann zeta function. Similarly, for the Selberg zeta function of a nonuniform lattice Γ of a rank one symmetric space, an analogue of the Euler constant which we call *the Euler-Selberg constant* can be defined. We establish a certain expression of the Euler-Selberg constant which is similar to the one γ possesses as the sum over the hyperbolic conjugacy classes of Γ. As an application, we give a numerical computation of the Euler-Selberg constant when $\Gamma = SL_2(\mathbb{Z})$.

Introduction

Let G be a connected non-compact semisimple Lie group with finite center, K a maximal compact subgroup of G, and Γ a discrete subgroup of G such that $\Gamma\backslash G/K$ is not necessarily compact but its volume is finite. We denote by ζ_Γ (a shifted ratio of the Selberg zeta function Z_Γ, see (1.1)) the function defined by the Euler product over the primitive hyperbolic classes of Γ as follows;

$$\zeta_\Gamma(s) = \prod_{\delta\in\mathrm{Prim}(\Gamma)} (1 - N(\delta)^{-s})^{-1} \qquad \mathrm{Re}\, s > 2\rho_0,$$

where $\mathrm{Prim}(\Gamma)$ denotes a set of primitive hyperbolic conjugacy classes of Γ, $N(\delta)$ is the norm of δ, and $2\rho_0$ is an explicitly determined constant depending only on the structure of G/K.

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It is known that $\zeta_\Gamma(s)$ has a simple pole at $s = 2\rho_0$. Hence the Laurent expansion of $\zeta_\Gamma(s)$ around $s = 2\rho_o$ is written in the form

$$\zeta_\Gamma(s) = \frac{C}{s - 2\rho_0} + \gamma_\Gamma + \sum_{n=1}^{\infty} \gamma_\Gamma^{(n)} (s - 2\rho_0)^n. \tag{0.1}$$

We call γ_Γ *the Euler-Selberg constant* (of the second kind).

Recall the Riemann zeta function $\zeta(s)$;

$$\zeta(s) = \prod_{p;\text{prime}} (1 - p^{-s})^{-1} \qquad \mathrm{Re}\, s > 1.$$

It is well-known that $\zeta(s)$ has a simple pole at $s = 1$, so its Laurent expansion at $s = 1$ is written as

$$\zeta(s) = \frac{1}{s-1} + \gamma + \sum_{n=1}^{\infty} \gamma^{(n)} (s - 1)^n,$$

where the coefficient γ is known as the Euler constant

$$\gamma = \lim_{x \to \infty} \left(\sum_{n<x} \frac{1}{n} - \log x \right) = 0.57721\ldots.$$

In the previous paper [HIKW], we establish certain interesting expressions not only of γ_Γ but of the higher coefficients $\gamma_\Gamma^{(n)}$'s for the cases of the compact Riemann surfaces. The aim of the present paper is to generalize the result of [HIKW] to non-compact Riemannian locally symmetric spaces (see Theorem 1.1).

1 Preliminaries

Let G be a connected non-compact semisimple Lie group with finite center, and K a maximal compact subgroup of G. We put $r = \dim(G/K)$. We denote by $\mathfrak{g}$, $\mathfrak{k}$ the Lie algebras of G, K respectively and $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ a Cartan decomposition with respect to the Cartan involution θ. Let $\mathfrak{a}_\mathfrak{p}$ be a maximal abelian subspace of $\mathfrak{p}$. Throughout this paper we assume that $\mathrm{rank}(G/K) = 1$, that is, $\dim \mathfrak{a}_\mathfrak{p} = 1$. We extend $\mathfrak{a}_\mathfrak{p}$ to a θ-stable maximal abelian subalgebra $\mathfrak{a}$ of $\mathfrak{g}$, so that $\mathfrak{a} = \mathfrak{a}_\mathfrak{p} + \mathfrak{a}_\mathfrak{k}$, where $\mathfrak{a}_\mathfrak{p} = \mathfrak{a} \cap \mathfrak{p}$ and $\mathfrak{a}_\mathfrak{k} = \mathfrak{a} \cap \mathfrak{k}$. We put $A = \exp \mathfrak{a}$, $A_\mathfrak{p} = \exp \mathfrak{a}_\mathfrak{p}$ and $A_\mathfrak{k} = \exp \mathfrak{a}_\mathfrak{k}$.

We denote by $\mathfrak{g}^\mathbb{C}$, $\mathfrak{a}^\mathbb{C}$ the complexification of $\mathfrak{g}$, $\mathfrak{a}$ respectively. Let Φ be the set of roots of $(\mathfrak{g}^\mathbb{C}, \mathfrak{a}^\mathbb{C})$, Φ^+ the set of positive roots in Φ, $P_+ = \{\alpha \in \Phi^+ | \alpha \not\equiv 0 \text{ on } \mathfrak{a}_\mathfrak{p}\}$, and $P_- = \Phi^+ - P_+$. We put $\rho = 1/2 \sum_{\alpha \in P_+} \alpha$. For $h \in A$ and linear form λ on $\mathfrak{a}$, we denote by ξ_λ the character of $\mathfrak{a}$ given by $\xi_\lambda(h) = \exp \lambda(\log h)$. Let Σ be the set of restrictions to $\mathfrak{a}_\mathfrak{p}$ of the elements of

P_+. Then the set Σ is either of the form $\{\beta\}$ or $\{\beta, 2\beta\}$ with some element $\beta \in \Sigma$. We fix an element $H_0 \in \mathfrak{a}_{\mathfrak{p}}$ such that $\beta(H_0) = 1$, and put $\rho_0 = \rho(H_0)$.

Let Γ be a discrete subgroup of G such that $\Gamma\backslash G/K$ is not necessarily compact but has a finite volume. We denote by $C(\Gamma)$ a complete set of representatives of Γ-conjugacy classes of semisimple elements in Γ, $\mathrm{Prim}(\Gamma)$ a set of primitive hyperbolic conjugacy classes of Γ, and $Z(\Gamma)$ a center of Γ. For $\gamma \in C(\Gamma)$, we denote by δ_γ is the element of $\mathrm{Prim}(\Gamma)$ such that $\gamma = \delta_\gamma^j$, with some integer $j \geq 1$, $h(\gamma)$ the element of A which is conjugate to γ, and $h_{\mathfrak{p}}(\gamma)$, $h_{\mathfrak{k}}(\gamma)$ the elements of $A_{\mathfrak{p}}$, $A_{\mathfrak{k}}$ respectively such that $h(\gamma) = h_{\mathfrak{p}}(\gamma)h_{\mathfrak{k}}(\gamma)$. Let $N(\gamma)$ be a norm of γ given by $N(\gamma) = \exp(\beta(\log(h_{\mathfrak{p}}(\gamma))))$, and $D(\gamma)$ the function defined by

$$D(\gamma) = N(\gamma)^{2\rho_0} \prod_{\alpha \in P_+} |1 - \xi_\alpha(h(\gamma))^{-1}|.$$

We assume that Γ has no elements of finite order, other than those in $Z(\Gamma)$. Let d be the number of equivalence classes of Γ-cuspidal minimal parabolic subgroups of G, and $M(s)$ the scattering matrix with the determinant $\Psi(s)$. We denote by $\mu(s)$ the Plancherel measure of G/K. Let λ_j be the eigenvalue of the Laplacian on $\Gamma\backslash G/K$ such that $0 = \lambda_0 < \lambda_1 < \lambda_2 < \ldots$, and n_j the multiplicity of λ_j.

The Selberg zeta function $Z_\Gamma(s)$ of Γ is defined by

$$Z_\Gamma(s) = \prod_{\delta \in \mathrm{Prim}(\Gamma)} \prod_{\lambda \in L} \left(1 - \xi_\lambda(h(\delta))^{-1} N(\delta)^{-s}\right)^{4\kappa m_\lambda} \qquad \mathrm{Re}\, s > 2\rho_0, \tag{1.1}$$

where L is the semi-lattice of linear forms on $\mathfrak{a}$ given by $L = \left\{\sum_{i=1}^l m_i\alpha_i | \alpha_i \in P_+, m_i \in \mathbb{Z}_{\geq 0}\right\}$, m_λ is the number of distinct t-tuples $(m_1, \ldots, m_l)$ such that $\lambda = \sum_{i=1}^l m_i\alpha_i \in L$, and $\kappa > 0$ is an integer (see [GW]). If $G = SL_2(\mathbb{R})$, L can be identified to $\mathbb{Z}_{\geq 0}$, and κ can be taken 1.

It is easy to see that the logarithmic derivative of $Z_\Gamma(s)$ is given by

$$\frac{Z'_\Gamma(s)}{Z_\Gamma(s)} = 4\kappa \sum_{\gamma \in C(\Gamma) - Z(\Gamma)} \log N(\delta_\gamma) D(\gamma)^{-1} N(\gamma)^{2\rho_0 - s} \qquad \mathrm{Re}\, s > 2\rho_0. \tag{1.2}$$

This function is analytically continued to the whole complex plane as a meromorphic function by

$$\begin{aligned}
\frac{Z'_\Gamma(s)}{Z_\Gamma(s)} &= \sum_{j=0}^{\infty} n_j \left\{ \frac{H(i(s - \rho_0 + i\nu_j))}{s - \rho_0 + i\nu_j} + \frac{H(i(s - \rho_0 - i\nu_j))}{s - \rho_0 - i\nu_j} \right\} \\
&\quad - i\mathrm{vol}(\Gamma\backslash G)[Z(\Gamma)] \sum_{k=0}^{\infty} d_k \frac{H(i(s - \rho_0 + ir_k))}{s - \rho_0 + ir_k} \\
&\quad - \frac{1}{2}(d - \mathrm{tr}(M(0))) \frac{H(i(s - \rho_0))}{s - \rho_0} \\
&\quad - d \sum_{k=0}^{\infty} \frac{H(i(s - \rho_0 + k))}{s - \rho_0 + k} + \sum_{k=l+1}^{\infty} b_k \frac{H(i(s - \rho_0 + q_k))}{s - \rho_0 + q_k}.
\end{aligned} \tag{1.3}$$

Here the function H is defined in [GW]. This is an entire function and has the following property; For any integer $n \geq 1$, there exists $c_n > 0$ such that

$$\begin{aligned} |H(s)| &\leq c_n |s|^{-n} && \text{if} \quad \operatorname{Im} s \geq 0 \\ &\leq c_n |s|^{-n} \exp(\varepsilon |\operatorname{Im} s|) && \text{if} \quad \operatorname{Im} s < 0. \end{aligned} \tag{1.4}$$

The number ν_j is defined by the relation $\lambda_j = \rho_0^2 + \nu_j^2$, r_k's are the poles in the upper half plane of $\mu(s)$, d_k is the residue at $s = r_k$, q_k's are the poles of $\Psi'(s)/\Psi(s)$, and b_k is the residue at $s = q_k$. It is known that there exist positive integers N_1, N_2 such that

$$\sum_{k \geq 1} d_k |r_k|^{-N_1} < \infty \quad , \quad \sum_{k \geq 1} b_k |q_k|^{-N_2} < \infty. \tag{1.5}$$

The number κ is the integer such that, for any $k \geq 0$, $i\mathrm{vol}(\Gamma\backslash G) d_k \kappa$ is an integer multiple of the Euler-Poincare characteristic E of $\Gamma\backslash G/K$. We denote by e_k the integer such that $i\mathrm{vol}(\Gamma\backslash G) d_k \kappa = e_k E$. We choose H such that $H(0) = 4\kappa$.

The following functional equation for $Z_\Gamma(s)$ is due to [GW];

$$\begin{aligned} Z_\Gamma(2\rho_0 - s) = Z_\Gamma(s) &\left(\frac{\Gamma(1 - s + \rho_0)}{\Gamma(1 + s - \rho_0)} \right)^{4\kappa d} [\Psi(\rho_0 - s)]^{4\kappa} \prod_{k=1}^{l} \left(\frac{s - \rho_0 - q_k}{-s + \rho_0 - q_k} \right)^{4\kappa b_k} \\ &\exp\left[4\kappa \mathrm{vol}(\Gamma\backslash G)[Z(\Gamma)] \int_0^{s-\rho_0} \mu(it) dt + \int_{\rho_0}^{s} J(t) dt \right]. \end{aligned} \tag{1.6}$$

The function $J(t)$ is given in [GW]. The contribution of J appears only in the case when $\Gamma\backslash G/K$ is not compact. It is known that $J(t)$ is a polynomial in the case $G \neq SU(2n, 1)$. However, in the case $G = SU(2n, 1)$, little is known about J. This may cause several complicated problems. Throughout this paper, to avoid these technical difficulties, we exclude the case $G = SU(2n, 1)$ when Γ is a non-uniform lattice.

Because of (1.3), the point $s = 2\rho_0$ is a simple pole of the logarithmic derivative of $Z_\Gamma(s)$. Hence $Z_\Gamma'(s)/Z_\Gamma(s)$ can be expanded as

$$\frac{Z_\Gamma'(s)}{Z_\Gamma(s)} = \frac{4\kappa}{s - 2\rho_0} + \tilde{\gamma}_\Gamma + \sum_{n=1}^{\infty} \tilde{\gamma}_\Gamma^{(n)} (s - 2\rho_0)^n$$

around $s = 2\rho_0$. We call the coefficient $\tilde{\gamma}_\Gamma$ *the Euler-Selberg constant* (of the first kind) and $\tilde{\gamma}_\Gamma^{(n)}$ *the higher Euler-Selberg constant.* The main result of the present paper is the following expressions of the (higher) Euler-Selberg constants of the first kind $\tilde{\gamma}_\Gamma$ and $\tilde{\gamma}_\Gamma^{(n)}$'s.

Theorem 1.1. *We have*

$$\tilde{\gamma}_\Gamma = 4\kappa \lim_{x\to\infty}\left\{ \sum_{\substack{\gamma\in C(\Gamma)-Z(\Gamma)\\ N(\gamma)<x}} \log N(\delta_\gamma)D(\gamma)^{-1} - \log x \right\}, \tag{1.7}$$

$$\tilde{\gamma}_\Gamma^{(n)} = \frac{(-1)^n}{n!}4\kappa \lim_{x\to\infty}\left\{ \sum_{\substack{\gamma\in C(\Gamma)-Z(\Gamma)\\ N(\gamma)<x}} \log N(\delta_\gamma)(\log N(\gamma))^n D(\gamma)^{-1} - \frac{(\log x)^{n+1}}{n+1} \right\}. \tag{1.8}$$

As a corollary of Theorem 1.1, we obtain the following expressions of the Euler-Selberg constants of the second kind γ_Γ defined in (0.1).

Corollary 1.2. *We have*

$$\gamma_\Gamma = -C \lim_{x\to\infty}\left\{ \sum_{\substack{\gamma\in C(\Gamma)-Z(\Gamma)\\ N(\gamma)<x}} \frac{\log N(\delta_\gamma)}{N(\gamma)^{2\rho_0}} - \log x \right\},$$

where C is the residue at $s = 2\rho_0$ of $\zeta_\Gamma(s)$ defined in (0.1).

Proof. Let F_Γ be the function given by

$$\zeta_\Gamma(s)^{4\kappa} = F_\Gamma(s)/Z_\Gamma(s). \tag{1.9}$$

For instance, in the case $G = SL_2(\mathbb{R})$, $F_\Gamma(s)$ coincides $Z_\Gamma(s+1)$. Because the singularity at $s = 2\rho_0$ of $1/Z_\Gamma(s)$ comes from that of $\zeta_\Gamma(s)$, $F_\Gamma(s)$ is holomorphic at $s = 2\rho_0$. Observing the coefficients of the Laurent expansions of the logarithmic derivatives of (1.9), it is easy to see that

$$4\kappa\frac{\gamma_\Gamma}{C} = \frac{F_\Gamma'(2\rho_0)}{F_\Gamma(2\rho_0)} - \tilde{\gamma}_\Gamma, \qquad 4\kappa\Big(2\frac{\gamma_\Gamma^{(1)}}{C} - \frac{\gamma_\Gamma^2}{C^2}\Big) = \Big(\frac{d}{ds}\frac{F_\Gamma'}{F_\Gamma}\Big)(2\rho_0) - \tilde{\gamma}_\Gamma^{(1)},$$
$$4\kappa\Big(3\frac{\gamma_\Gamma^{(2)}}{C} - 3\frac{\gamma_\Gamma^{(1)}\gamma_\Gamma}{C^2} + \frac{\gamma_\Gamma^3}{C^3}\Big) = \frac{1}{2}\Big(\frac{d^2}{ds^2}\frac{F_\Gamma'}{F_\Gamma}\Big)(2\rho_0) - \tilde{\gamma}_\Gamma^{(2)}, \ldots. \tag{1.10}$$

By Theorem 1.1 and

$$\frac{F_\Gamma'(s)}{F_\Gamma(s)} = 4\kappa \sum_{\gamma\in C(\Gamma)-Z(\Gamma)} \log N(\delta_\gamma)\big(D(\gamma)^{-1} - N(\gamma)^{-2\rho_0}\big)N(\gamma)^{2\rho_0 - s} \qquad \mathrm{Re}\, s \geq 2\rho_0,$$

we obtain the desired result. □

Remark 1.1. Using the relations (1.10), in principle similar expressions of $\gamma_\Gamma^{(n)}$'s can be obtained inductively.

Example 1.1. *The case* $G = SL_2(\mathbb{R})$, $K = SO(2)$ (when Γ is co-compact, see [HIKW]): In this case, $r = 2$, $\rho_0 = \frac{1}{2}$, $\mathrm{Hyp}(\Gamma)$ is the set of the hyperbolic conjugacy classes of Γ, $Z(\Gamma) = \{1\}$, $N(\gamma)$ is the square of the larger eigenvalue of γ. The Selberg zeta function Z_Γ is defined by

$$Z_\Gamma(s) = \prod_{\delta \in \mathrm{Prim}(\Gamma)} \prod_{n=0}^{\infty} (1 - N(\delta)^{-s-n}) \qquad \mathrm{Re}\, s > 1.$$

Then the following expression of the Euler-Selberg constant $\tilde{\gamma}_\Gamma$ is obtained in [HIKW];

$$\tilde{\gamma}_\Gamma = \lim_{x \to \infty} \left\{ \sum_{\substack{\gamma \in \mathrm{Hyp}(\Gamma) \\ N(\gamma) < x}} \frac{\log N(\delta_\gamma)}{N(\gamma) - 1} - \log x \right\}.$$

Since $F_\Gamma(s) = Z_\Gamma(s+1)$ (see (1.9)), it is easy to see that

$$C = \frac{Z_\Gamma(2)}{Z'_\Gamma(1)}.$$

Hence, by Colorally 1.2, we have

$$\gamma_\Gamma = -\frac{Z_\Gamma(2)}{Z'_\Gamma(1)} \lim_{x \to \infty} \left\{ \sum_{\substack{\gamma \in \mathrm{Hyp}(\Gamma) \\ N(\gamma) < x}} \frac{\log N(\delta_\gamma)}{N(\gamma)} - \log x \right\}.$$

Remark 1.2. When $G = SL_2(\mathbb{R})$, several studies related to $\tilde{\gamma}_\Gamma$ and $\tilde{\gamma}_\Gamma^{(n)}$ have been made as follows;

1) the power sums of λ_j's: (see [HIKW])

When Γ is a co-compact torsion free discrete subgroup of $SL_2(\mathbb{R})$, the values at the integer points of $\zeta_\Delta(s) = \sum_{j=1}^{\infty} n_j \lambda_j^{-s}$ are expressed using $\tilde{\gamma}_\Gamma$ and $\tilde{\gamma}_\Gamma^{(n)}$'s. For instance,

$$\zeta_\Delta(2) = 2\tilde{\gamma}_\Gamma - \tilde{\gamma}_\Gamma^{(1)} + 2(g-1)\zeta(2) - 3,$$

where g is the genus of the compact Riemann surface $\Gamma \backslash G/K$.

2) a relation with the Arakelov geometry:

We denote by $X_0(N)$ the modular curve associated to the congruence subgroup $\Gamma_0(N)$, and $\mathcal{X}_0(N)$ the minimal regular model of $X_0(N)$. Let ω_N be the relative dualizing sheaf of $\mathcal{X}_0(N)$ equipped with the Arakelov metric, and ω_N^2 the arithmetic self-intersection number of ω_N. According to [AU], when N is square-free and $\gcd(N, 6) = 1$, ω_N^2 is bounded as

$$\omega_N^2 \leq -\frac{8\pi(g_N - 1)}{\mathrm{vol}(X_0(N))} \tilde{\gamma}_{\Gamma_0(N)} + g_N \left(2\log N + \sum_{\substack{p;\mathrm{prime} \\ p|N}} \frac{p+1}{p-1} \log p \right) + o(g_N \log N) \quad \text{as} \quad N \to \infty,$$

where g_N is the genus of $X_0(N)$.

Remark 1.3. (numerical computations) When $\Gamma = SL_2(\mathbb{Z})$, by the relation between hyperbolic conjugacy classes of $SL_2(\mathbb{Z})$ and equivalence classes of indefinite binary quadratic forms (see, e.g. [Sar]), $\tilde{\gamma}_{SL_2(\mathbb{Z})}$ can be written as

$$\tilde{\gamma}_{SL_2(\mathbb{Z})} = \lim_{T\to\infty}\left\{\sum_{t=3}^{T} \sum_{\substack{u;u^2|t^2-4 \\ d(t,u)\equiv 0,1 \bmod 4}} h\big(d(t,u)\big)\frac{2\log\varepsilon_0(t,u)}{\varepsilon(t)^2-1} - 2\log T\right\},$$

where

$$d(t,u) = \frac{t^2-4}{u^2},$$
$$\varepsilon(t) = \frac{1}{2}(t+\sqrt{t^2-4}) = \frac{1}{2}\big(t+u\sqrt{d(t,u)}\big),$$
$$\varepsilon_0(t,u) = \min\Big\{\frac{1}{2}\big(t_0+u_0\sqrt{d(t,u)}\big) \,\Big|\, \big(\frac{1}{2}(t_0+u_0\sqrt{d(t,u)})\big)^k = \varepsilon(t) \quad \text{for some } k\geq 1\Big\},$$

and $h(d)$ is the class number of the binary quadratic form with the determinant d. Actually calculating

$$\gamma(T) := \sum_{t=3}^{T} \sum_{\substack{u;u^2|t^2-4 \\ d(t,u)\equiv 0,1 \bmod 4}} h\big(d(t,u)\big)\frac{2\log\varepsilon_0(t,u)}{\varepsilon(t)^2-1} - 2\log T$$

for $T \leq 100,000$ by computer (the algorithm of calculating $h(d)$ is due to [Wada]), the following data appears.

T	$\gamma(T)$				
100	-2.681 ...	1,000	-2.7124 ...	10,000	-2.71538 ...
200	-2.697 ...	2,000	-2.7137 ...	20,000	-2.71546 ...
300	-2.703 ...	3,000	-2.7141 ...	30,000	-2.71552 ...
400	-2.704 ...	4,000	-2.7146 ...	40,000	-2.71551 ...
500	-2.711 ...	5,000	-2.7149 ...	50,000	-2.71556 ...
600	-2.711 ...	6,000	-2.7152 ...	60,000	-2.71551 ...
700	-2.713 ...	7,000	-2.7150 ...	70,000	-2.71554 ...
800	-2.710 ...	8,000	-2.7152 ...	80,000	-2.71557 ...
900	-2.712 ...	9,000	-2.7157 ...	90,000	-2.71558 ...
				100,000	-2.71555 ...

According to the data above, we conclude that

$$\tilde{\gamma}_{SL_2(\mathbb{Z})} = -2.715\ldots.$$

Example 1.2. *The case $G = SL_2(\mathbb{C})$, $K = SU(2)$:* In this case, $r=3$, $\rho_0 = 1$, $C(\Gamma)$ is the set of the hyperbolic or loxodromic conjugacy classes of Γ, $Z(\Gamma) = \{1\}$, $N(\gamma) = |a(\gamma)|^2$, where $a(\gamma)$ is the eigenvalue of γ such that $|a(\gamma)|>1$, $m(\gamma)$ is the order of the torsion of the centralizer of γ. The Selberg zeta function Z_Γ is defined as

$$Z_\Gamma(s)=\prod_{\delta\in\mathrm{Prim}(\Gamma)}\prod_{\substack{k\equiv l\\ (\mathrm{mod}\ m(\delta))}}(1-a(\delta)^{-2k}\overline{a(\delta)}^{-2l}N(\delta)^{-s})\qquad \mathrm{Re}\, s>2.$$

Then the Euler-Selberg constant is given as

$$\tilde{\gamma}_\Gamma=\lim_{x\to\infty}\left\{\sum_{\substack{\gamma\in C(\Gamma)-\{1\}\\ N(\gamma)<x}}\frac{\log N(\delta_\gamma)}{m(\gamma)|a(\gamma)^2-1|^2}-\log x\right\}.$$

2 The proof of Theorem 1.1

2.1 Some Lemmas

We prepare some lemmas. First, related to the prime geodesic theorem, we recall the following estimate (see, for example [DeG], [GW]);

$$\psi(x)=:\sum_{\substack{\gamma\in C(\Gamma)-Z(\Gamma)\\ N(\gamma)<x}}\log N(\delta_\gamma)=\frac{x^{2\rho_0}}{2\rho_0}+O(x^{2\rho_0-\delta})\quad\text{as}\quad x\to\infty. \tag{2.1}$$

Based on this fact, we show the following lemma.

Lemma 2.1. *For any $n\geq 0$, $m\geq 1$, we have*

$$\sum_{\substack{\gamma\in C(\Gamma)-Z(\Gamma)\\ N(\gamma)<x}}\log N(\delta_\gamma)(\log N(\gamma))^n D(\gamma)^{-1}=\frac{(\log x)^{n+1}}{n+1}+A_n+O(x^{-\delta})\quad as\quad x\to\infty, \tag{2.2}$$

$$\sum_{\substack{\gamma\in C(\Gamma)-Z(\Gamma)\\ N(\gamma)<x}}\log N(\delta_\gamma)D(\gamma)^{-1}N(\gamma)^m=\frac{x^m}{m}+O(x^{m-\delta'})\quad as\quad x\to\infty, \tag{2.3}$$

where $\delta,\delta'>0$ and $A_n>0$ are some constants.

Proof. We first study

$$\hat{\psi}_n(x)=:\sum_{\substack{\gamma\in C(\Gamma)-Z(\Gamma)\\ N(\gamma)<x}}\log N(\delta_\gamma)(\log N(\gamma))^n N(\gamma)^{-2\rho_0},$$

$$\check{\psi}_m(x)=:\sum_{\substack{\gamma\in C(\Gamma)-Z(\Gamma)\\ N(\gamma)<x}}\log N(\delta_\gamma)N(\gamma)^{-2\rho_0+m}.$$

We can write $\hat{\psi}_n(x)$ and $\check{\psi}_m(x)$ as the Stieltjes integral;

$$\hat{\psi}_n(x) = \int_\tau^x (\log t)^n t^{-2\rho_0} d\psi(t), \qquad \check{\psi}_m(x) = \int_\tau^x t^{-2\rho_0+m} d\psi(t),$$

where $\tau = \min\{N(\gamma) | \gamma \in C(\Gamma) - Z(\Gamma)\}$.

Calculating the right hand side of each expression using (2.1), we obtain

$$\hat{\psi}_n(x) = \frac{(\log x)^{n+1}}{n+1} + C_n + O(x^{-\delta}) \quad \text{as} \quad x \to \infty,$$
$$\check{\psi}_m(x) = \frac{x^m}{m} + O(x^{m-\delta'}) \quad \text{as} \quad x \to \infty,$$

where $\delta, \delta' > 0$ and $C_n > 0$ are some constants.

Now we consider the difference between $N(\gamma)^{-2\rho_0}$ and $D(\gamma)^{-1}$;

$$\begin{aligned} |N(\gamma)^{-2\rho_0} - D(\gamma)^{-1}| =& N(\gamma)^{-2\rho_0} \Big|1 - \prod_{\alpha \in P_+} |1 - \xi_\alpha(h(\gamma))^{-1}|^{-1}\Big| \\ \leq& N(\gamma)^{-2\rho_0} \Big(\sum_{\lambda \in L'} |\xi_\lambda(h(\gamma))|^{-1}\Big) / \prod_{\alpha \in P_+} |1 - \xi_\alpha(h(\gamma))^{-1}| \\ \leq& C N(\gamma)^{-2\rho_0 - \varepsilon}, \end{aligned}$$

where L' is some subset of L, and $C, \varepsilon > 0$ are constants. Hence it is easy to see that

$$\Bigg| \sum_{\substack{\gamma \in C(\Gamma) - Z(\Gamma) \\ N(\gamma) < x}} \log N(\delta_\gamma) (\log N(\gamma))^n D(\gamma)^{-1} - \hat{\psi}_n(x) \Bigg| = O(1) \quad \text{as} \quad x \to \infty,$$

$$\Bigg| \sum_{\substack{\gamma \in C(\Gamma) - Z(\Gamma) \\ N(\gamma) < x}} \log N(\delta_\gamma) D(\gamma)^{-1} N(\gamma)^m - \check{\psi}_m(x) \Bigg| = O(x^{m-\varepsilon}) \quad \text{as} \quad x \to \infty.$$

Thus we obtain the desired estimates. □

We also need to have a behavior of $Z'_\Gamma(s)/Z_\Gamma(s)$ in the strip $\{s \in \mathbb{C} | \rho_0 < \operatorname{Re} s \leq 2\rho_0\}$.

Lemma 2.2. *Assume that* $|\operatorname{Im} s|$ *is sufficiently large and* $\operatorname{Re} s \geq \rho_0 + \delta$ *for some fixed constant* δ *such that* $0 < \delta < \rho_0$. *Then we have*

$$\left| \frac{Z'_\Gamma(s)}{Z_\Gamma(s)} \right| = O\left(|\operatorname{Im} s|^{r \max(0, 2 + \frac{\delta - \operatorname{Re} s}{\rho_0})} \right).$$

Remark 2.1. In the case of a compact Riemann surface, we have a sharper estimate. See [Hej], Chap2, Prop 6.7.

Proof. We put $s = \sigma + iT$ and assume $\sigma > \rho_0$. We estimate each term of the expression (1.3) of $Z'_\Gamma(s)/Z_\Gamma(s)$.

First we estimate the term

$$\sum_{j=0}^{\infty} n_j \left\{ \frac{H(i(s-\rho_0+i\nu_j))}{s-\rho_0+i\nu_j} + \frac{H(i(s-\rho_0-i\nu_j))}{s-\rho_0-i\nu_j} \right\}.$$

We divide this sum into the two parts;

$$\sum_{|\nu_j|<2T} + \sum_{|\nu_j|\geq 2T} .$$

We denote by $S_1(T), S_2(T)$ the corresponding sum respectively. Using the property (1.4) of H, we have

$$\begin{aligned} |S_1(T)| &\leq \sum_{|\nu_j|<2T} n_j c_m \{ |(\sigma-\rho_0)+i(T+\nu_j)|^{-(m+1)} + |(\sigma-\rho_0)+i(T-\nu_j)|^{-(m+1)} \} \\ &\leq \sum_{|\nu_j|<2T} n_j \{ CT^{-(m+1)} + C'(\sigma-\rho_0)^{-(m+1)} \}. \end{aligned}$$

On account of the Weyl law, we have

$$\sum_{|\nu_j|<T} n_j = O(T^r) \quad \text{as} \quad T \to \infty. \tag{2.4}$$

Hence

$$S_1(T) = O(T^r) \quad \text{as} \quad T \to \infty.$$

Similarly, we have $S_2(T) = O(1)$.

On the other hand, using (1.4), it is easy to see that the other terms are dominated by $S_1(T)$. Thus we have $|Z'_\Gamma(s)/Z_\Gamma(s)| = O(T^r)$ as T tends to infinity.

For $\sigma > 2\rho_0$, it is clear that the series of the right hand side of (1.2) converges absolutely. Hence we have

$$\left| \frac{Z'_\Gamma(s)}{Z_\Gamma(s)} \right| = \begin{cases} O(1) & \text{if} \quad \sigma \geq 2\rho_0 + \delta \\ O(T^r) & \text{if} \quad \rho_0 + \delta \leq \sigma < 2\rho_0 + \delta. \end{cases}$$

Thus the desired conclusion follows from Landau [Lan], Satz 405. □

We recall the following formula. See, for example, [EMOT].

Lemma 2.3. *If $y > 0$ and $k \geq 0$, then we have*

$$\frac{1}{2\pi i} \lim_{T\to\infty} \int_{c-iT}^{c+iT} \frac{y^s}{s^{k+1}} ds = \begin{cases} \frac{1}{k!}(\log y)^k & (y > 1) \\ 0 & (0 < y \leq 1). \end{cases}$$

□

The following formula can be proved easily by induction.

Lemma 2.4. *For any $n \geq 0$, $M \geq 1$, we have*

$$\frac{1}{s^{n+1}(s+1)\dots(s+M)} = \sum_{l=1}^{n+1} v_l s^{-l} + \sum_{k=1}^{M} w_k \frac{1}{s+k},$$

where

$$v_l = \begin{cases} \sum_{k=1}^{M} \frac{(-1)^{n-l}}{k!(M-k)!k^{n-l+1}} & \text{if } 1 \leq l \leq n, \\ \frac{1}{M!} & \text{if } l = n+1, \end{cases}$$

$$w_k = \frac{(-1)^n}{k!(M-k)!k^n}.$$

□

2.2 The proof of the case $n = 0$

We put $N = \max(r, N_1, N_2) + 1$, where N_1 and N_2 are given in (1.5).

We calculate the following integral in two ways.

$$I(\tilde{\gamma}_\Gamma) = \frac{1}{2\pi i} \lim_{T \to \infty} \int_{c-iT}^{c+iT} \frac{Z'_\Gamma(s+2\rho_0)}{Z_\Gamma(s+2\rho_0)} \frac{x^s}{s(s+1)\dots(s+N)} ds,$$

where $c > 0$ is a constant.

First we calculate $I(\tilde{\gamma}_\Gamma)$ by using the series expression (1.2) of $Z'_\Gamma(s)/Z_\Gamma(s)$.

$$\begin{aligned} I(\tilde{\gamma}_\Gamma) &= 4\kappa \sum_{\gamma \in C(\Gamma) - Z(\Gamma)} \log N(\delta_\gamma) D(\gamma)^{-1} \frac{1}{2\pi i} \lim_{T \to \infty} \int_{c-iT}^{c+iT} \frac{(xN(\gamma)^{-1})^s}{s(s+1)\dots(s+N)} ds \\ &= 4\kappa \sum_{\gamma \in C(\Gamma) - Z(\Gamma)} \log N(\delta_\gamma) D(\gamma)^{-1} \frac{1}{2\pi i} \lim_{T \to \infty} \int_{c-iT}^{c+iT} \sum_{k=0}^{N} \frac{1}{k!(N-k)!} \frac{(xN(\gamma)^{-1})^s}{s+k} ds \\ &= 4\kappa \sum_{k=0}^{N} \frac{1}{k!(N-k)!} \sum_{\gamma \in C(\Gamma) - Z(\Gamma)} \log N(\delta_\gamma) D(\gamma)^{-1} \frac{1}{2\pi i} \lim_{T \to \infty} \int_{c-iT}^{c+iT} \frac{(xN(\gamma)^{-1})^s}{s+k} ds. \end{aligned}$$

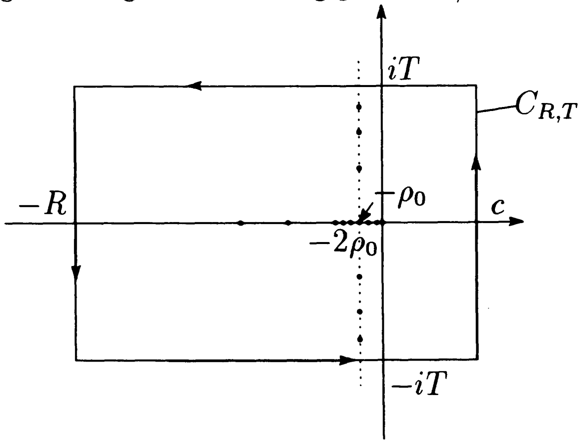
By Lemma 2.3, we have

$$I(\tilde{\gamma}_\Gamma) = 4\kappa \sum_{k=0}^{N} \frac{1}{k!(N-k)!} x^{-k} \sum_{\substack{\gamma \in C(\Gamma) - Z(\Gamma) \\ N(\gamma) < x}} \log N(\delta_\gamma) D(\gamma)^{-1} N(\gamma)^k.$$

Using Lemma 2.1, we obtain

$$I(\tilde{\gamma}_\Gamma) = \frac{4\kappa}{N!}(\log x + A_0) + 4\kappa \sum_{k=1}^{N} \frac{1}{k!(N-k)!}\frac{1}{k} + o(1) \quad \text{as} \quad x \to \infty. \tag{2.5}$$

On the other hand, we use the residue theorem to calculate $I(\tilde{\gamma}_\Gamma)$ by considering the integral along the following path $C_{R,T}$.



Observing (1.3), the poles and their residues of $Z'_\Gamma(s+2\rho_0)/Z_\Gamma(s+2\rho_0)$ are as follows;

poles	residues
$s = -\rho_0 \pm i\nu_j$	$4\kappa n_j$
$s = 0$	4κ
$s = -\rho_0$	$2\kappa(d - \mathrm{tr}(M(0)))$
$s = -2\rho_0$	$4\kappa - 4e_0 E$
$s = -\rho_0 - k$	$4\kappa d$
$s = -\rho_0 + ir_k$	$4e_k E$
$s = -\rho_0 + q_k$	$4\kappa b_k$.

We calculate the residue of the integrand in $I(\tilde{\gamma}_\Gamma)$ at the poles above.
Since $s = -\rho_0 \pm i\nu_j$ are simple poles, it is immediate to see that

$$\begin{aligned}&\operatorname*{Res}_{s=-\rho_0\pm i\nu_j} \frac{Z'_\Gamma(s+2\rho_0)}{Z_\Gamma(s+2\rho_0)} \frac{x^s}{s(s+1)\dots(s+N)} ds \\ =& n_j \frac{x^{-\rho_0\pm i\nu_j}}{(-\rho_0 \pm i\nu_j)(-\rho_0 \pm i\nu_j + 1)\dots(-\rho_0 \pm i\nu_j + N)}.\end{aligned}$$

Using (2.4), the sum of the residues at $s = -\rho_0 \pm i\nu_j$ converges absolutely. In fact, we have

$$\sum_{|\nu_j|<T} \Big(\operatorname*{Res}_{s=-\rho_0+i\nu_j} + \operatorname*{Res}_{s=-\rho_0-i\nu_j} \Big) \frac{Z'_\Gamma(s+2\rho_0)}{Z_\Gamma(s+2\rho_0)} \frac{x^s}{s(s+1)\dots(s+N)} ds \overset{T\to\infty}{\longrightarrow} o(1) \quad \text{as} \quad x \to \infty. \tag{2.6}$$

Next we calculate the residue at $s=0$ (notice that $s=0$ is a double pole).

$$\begin{aligned}
&\frac{Z'_\Gamma(s+2\rho_0)}{Z_\Gamma(s+2\rho_0)}\frac{x^s}{s(s+1)\dots(s+N)}\\
=&\Big(\frac{4\kappa}{s}+\tilde{\gamma}_\Gamma+\tilde{\gamma}_\Gamma^{(1)}s+\dots\Big)\big(1+(\log x)s+\dots\big)\Big(\sum_{k=0}^{N}\frac{1}{k!(N-k)!}\frac{1}{s+k}\Big)\\
=&\Big(\frac{4\kappa}{s}+\tilde{\gamma}_\Gamma+\tilde{\gamma}_\Gamma^{(1)}s+\dots\Big)\big(1+(\log x)s+\dots\big)\left(\frac{1}{s}+\sum_{l=0}^{\infty}\Big(\sum_{k=1}^{N}\frac{(-1)^l}{k!(N-k)!}\frac{1}{k^{l+1}}\Big)s^l\right)\\
=&4\kappa s^{-2}+\left(4\kappa\sum_{k=1}^{N}\frac{1}{k!(N-k)!}\frac{1}{k}+\frac{4\kappa\log x+\tilde{\gamma}_\Gamma}{N!}\right)s^{-1}+\dots.
\end{aligned}$$

Thus the residue at $s=0$ is given by

$$4\kappa\sum_{k=1}^{N}\frac{1}{k!(N-k)!}\frac{1}{k}+\frac{4\kappa\log x+\tilde{\gamma}_\Gamma}{N!}. \tag{2.7}$$

The contribution of the other poles are very small as we can show in the following argument.

Because of the existence of the factor $1/s(s+1)\dots(s+N)$, some of those poles may become double poles. If $s=-k$ is a double pole, then

$$\operatorname*{Res}_{s=-k}\frac{Z'_\Gamma(s+2\rho_0)}{Z_\Gamma(s+2\rho_0)}\frac{x^s}{s(s+1)\dots(s+N)}=O(x^{-k}\log x)\quad\text{as}\quad x\to\infty,$$

because the Talor expansion of x^s at $s=-k$ is

$$x^s=x^{-k}(1+(\log x)(s+k)+\frac{1}{2!}(\log x)^2(s+k)^2+\dots).$$

Since the number of the double poles is finite, the sum of the residues at the double poles tends to zero as $x\to\infty$.

On the other hand, since the number of the simple poles tends to infinite as $T,R\to\infty$, it remains to verify the convergence of the infinite sums. We estimate the sum of residues at $s=-\rho_0-k$.

$$\begin{aligned}
&\left|\sum_{k\le R-\rho_0}\operatorname*{Res}_{s=-\rho_0-k}\frac{Z'_\Gamma(s+2\rho_0)}{Z_\Gamma(s+2\rho_0)}\frac{x^s}{s(s+1)\dots(s+N)}\right|\\
=&\left|\sum_{k\le R-\rho_0}\frac{4\kappa d x^{-\rho_0-k}}{(-\rho_0-k)(-\rho_0-k+1)\dots(-\rho_0-k+N)}\right|\le Cx^{-\rho_0}\sum_{k\le R-\rho_0}k^{-(N+1)}.
\end{aligned}$$

Since the sum $\sum_{k\le R-\rho_0}k^{-(N+1)}$ converges as $R\to\infty$, the sum of residues at $s=-\rho_0-k$ tends to zero as $x\to\infty$. Using (1.5), the sums of the residues at $s=-\rho_0+ir_k$ and $s=-\rho_0+q_k$ tend to zero as $x\to\infty$.

Therefore we obtain

$$\frac{1}{2\pi i}\int_{C_{R,T}}\frac{Z'_\Gamma(s+2\rho_0)}{Z_\Gamma(s+2\rho_0)}\frac{x^s}{s(s+1)\dots(s+N)}ds$$
$$=4\kappa\sum_{k=1}^{N}\frac{1}{k!(N-k)!}\frac{1}{k}+4\kappa\log x+\tilde{\gamma}_\Gamma+o(1)\quad\text{as}\quad x\to\infty. \tag{2.8}$$

We estimate the integral along $C_{R,T}$. When we take a positive constant ε such that $\varepsilon<\rho_0$, we divide this integral into the following eight parts.

$$\int_{C_{R,T}}\frac{Z'_\Gamma(s+2\rho_0)}{Z_\Gamma(s+2\rho_0)}\frac{x^s}{s(s+1)\dots(s+N)}ds$$
$$=\int_{c-iT}^{c+iT}+\int_{c+iT}^{-\rho_0+\varepsilon+iT}+\int_{-\rho_0+\varepsilon+iT}^{-\rho_0-\varepsilon+iT}+\int_{-\rho_0-\varepsilon+iT}^{-R+iT}$$
$$+\int_{-R+iT}^{-R-iT}+\int_{-R-iT}^{-\rho_0-\varepsilon-iT}+\int_{-\rho_0-\varepsilon-iT}^{-\rho_0+\varepsilon-iT}+\int_{-\rho_0+\varepsilon-iT}^{c-iT}\frac{Z'_\Gamma(s+2\rho_0)}{Z_\Gamma(s+2\rho_0)}\frac{x^s}{s(s+1)\dots(s+N)}.$$

We denote by $I_1,\dots,I_8$ respectively the integrals appearing the right hand side. Since $I(\tilde{\gamma}_\Gamma)=1/2\pi i\lim_{T\to\infty}I_1$, it is enough to estimate $I_2,\dots,I_8$.

Estimate of I_5: By the functional equation (1.6), we have

$$I_5=\int_{-R+iT}^{-R-iT}\Big\{-\frac{Z'_\Gamma(-s)}{Z_\Gamma(-s)}+4\kappa d\Big(\frac{\Gamma'(1+\rho_0-s)}{\Gamma(1+\rho_0-s)}-\frac{\Gamma'(1-\rho_0+s)}{\Gamma(1-\rho_0+s)}\Big)$$
$$+\sum_{k=1}^{l}4\kappa b_k\Big(\frac{1}{s-\rho_0-q_k}-\frac{1}{\rho_0-s-q_k}\Big)+4\kappa\mathrm{vol}(\Gamma\backslash G)[Z(\Gamma)]\mu(i(s+\rho_0))$$
$$-4\kappa\frac{\Psi'(s-\rho_0)}{\Psi(s-\rho_0)}-J(s+2\rho_0)\Big\}\frac{x^s}{s(s+1)\dots(s+N)}ds.$$

Since, by (1.2),

$$\left|\frac{Z'_\Gamma(R+it)}{Z_\Gamma(R+it)}\right|=4\kappa\sum_{\gamma\in C(\Gamma)-Z(\Gamma)}\log N(\delta_\gamma)D(\gamma)^{-1}N(\gamma)^{2\rho_0-R}<\infty,$$

we have

$$\left|\int_{-R+iT}^{-R-iT}-\frac{Z'_\Gamma(-s)}{Z_\Gamma(-s)}\frac{x^s}{s(s+1)\dots(s+N)}ds\right|\overset{T,R\to\infty}{\longrightarrow}0.$$

We recall the following estimates from [Gan] and [GW],

$$\left|\frac{\Gamma'(s)}{\Gamma(s)}\right|=O(\log|s|),\qquad |\mu(is)|=O(|s|^{r-1}),$$
$$\left|\frac{\Psi'(s)}{\Psi(s)}\right|=O\big((\log|s|)^k\big),\qquad |J(s)|=O(|s|^{k'})\quad\text{as}\quad|s|\to\infty, \tag{2.9}$$

where $k, k' \leq r$. Hence it is easy to see that the other terms of I_5 tends to zero as $T, R \to \infty$. Thus we obtain

$$I_5 \overset{T,R\to\infty}{\longrightarrow} 0. \tag{2.10}$$

Estimate of I_2 and I_8: By Lemma 2.2, we have

$$|I_2| \leq \int_c^{-\rho_0+\varepsilon} O\Big(T^{r\max(0,\frac{\delta-t}{\rho_0})}\Big)\frac{x^t}{T^{N+1}}dt = O\Big(T^{r-(N+1)}\Big) \quad \text{as} \quad T \to \infty \tag{2.11}$$

for a fixed $\delta < \varepsilon$. Similarly we have

$$I_8 \overset{T\to\infty}{\longrightarrow} 0. \tag{2.12}$$

Estimate of I_4 and I_6: Using the functional equation (1.6), we have

$$\begin{aligned} I_4 = &\int_{-\rho_0-\varepsilon+iT}^{-R+iT} \Big\{-\frac{Z'_\Gamma(-s)}{Z_\Gamma(-s)} + 4\kappa d\Big(\frac{\Gamma'(1+\rho_0-s)}{\Gamma(1+\rho_0-s)} - \frac{\Gamma'(1-\rho_0+s)}{\Gamma(1-\rho_0+s)}\Big) \\ &+ \sum_{k=1}^{l} 4\kappa b_k\big(\frac{1}{s-\rho_0-q_k} - \frac{1}{\rho_0-s-q_k}\big) + 4\kappa \mathrm{vol}(\Gamma\backslash G)[Z(\Gamma)]\mu(i(s+\rho_0)) \\ &- 4\kappa\frac{\Psi'(s-\rho_0)}{\Psi(s-\rho_0)} - J(s+2\rho_0)\Big\}\frac{x^s}{s(s+1)\dots(s+N)}ds. \end{aligned}$$

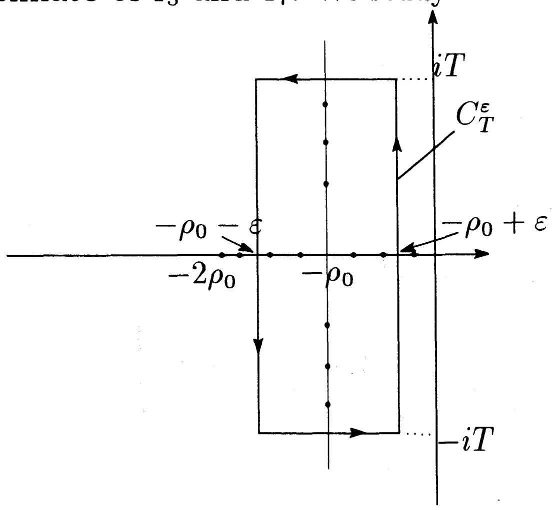
By Lemma 2.2 and (2.9), it is easy to verify that

$$I_4 \overset{R,T\to\infty}{\longrightarrow} 0. \tag{2.13}$$

Similarly

$$I_6 \overset{R,T\to\infty}{\longrightarrow} 0. \tag{2.14}$$

Estimate of I_3 and I_7: We study the following contour C_T^ε.



Using the same argument to obtain (2.6), the sum of residues at the poles inside this contour converges to zero as $x \to \infty$. Then we have

$$I_3 + I_7 + \int_{-\rho_0+\varepsilon-iT}^{-\rho_0+\varepsilon+iT} + \int_{-\rho_0-\varepsilon+iT}^{-\rho_0-\varepsilon-iT} \frac{Z'_\Gamma(s+2\rho_0)}{Z_\Gamma(s+2\rho_0)} \frac{x^s}{s(s+1)\dots(s+N)} ds = o(1) \quad \text{as} \quad x \to \infty.$$

Let I_9, I_{10} be the third and the fourth integrals of the left hand side of the above respectively. In order to estimate I_3 and I_7, it is enough to estimate I_9 and I_{10}.

When we take a sufficiently large number $T'(< T)$, I_9 is written as

$$I_9 = \int_{T'}^{T} + \int_{-T'}^{T'} + \int_{-T}^{-T'} \frac{Z'_\Gamma(s+2\rho_0)}{Z_\Gamma(s+2\rho_0)} \frac{x^s}{s(s+1)\dots(s+N)} ds.$$

By Lemma 2.2, the first and the third integrals above are bounded by

$$\int_{T'}^{T} O(t^r) \frac{x^{-\rho_0+\varepsilon}}{t^{N+1}} dt \overset{T\to\infty}{\longrightarrow} O(x^{-\rho_0+\varepsilon}) \quad \text{as} \quad x \to \infty.$$

Also it is clear to see that the second integral turns to be $O(x^{-\rho_0+\varepsilon})$. Hence it follows that

$$I_9 \overset{T\to\infty}{\longrightarrow} O(x^{-\rho_0+\varepsilon}) \quad \text{as} \quad x \to \infty. \tag{2.15}$$

By the functional equation (1.6), we see that

$$\begin{aligned} I_{10} = \int_{-\rho_0-\varepsilon+iT}^{-\rho_0-\varepsilon-iT} & \Big\{ -\frac{Z'_\Gamma(-s)}{Z_\Gamma(-s)} + 4\kappa d \Big(\frac{\Gamma'(1+\rho_0-s)}{\Gamma(1+\rho_0-s)} - \frac{\Gamma'(1-\rho_0+s)}{\Gamma(1-\rho_0+s)} \Big) \\ & + \sum_{k=1}^{l} 4\kappa b_k \Big(\frac{1}{s-\rho_0-q_k} - \frac{1}{\rho_0-s-q_k} \Big) + 4\kappa \mathrm{vol}(\Gamma\backslash G)[Z(\Gamma)]\mu(i(s+\rho_0)) \\ & - 4\kappa \frac{\Psi'(s-\rho_0)}{\Psi(s-\rho_0)} - J(s+2\rho_0) \Big\} \frac{x^s}{s(s+1)\dots(s+N)} ds. \end{aligned}$$

The argument similar to (2.15) and the estimates (2.9) yield

$$I_{10} \overset{T\to\infty}{\longrightarrow} O(x^{-\rho_0-\varepsilon}) \quad \text{as} \quad x \to \infty. \tag{2.16}$$

Therefore we have

$$I_3 + I_7 \overset{T\to\infty}{\longrightarrow} o(1) \quad \text{as} \quad x \to \infty.$$

Combining (2.5), (2.8) and the estimates of I_i's, we obtain

$$\tilde{\gamma}_\Gamma = 4\kappa A_0 = 4\kappa \lim_{x\to\infty} \Bigg\{ \sum_{\substack{\gamma \in C(\Gamma)-Z(\Gamma) \\ N(\gamma)<x}} \log N(\delta_\gamma) D(\gamma)^{-1} - \log x \Bigg\}.$$

This completes the proof of the case $n = 0$. □

2.3 The case $n \geq 1$

We want to obtain the expression (1.8) of $\tilde{\gamma}_\Gamma^{(n)}$ by calculating the following integral in two ways (similar to the proof of the case $n = 0$, one of these is to use the series expression of $Z'_\Gamma(s)/Z_\Gamma(s)$, another is to use the residue theorem).

$$\frac{1}{2\pi i} \lim_{T\to\infty} \int_{c-iT}^{c+iT} \frac{Z'_\Gamma(s+2\rho_0)}{Z_\Gamma(s+2\rho_0)} \frac{x^s}{s^{n+1}} ds.$$

However, in the case $1 \leq n < N$, we have several difficulties about the convergence of the sum of the residues and the integral along $C_{R,T}$. Hence, to assure the convergence of them, we need to study the following integrals.

$$J(\tilde{\gamma}_\Gamma^{(n)}) := \frac{1}{2\pi i} \lim_{T\to\infty} \int_{c-iT}^{c+iT} \frac{Z'_\Gamma(s+2\rho_0)}{Z_\Gamma(s+2\rho_0)} \frac{x^s}{s^{n+1}(s+1)\dots(s+N-n)} ds \quad \text{for} \quad 1 \leq n < N,$$

$$K(\tilde{\gamma}_\Gamma^{(n)}) := \frac{1}{2\pi i} \lim_{T\to\infty} \int_{c-iT}^{c+iT} \frac{Z'_\Gamma(s+2\rho_0)}{Z_\Gamma(s+2\rho_0)} \frac{x^s}{s^{n+1}} ds \quad \text{for} \quad n \geq N.$$

The same results (1.8) can be established by both calculations. Since these calculations are quite similar to each other, we give only the calculation of $J(\tilde{\gamma}_\Gamma^{(n)})$.

Using (1.2), we have

$$\begin{aligned} J(\tilde{\gamma}_\Gamma^{(n)}) &= 4\kappa \sum_{\gamma\in C(\Gamma)-Z(\Gamma)} \log N(\delta_\gamma) D(\gamma)^{-1} \frac{1}{2\pi i} \lim_{T\to\infty} \int_{c-iT}^{c+iT} \frac{(xN(\gamma)^{-1})^s}{s^{n+1}(s+1)\dots(s+N-n)} ds \\ &= 4\kappa \sum_{\gamma\in C(\Gamma)-Z(\Gamma)} \log N(\delta_\gamma) D(\gamma)^{-1} \frac{1}{2\pi i} \lim_{T\to\infty} \int_{c-iT}^{c+iT} \left\{ \sum_{l=1}^{n+1} \frac{v_l}{s} + \sum_{k=1}^{N-n} \frac{w_k}{s+k} \right\} (xN(\gamma)^{-1})^s ds. \end{aligned}$$

By Lemma 2.3, we obtain

$$\begin{aligned} J(\tilde{\gamma}_\Gamma^{(n)}) = 4\kappa \sum_{l=0}^{n} \frac{v_{n-l+1}}{(n-l)!} \sum_{\substack{\gamma\in C(\Gamma)-Z(\Gamma) \\ N(\gamma)<x}} \log N(\delta_\gamma) D(\gamma)^{-1} (\log (xN(\gamma)^{-1}))^{n-l} \\ + 4\kappa \sum_{k=1}^{N-n} w_k x^{-k} \sum_{\substack{\gamma\in C(\Gamma)-Z(\Gamma) \\ N(\gamma)<x}} \log N(\delta_\gamma) D(\gamma)^{-1} N(\gamma)^k. \end{aligned}$$

The binomial theorem shows

$$(\log (xN(\gamma)^{-1}))^{n-l} = \sum_{m=0}^{n-l} \binom{n-l}{m} (-1)^m (\log N(\gamma))^m (\log x)^{n-l-m}.$$

Hence we have

$$J(\tilde{\gamma}_\Gamma^{(n)}) = 4\kappa \sum_{l=0}^{n}\sum_{m=0}^{n-l} \frac{(-1)^m v_{n-l+1}}{m!(n-l-m)!}(\log x)^{n-l-m} \sum_{\substack{\gamma\in C(\Gamma)-Z(\Gamma)\\ N(\gamma)<x}} \log N(\delta_\gamma)(\log N(\delta_\gamma))^m D(\gamma)^{-1}$$

$$+ 4\kappa \sum_{k=1}^{N-n} w_k x^{-k} \sum_{\substack{\gamma\in C(\Gamma)-Z(\Gamma)\\ N(\gamma)<x}} \log N(\delta_\gamma) D(\gamma)^{-1} N(\gamma)^k.$$

By Lemma 2.1, we conclude

$$J(\tilde{\gamma}_\Gamma^{(n)}) = 4\kappa \sum_{l=0}^{n} \frac{v_{n-l+1}}{(n-l+1)!}(\log x)^{n-l+1} + 4\kappa \sum_{l=0}^{n}\sum_{m=0}^{n-l} \frac{(-1)^m v_{n-l+1}}{m!(n-l-m)!}(\log x)^{n-l-m} A_m$$

$$+ 4\kappa \sum_{k=1}^{N-n} \frac{w_k}{k} + o(1) \quad \text{as} \quad x\to\infty. \tag{2.17}$$

W consider again the same contour $C_{R,T}$ as we used before. Using a similar argument in the proof of the case $n=0$, all residues at the poles, except at $s=0$, converge to zero as $x\to\infty$. Thus it is enough to calculate only the residue at $s=0$.

We study the Laurent expansion around $s=0$. The Laurent expansion of $1/s^{n+1}(s+1)\dots(s+N-n)$ is written by

$$\frac{1}{s^{n+1}(s+1)\dots(s+N-n)} = \sum_{l=-(n+1)}^{\infty} v'_l s^l,$$

where

$$v'_l = \begin{cases} v_{-l} & \text{if} \quad -(n+1)\le l\le -1 \\ \sum_{k=1}^{N-n}(-1)^l k^{-(l+1)} w_k & \text{if} \quad l\ge 0 \end{cases}$$

(see, Lemma 2.4 for the definition of v_k and w_k). Hence the Laurent expansion of the integrand of $J(\tilde{\gamma}_\Gamma^{(n)})$ is written as

$$\frac{Z'_\Gamma(s+2\rho_0)}{Z_\Gamma(s+2\rho_0)} \frac{x^s}{s^{n+1}(s+1)\dots(s+N-n)}$$

$$=\Big(\sum_{m=0}^{\infty} \tilde{\gamma}_\Gamma^{(m-1)} s^{m-1}\Big)\Big(\sum_{i=0}^{\infty} \frac{(\log x)^i}{i!} s^i\Big)\Big(\sum_{l=0}^{\infty} v'_{l-(n+1)} s^{l-(n+1)}\Big)$$

$$= \sum_{l,m,i=0}^{\infty} \tilde{\gamma}_\Gamma^{(m-1)} \frac{(\log x)^i}{i!} v'_{l-(n+1)} s^{(l+m+i)-(n+2)}$$

$$=\sum_{i=0}^{\infty}\Big\{\sum_{l=0}^{i}\sum_{m=0}^{i-l} \tilde{\gamma}_\Gamma^{(m-1)} \frac{(\log x)^{i-l-m}}{(i-l-m)!} v'_{l-(n+1)}\Big\} s^{i-(n+2)}.$$

It follows that the residue at $s=0$ is calculated as

$$
\begin{aligned}
&\operatorname*{Res}_{s=0} \frac{Z'_\Gamma(s+2\rho_0)}{Z_\Gamma(s+2\rho_0)} \frac{x^s}{s^{n+1}(s+1)\dots(s+N-n)} \\
=&\sum_{l=0}^{n+1}\sum_{m=0}^{n-l+1} \tilde{\gamma}_\Gamma^{(m-1)} \frac{(\log x)^{n-l-m+1}}{(n-l-m+1)!} v'_{l-(n+1)} \\
=&\sum_{l=0}^{n}\sum_{m=0}^{n-l+1} \tilde{\gamma}_\Gamma^{(m-1)} \frac{(\log x)^{n-l-m+1}}{(n-l-m+1)!} v'_{l-(n+1)} + \tilde{\gamma}_\Gamma^{(-1)} v'_0 \\
=&\sum_{l=0}^{n}\sum_{m=0}^{n-l} \tilde{\gamma}_\Gamma^{(m)} \frac{(\log x)^{n-l-m}}{(n-l-m)!} v'_{l-(n+1)} + \tilde{\gamma}_\Gamma^{(-1)} \sum_{l=0}^{n} \frac{(\log x)^{n-l+1}}{(n-l+1)!} v'_{l-(n+1)} + \tilde{\gamma}_\Gamma^{(-1)} v'_0 \\
=&\sum_{l=0}^{n}\sum_{m=0}^{n-l} \tilde{\gamma}_\Gamma^{(m)} \frac{(\log x)^{n-l-m}}{(n-l-m)!} v'_{l-(n+1)} + 4\kappa \sum_{l=0}^{n} \frac{(\log x)^{n-l+1}}{(n-l+1)!} v'_{l-(n+1)} + 4\kappa \sum_{k=1}^{N-n} \frac{w_k}{k}.
\end{aligned}
$$

Therefore we have

$$
\begin{aligned}
&\frac{1}{2\pi i}\int_{C_{R,T}} \frac{Z'_\Gamma(s+2\rho_0)}{Z_\Gamma(s+2\rho_0)} \frac{x^s}{s^{n+1}(s+1)\dots(s+N-n)} ds \\
=&\sum_{l=0}^{n}\sum_{m=0}^{n-l} \tilde{\gamma}_\Gamma^{(m)} \frac{(\log x)^{n-l-m}}{(n-l-m)!} v_{n-l+1} + 4\kappa \sum_{l=0}^{n} \frac{(\log x)^{n-l+1}}{(n-l+1)!} v_{n-l+1} \\
&+4\kappa \sum_{k=1}^{N-n} \frac{w_k}{k} + o(1) \quad \text{as} \quad x \to \infty. \qquad (2.18)
\end{aligned}
$$

Using a similar argument as we made before, the estimate of the integral along $C_{R,T}$ excluding the line $\operatorname{Re} s = c$ tends to zero as $R, T \to \infty$ and $x \to \infty$.

From (2.17) and the result above, we have

$$
\begin{aligned}
&4\kappa \sum_{l=0}^{n}\sum_{m=0}^{n-l} \frac{(-1)^m v_{n-l+1}}{m!(n-l-m)!} (\log x)^{n-l-m} A_m \\
=&\sum_{l=0}^{n}\sum_{m=0}^{n-l} \frac{v_{n-l+1}}{(n-l-m)!} (\log x)^{n-l-m} \tilde{\gamma}_\Gamma^{(m)} + o(1) \quad \text{as} \quad x \to \infty.
\end{aligned}
$$

Thus we obtain the following equality recursively.

$$
\begin{aligned}
\tilde{\gamma}_\Gamma^{(n)} =& \frac{(-1)^n}{n!} 4\kappa A_n \\
=& \frac{(-1)^n}{n!} 4\kappa \lim_{x\to\infty} \left\{ \sum_{\substack{\gamma \in C(\Gamma)-Z(\Gamma) \\ N(\gamma)<x}} \log N(\delta_\gamma) (\log N(\delta_\gamma))^n D(\gamma)^{-1} - \frac{(\log x)^{n+1}}{n+1} \right\}.
\end{aligned}
$$

This completes the proof of Theorem 1.1. □

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