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On change of polarization

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We consider here the following setting.

Let G be a real reductive linear Lie group which is contained in the complexification $G_{\mathbb{C}}$. We fix a maximal compact subgroup K of G and let θ be the corresponding Cartan involution. We denote by $\mathfrak{g}_0$ (resp. $\mathfrak{k}_0$)the Lie algebra of G (resp. K) and denote by $\mathfrak{g}$ (resp. $\mathfrak{k}$) its complexification. We denote also by the same letter θ the complexified Cartan involution on $\mathfrak{g}$. We denote by σ he complex conjugation on $\mathfrak{g}$ with respect to $\mathfrak{g}_0$.

Definition 1. *Assume that a parabolic subalgebra $\mathfrak{q}$ has a Levi decomposition $\mathfrak{q} = \mathfrak{l} + \mathfrak{u}$ such that $\mathfrak{l}$ is stable under θ and σ. Such a Levi decomposition is called an orderly Levi decomposition.*

A θ-stable or σ-stable parabolic subalgebra has a unique orderly Levi decomposition. In fact, if $\mathfrak{q}$ is θ (resp. σ)-stable, then $\mathfrak{l} = \mathfrak{q} \cap \sigma(\mathfrak{q})$ (resp. $\mathfrak{l} = \mathfrak{q} \cap \theta(\mathfrak{q})$).

Let $\mathfrak{q}$ be a parabolic subalgebra of $\mathfrak{g}$ with an orderly Levi decomposition $\mathfrak{q} = \mathfrak{l}+\mathfrak{u}$. We fix a θ and σ-stable Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{l}$ and a Weyl group invariant non-degenerate bilinear form $\langle\ ,\ \rangle$. Let L be the corresponding Levi subgroup in G to $\mathfrak{l}$.

We denote by ${}^u\mathcal{R}^{\mathfrak{g},K}_{\mathfrak{q},L\cap K}$ the right adjoint functor of the forgetful functor of the category of $(\mathfrak{g}, K)$-modules to the category of $(\mathfrak{q}, L\cap K)$-modules. Introducing trivial $\mathfrak{u}$-action, we regard an $(\mathfrak{l}, L\cap K)$-module as a $(\mathfrak{q}, L\cap K)$-module. So, we also regard ${}^u\mathcal{R}^{\mathfrak{g},K}_{\mathfrak{q},L\cap K}$ as a functor of the category of $(\mathfrak{l}, L\cap K)$-modules to the category of $(\mathfrak{g}, K)$-modules. We denote by $({}^u\mathcal{R}^{\mathfrak{g},K}_{\mathfrak{q},L\cap K})^i$ the i-th right derived functor. (See [Knapp-Vogan] p671)

Let V be a finite dimensional semisimple $\mathfrak{l}$-module. We denote by $\delta(V)$ a one-dimensional representation of $\mathfrak{l}$ defined by $\delta(V)(X) = \frac{1}{2}\mathrm{tr}(X|_V)$.

We consider two extreme cases. Let Z be a Harish-Chandra $(\mathfrak{l}, L\cap K)$-module with the infinitesimal character $\lambda+\delta(\mathfrak{n})$.

(1) (*Hyperbolic case*) If $\mathfrak{q}$ is stable under the complex conjugation of $\mathfrak{g}$ with respect to G, there is a parabolic subgroup $Q=LU$ whose complexified Lie algebra is $\mathfrak{q}$ and whose nilradical is U. In this case, we have $({}^{u}\mathcal{R}^{\mathfrak{g},K}_{\mathfrak{q},L\cap K})^{i}(Z)=0$ for all $i>0$. In fact, $({}^{u}\mathcal{R}^{\mathfrak{g},K}_{\mathfrak{q},L\cap K})^{0}(Z)$ is nothing but the unnormalized parabolic induction ${}^{u}\mathsf{Ind}^{G}_{Q}(Z)$.

We clarify the definition of the parabolic induction. ${}^{u}\mathsf{Ind}^{G}_{Q}(Z)$ (or we also write ${}^{u}\mathsf{Ind}(Q\uparrow G;Z)$) is the K-finite part of

$$\{f\in C^{\infty}(G)\otimes H \mid f(g\ell n)=\pi(\ell^{-1})f(g) \quad (g\in G,\ell\in L,n\in U)\}.$$

Here, (π,H) is any Hilbert globalization of Z. If Z is unitarizable, so is ${}^{u}\mathsf{Ind}(Q\uparrow G;Z\otimes\mathbb{C}_{\delta(\mathfrak{n})})$ (unitary induction).

(2) (*Elliptic case*) Assume $\mathfrak{q}$ is θ-stable and put $S=\dim(\mathfrak{u}\cap\mathfrak{k})$. In this case, ${}^{u}\mathcal{R}^{\mathfrak{g},K}_{\mathfrak{q},L\cap K})^{S}(Z)$ is essentilally "a usuall cohomological induction". We stress that usually the chomological induction in the elliptic case is normalized by the $2\delta(\mathfrak{u})$-shift of the parameter (cf. [Vogan(green)], [Knapp-Vogan] Chapter V).

We call Z weakly good (or λ is in the weakly good range), if $\mathrm{Re}\langle\lambda,\alpha\rangle\geqslant 0$ holds for each root α of $\mathfrak{h}$ in $\mathfrak{u}$. We call Z integrally good (resp. weakly integrally good), if $\langle\lambda,\alpha\rangle>0$ (resp. $\langle\lambda,\alpha\rangle\geqslant 0$) holds for each root α of $\mathfrak{h}$ in $\mathfrak{u}$ such that $2\frac{\langle\lambda,\alpha\rangle}{\langle\alpha,\alpha\rangle}\in\mathbb{Z}$.

Theorem 2. *([Vogan 1988] Theorem2.6)*

(1) If Z is weakly integrally good, then $({}^{n}\mathcal{R}^{\mathfrak{g},K}_{\mathfrak{q},L\cap K})^{i}=0$ for $i\neq S$.

(2) If Z is irreducible and weakly integrally good, $({}^{n}\mathcal{R}^{\mathfrak{g},K}_{\mathfrak{q},L\cap K})^{S}(Z)$ is irreducible or zero.

(3) If Z is irreducible and integrally good, $({}^{n}\mathcal{R}^{\mathfrak{g},K}_{\mathfrak{q},L\cap K})^{S}(Z)$ is irreducible.

(4) , If Z is unitarizable and weakly good, $({}^{n}\mathcal{R}^{\mathfrak{g},K}_{\mathfrak{q},L\cap K})^{S}(Z)$ is unitarizable.

Definition 3. *A pair $(\mathfrak{p},\mathfrak{q})$ is called a $\sigma\theta$ pair of parabolic subalgebras, if it satisfies the following conditions (S1-2)*

(S1) $\mathfrak{q}$ (resp. $\mathfrak{p}$) is a θ-stable (resp. σ-stable) parabolic subalgebra of $\mathfrak{g}$.

(S2) There exists a θ and σ-stable Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ such that $\mathfrak{h}\subseteq\mathfrak{p}\cap\mathfrak{q}$.

Hereafter, we fix a $\sigma\theta$ pair $(\mathfrak{p},\mathfrak{q})$. Let $\mathfrak{h}$ be any θ and σ-stable Cartan subalgebra of $\mathfrak{g}$ contained in $\mathfrak{p}\cap\mathfrak{q}$. For $\alpha\in\Delta(\mathfrak{g},\mathfrak{h}))$, we denote by $\mathfrak{g}_{\alpha}$ (resp. s_{α}) the root space (resp. the reflection) corresponding to α. Since $\mathfrak{h}$ is θ-stable, θ and σ induce actions on $\Delta(\mathfrak{g},\mathfrak{h})$. We easily see $\theta\alpha = -\sigma\alpha$ for any $\alpha \in \Delta(\mathfrak{g},\mathfrak{h})$.

For a subspace U in $\mathfrak{g}$, we denote by $\Delta(U)$ the set of roots in $\Delta(\mathfrak{g},\mathfrak{h})$ whose root space is contained in U. We put

$$\mathfrak{m} = \mathfrak{h} + \sum_{\alpha\in\Delta(\mathfrak{p})\cap(-\Delta(\mathfrak{p}))} \mathfrak{g}_\alpha, \quad \mathfrak{n} = \sum_{\alpha\in\Delta(\mathfrak{p})-\Delta(\mathfrak{m})} \mathfrak{g}_\alpha, \quad \bar{\mathfrak{n}} = \sum_{\alpha\in\Delta(\mathfrak{n})} \mathfrak{g}_{-\alpha},$$

$$\mathfrak{l} = \mathfrak{h} + \sum_{\alpha\in\Delta(\mathfrak{q})\cap(-\Delta(\mathfrak{q}))} \mathfrak{g}_\alpha, \quad \mathfrak{u} = \sum_{\alpha\in\Delta(\mathfrak{q})-\Delta(\mathfrak{l})} \mathfrak{g}_\alpha, \quad \bar{\mathfrak{u}} = \sum_{\alpha\in\Delta(\mathfrak{u})} \mathfrak{g}_{-\alpha}.$$

We immediately see $\mathfrak{q} = \mathfrak{l} + \mathfrak{u}$ (resp. $\mathfrak{p} = \mathfrak{m} + \mathfrak{n}$) is an orderly Levi decomposition of $\mathfrak{q}$ (resp. $\mathfrak{p}$) and the nilradical satisfies $\sigma(\mathfrak{u}) = \bar{\mathfrak{u}}$ (resp. $\theta(\mathfrak{n}) = \bar{\mathfrak{n}}$). Moreover, $\bar{\mathfrak{u}}$ (resp. $\bar{\mathfrak{n}}$) is the opposite nilradical to $\mathfrak{u}$ (resp. $\mathfrak{n}$).

We denote by $L_{\mathbb{C}}$, $P_{\mathbb{C}}$, and $M_{\mathbb{C}}$ the analytic subgroups of $G_{\mathbb{C}}$ with respect to $\mathfrak{l}$, $\mathfrak{p}$, and $\mathfrak{m}$, respectively. We put $L = L_{\mathbb{C}} \cap G$, $P = P_{\mathbb{C}} \cap G$, $M = M_{\mathbb{C}} \cap G$.

We easily have:

Proposition 4. *Under the above setting, we have the followings.*

(S3) $\mathfrak{l}\cap\mathfrak{p}$ *is a parabolic subalgebra of* $\mathfrak{l}$ *and* $L\cap P$ *is a parabolic subgroup of* L.

(S4) $\mathfrak{m}\cap\mathfrak{q}$ *is a parabolic subalgebra of* $\mathfrak{m}$.

(S5) $\mathfrak{l}\cap\mathfrak{m}$ *is a* θ *and* σ*-stable Levi subalgebra of the both* $\mathfrak{l}\cap\mathfrak{p}$ *and* $\mathfrak{m}\cap\mathfrak{q}$.

The followings is the main subject of my talk.

Conjecture 5. *Let* Z *be a Harish-Chandra* $(\mathfrak{l}\cap\mathfrak{m}, L\cap M\cap K)$*-module with an infinitesimal character* $\lambda \in \mathfrak{h}^*$. *assume* $\langle\lambda - \delta(\mathfrak{u}\cap\mathfrak{m}) - \delta(\mathfrak{n}), \alpha\rangle \geqslant 0$ *for all* $\alpha \in \Delta(\mathfrak{u})$ *such that* $2\frac{\langle\lambda-\delta(\mathfrak{u}\cap\mathfrak{m})-\delta(\mathfrak{n}),\alpha\rangle}{\langle\alpha,\alpha\rangle} \in \mathbb{Z}$.

Then, we have

$$(*) \qquad {}^u\mathrm{Ind}_P^G\left(\left({}^u\mathcal{R}^{\mathfrak{m},M\cap K}_{\mathfrak{q}\cap\mathfrak{m},L\cap M\cap K}\right)^{\dim \mathfrak{u}\cap\mathfrak{m}\cap\mathfrak{k}}(Z)\right) \cong \left({}^u\mathcal{R}^{\mathfrak{g},K}_{\mathfrak{q},L\cap K}\right)^{\dim \mathfrak{u}\cap\mathfrak{k}}\left({}^u\mathrm{Ind}^L_{P\cap L}(Z\otimes\mathbb{C}_{2\delta(\mathfrak{u}\cap\bar{\mathfrak{n}})})\right)$$

The above conjecture can be regard as a generalization of the transfer thorem ([Knapp-Vogan 1995] Theorem 11.87, [Schmid 1988] p361) for standard modules.

For a Harish-Chandra $(\mathfrak{g}, K)$-module V, we denote by $[V]$ the distribution character of V. In [Matumoto 2002], we proved the following weaker version.

Theorem 6. *Let Z be a Harish-Chandra $(\mathfrak{l} \cap \mathfrak{m}, L \cap M \cap K)$-module with an infinitesimal character $\lambda \in \mathfrak{h}^*$. Assume $\langle \lambda - \delta(\mathfrak{u} \cap \mathfrak{m}) - \delta(\mathfrak{n}), \alpha \rangle \geqslant 0$ for all $\alpha \in \Delta(\mathfrak{u})$ such that $2\frac{\langle \lambda - \delta(\mathfrak{u}\cap\mathfrak{m}) - \delta(\mathfrak{n}), \alpha\rangle}{\langle \alpha, \alpha \rangle} \in \mathbb{Z}$. Then, we have*

$$(**) \qquad [{}^u\mathrm{Ind}_P^G(({}^u\mathcal{R}^{\mathfrak{m}, M\cap K}_{\mathfrak{q}\cap\mathfrak{m}, L\cap M\cap K})^{\dim \mathfrak{u}\cap\mathfrak{m}\cap\mathfrak{k}}(Z))] = [({}^u\mathcal{R}^{\mathfrak{g},K}_{\mathfrak{q}, L\cap K})^{\dim \mathfrak{u}\cap\mathfrak{k}}({}^u\mathrm{Ind}^L_{P\cap L}(Z \otimes \mathbb{C}_{2\delta(\mathfrak{u}\cap\bar{\mathfrak{n}})}))].$$

Another weaker version of the conjecture is:

Theorem 7. *Let Z be a Harish-Chandra $(\mathfrak{l} \cap \mathfrak{m}, L \cap M \cap K)$-module with an infinitesimal character $\lambda \in \mathfrak{h}^*$ which is cohomologically induced from a finite-dimensional repressentation. Assume $\langle \lambda - \delta(\mathfrak{u} \cap \mathfrak{m}) - \delta(\mathfrak{n}), \alpha \rangle \geqslant 0$ for all $\alpha \in \Delta(\mathfrak{u})$ such that $2\frac{\langle \lambda - \delta(\mathfrak{u}\cap\mathfrak{m}) - \delta(\mathfrak{n}), \alpha\rangle}{\langle \alpha, \alpha \rangle} \in \mathbb{Z}$. Then Conjecture 5 holds for Z.*

The main ingredients of the proof of Theorem 7 are:

(1) The resolution of finite dimensional irreducible representation by standard module. ([Johnson 1984])

(2) The dimension of the space of intertwing operators between "adjacent standard representations" is at most one.

References

[Knapp-Vogan 1995] A. W. Knapp, "Cohomological Induction and Unitary Representations" Princeton Mathematical series **45**, Princeton University Press, Lawrenceville, New Jersey, 1995.

[Johnson 1984] Joseph, F. Johnson, Lie algebra cohomology and the resolution of certain Harish-Chndra modules, *Math. Ann.* **267** (1984), 377-393.

[Matumoto 2002] Hisayosi Matumoto, On the representations of $Sp(p,q)$ and $SO^*(2n)$ unitarily induced from derived functor modules, preprint, UTMS 2002-12, arXive: math.RT/0203107.

[Schmid 1988] W. Schmid, Geometric constructions of representations, *Adv. Stud. in Pure Math.* vol. 14, Kinokuniya Book Store, 349-368, 1988.

[Vogan (green)] D. A. Vogan Jr., "Representations of Real reductive Lie Groups", *Progress in Mathematics,* Birkhäuser, 1982.

[Vogan 1988] D. A. Vogan Jr., Irreducibilities of discrete series representations for semisimple symmetric spaces, *Adv. Stud. in Pure Math.* vol. 14, Kinokuniya Book Store, 1988, 381-417.