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Integral switching engine for special canonical Clebsch-Gordan coefficients for $\mathfrak{gl}_3$

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0 Introduction

The authors have been interested in explicit formulae for certain spherical functions on real semisimple Lie groups of split rank 2. Because our problems have been to consider the case of representations with non-trivial minimal K-types, the explicit realization of the irreducible finite dimensional representation of K was crucial in computation. Unfortunately this problem is by no means easy except for $U(2)$ and $SU(2)$, though we have Gelfand-Tsetlin basis for K of type A or BD, and for general case the canonical basis or/and crystal basis. Even if these basis makes some problem *effectively computable*, the real computation is still difficult.

Around mid 80's, Gelfand and Zelvinsky found a relation between the Gelfand-Tsetlin basis and the canonical basis in the representation spaces of $\mathfrak{gl}_3$. Though the canonical basis is investigated by Kashiwara and Lustzig for general quantum groups of classical type, the relation of them with the Gelfand-Tsetlin basis is not known except for this special case, i.e. $\mathfrak{gl}_3$-case.

We utilize this result to formulate Theorem 1 in §2, which gives the explicit formulae for the projectors from the tensor product $V \otimes V_{(1,0,0)}$ of an irreducible representation V of $\mathfrak{gl}_3$ and the standard represenation, to its irreducible components. Once one can find the 'right formulae', the proof is given by direct computation.

In §3, we also give the explicit projectors from $V \otimes V_{(2,0,0)}$ to its (generically 6) irreducible components (Theorem 2 in §3).

Our result in this paper is just simple computation, but it might have application for investigation of spherical functions with non-trivial K-types, and also it might contain some suggestion for general investigation of canonical Clebsch-Gordan coefficients.

Notations For a Gelfand-Tsetlin pattern (which simply we may call G-pattern)

$$M = \begin{pmatrix} \mathbf{m}_3 \\ \mathbf{m}_2 \\ \mathbf{m}_1 \end{pmatrix} = \begin{pmatrix} m_{13}\, m_{23}\, m_{33} \\ m_{12}\, m_{22} \\ m_{11} \end{pmatrix}$$

of degree 3, we define

$$M \begin{pmatrix} i_{13}\, i_{23}\, i_{33} \\ i_{12}\, i_{22} \\ i_{11} \end{pmatrix} = \begin{pmatrix} m_{13}+i_{13}, \ m_{23}+i_{23}, \ m_{33}+i_{33} \\ m_{12}+i_{12}, \ m_{22}+i_{22} \\ m_{11}+i_{11} \end{pmatrix}.$$

If the vector $(i_{13},\ i_{23},\ i_{33})$ is zero, we omit the top row in the left hand side of the above defining equality. So the left hand side is written as

$$M \begin{pmatrix} i_{12}\, i_{22} \\ i_{11} \end{pmatrix}.$$

A convenient symbol is $M[k]$, which is defined by

$$M \begin{pmatrix} k, -k \\ 0 \end{pmatrix}.$$

This means that it causes a 'twist' of weight k at the second row $\mathbf{m}_2$ in M.

If any of the above shifts M' of M violates the conditions of Gelfand-Tsetlin pattern, i.e. if either

$$m'_{13} \geq m'_{12} \geq m'_{23} \geq m'_{22} \geq m'_{33}$$

or

$$m'_{12} \geq m'_{11} \geq m'_{22}$$

is not satisfied, then the corresponding vectors $f(M), f'(M)$ in the canonical basis should be zero.

Functions in M. We set

$$\delta(M) = m_{12} + m_{22} - m_{11} - m_{23}.$$

Let $\chi_+(M)$, $\chi_-(M)$ be the characteric functions of the sets $\{M|\delta(M) > 0\}$, $\{M|\delta(M) < 0\}$ respectively. More generally we introduce functions $\chi_\pm^{(i)}(M)$ by

$$\chi_+^{(i)}(M) = \begin{cases} 1, & \delta(M) > i; \\ 0, & \delta(M) \leq i. \end{cases} \qquad \chi_-^{(i)}(M) = \begin{cases} 1, & \delta(M) < -i; \\ 0, & \delta(M) \geq -i. \end{cases}$$

Then we have $\chi_+(M) = \chi_+^{(0)}(M)$ and $\chi_-(M) = \chi_-^{(i)}(M)$.

We introduce 'piecewise-linear' functions $C_1(M), \bar{C}_1(M), C_2(M)$, by

$$C_1(M) = \begin{cases} m_{11} - m_{22}, & \text{if } \delta(M) \geq 0; \\ m_{12} - m_{23}, & \text{if } \delta(M) \leq 0. \end{cases} \quad \bar{C}_1(M) = \begin{cases} m_{23} - m_{22}, & \text{if } \delta(M) \geq 0; \\ m_{12} - m_{11}, & \text{if } \delta(M) \leq 0. \end{cases}$$

and

$$C_2(M) = C_1(M)\bar{C}_1(M).$$

Another expression of $C_1(M)$ and $\bar{C}_1(M)$ is

$$C_1(M) = \mathrm{Min}\{m_{11}-m_{22}, m_{12}-m_{23}\}, \quad \bar{C}_1(M) = \mathrm{Min}\{m_{23}-m_{22}, m_{12}-m_{11}\}.$$

1 The result of Gelfand-Zelevinsky

There is nice basis in the representations spaces of quantum groups: Kashiwara called that 'crystal basis', Lustzig 'canonical basis'. These are developped in '90's.

However, around '85, Gelfand and Zelvinsky already investigated the case of (the classical Lie algebra) $\mathfrak{gl}_3$ in a very precise manner.

Firstly we recall the definition of the canonical basis in the sense of Gelfand and Zelvinsky. In the begining let us consider the case of the Lie algebra $\mathfrak{gl}_n$.

Definition A *weigtht* is an integral vector $\gamma = (\gamma_1, \cdots, \gamma_n) \in \mathbf{Z}^n$ of length n. A weight γ is *dominant* if $\gamma_1 \geq \gamma_2 \geq \cdots \geq \gamma_n$.

As is well-known, any irreducible representation of finite dimension V of $\mathfrak{gl}_n$ splits into weight subspaces:

$$V = \oplus_\gamma V(\gamma).$$

Here

$$V_\lambda(\gamma, \nu) = \{v \in V | E_{ii}v = \gamma_i v \text{ for all } i\} \neq \{0\}.$$

And there is the (unique) dominant weight λ s.t. $\lambda \geq \gamma$ in the lexigographical order. Therefore the representation V is labelled by λ, i.e. $V = V_\lambda$.

Now for another dominant weight ν, we set

$$V_\lambda = \{v \in V_\lambda(\gamma) | E_{i,i+1}^{\nu_i - \nu_{i+1}+1} v = 0, \text{ for } 1 \leq i \leq n-1\}.$$

Definition A basis B in V_λ is called *proper* if each of subspace $V_\lambda(\gamma, \nu)$ (for all possible γ, ν) is spanned by its subset, i.e.

$$V_\lambda(\gamma, \nu) = < B \cap V_\lambda(\gamma, \nu) > .$$

Theorem (Gelfand-Zelvinsky)
(i) Each irreducible finite dimensional representation of $\mathfrak{gl}_n$ has a proper basis.
(ii) In each irreducible finite dimensional representation of $\mathfrak{gl}_3$, there is only one proper basis up to scalar multiple. And this basis is called **canonical**.

Up to this point, the notion of the canonical basis has the ambiguity of scalar multiple. Gelfand and Zelvinsky normalized this scalar factor somehow to get the following formulae.

Let E_{ij} $(1 \le i, j \le 3)$ be the matrix unit of size 3 with 1 at the (i,j)-entry and 0 at other entries. If $i \neq j$, it is a generator of the root space of some root in $\mathfrak{gl}_3$ with respect to the Cartan subalgebra consisting of diagonal matrices. If $|i-j| = 1$, E_{ij} is a root vector of a simple root. There are 4 such simple root vectors E_{12}, E_{21}, E_{23} and E_{32}.

Proposition 1 (Gelfand-Zelevinsky) The action of simple root vectors on the canonical basis is given as follows.

$$
\begin{aligned}
E_{12}f(M) = \; & (m_{12} - m_{11}) f\left(M\left(\begin{smallmatrix} 0 & & 0 \\ & 1 & \end{smallmatrix}\right)\right) \\
& + (m_{23} - m_{22}) \chi_+(M) f\left(M\left(\begin{smallmatrix} 0 & & 0 \\ & 1 & \end{smallmatrix}\right)[-1]\right); \\
E_{21}f(M) = \; & (m_{11} - m_{22}) f\left(M\left(\begin{smallmatrix} 0 & & 0 \\ & -1 & \end{smallmatrix}\right)\right) \\
& + (m_{12} - m_{23}) \chi_-(M) f\left(M\left(\begin{smallmatrix} 0 & & 0 \\ & -1 & \end{smallmatrix}\right)[-1]\right); \\
E_{23}f(M) = \; & (m_{13} - m_{12}) f\left(M\left(\begin{smallmatrix} 1 & & 0 \\ & 0 & \end{smallmatrix}\right)\right) \\
& + (m_{13} - m_{12} - \delta(M)) \chi_-(M) f\left(M\left(\begin{smallmatrix} 1 & & 0 \\ & 0 & \end{smallmatrix}\right)[-1]\right); \\
E_{32}f(M) = \; & (m_{22} - m_{33}) f\left(M\left(\begin{smallmatrix} 0 & & -1 \\ & 0 & \end{smallmatrix}\right)\right) \\
& + (m_{22} - m_{33} + \delta(M)) \chi_+(M) f\left(M\left(\begin{smallmatrix} 0 & & -1 \\ & 0 & \end{smallmatrix}\right)[-1]\right).
\end{aligned}
$$

Remark We have

$$m_{23} - m_{22} = m_{12} - m_{11} - \delta(M), \text{ and } m_{12} - m_{23} = m_{11} - m_{22} + \delta(M).$$

hin the formulae of E_{12} and E_{21}.

Remark The formulae is the above proposition seems to be well-know among the specialist of quantum groups and canonical basis.

2 Tensor products with the standard representation

Generically the tensor product $V_{\mathbf{m_3}} \otimes V_{(1,0,0)}$ has three irreducible components: $V_{\mathbf{m_3}+(1,0,0)}$, $V_{\mathbf{m_3}+(0,1,0)}$ and $V_{\mathbf{m_3}+(0,0,1)}$. If either $\mathbf{m_3} + (0,1,0)$ or $\mathbf{m_3} + (0,0,1)$

is not dominant, the corresponding irreducible component does not occur. Thus for the dimension of the intertwing spaces, we have

$$\begin{aligned}
\dim_{\mathbf{C}}\ \mathrm{Hom}(V_{\mathbf{m_3}} \otimes V_{(1,0,0)}, V_{\mathbf{m_3}+(1,0,0)}) &= 1, \\
\dim_{\mathbf{C}}\ \mathrm{Hom}(V_{\mathbf{m_3}} \otimes V_{(1,0,0)}, V_{\mathbf{m_3}+(0,1,0)}) &\leq 1, \\
\dim_{\mathbf{C}}\ \mathrm{Hom}(V_{\mathbf{m_3}} \otimes V_{(1,0,0)}, V_{\mathbf{m_3}+(0,0,1)}) &\leq 1.
\end{aligned}$$

Let $P_{(1,0,0)}$ be a non-zero generator of the first space, which is unique up to scalar multiple. And let $P_{(0,1,0)}$ or $P_{(0,0,1)}$ also be the generator of the second or the third space respectively, if either space is non-zero. Our purpose in this section is to give explicit expression of these projectors $P_{(1,0,0)}$, $P_{(0,1,0)}$ and $P_{(0,0,1)}$ in terms of canonical basis.

2.1 The projectors for $V_{\mathbf{m}_3} \otimes V_{(1,0,0)}$

Let $V_{\mathbf{m}_3}$ be the representation of $\mathfrak{gl}_3$ with canonical basis $\{f(M)\}$, and let $V_{(1,0,0)}$ be the standard representation. To denote the canonical basis of the standard representation $V_{(1,0,0)}$, we supress the letter 'f' before G-patterns.

Theorem 1 Let $\{f'(M)\}$ be the canonical basis of the target representation.

(i) The projector $P_{(1,0,0)} : V_{\mathbf{m}_3} \otimes V_{(1,0,0)} \to V_{\mathbf{m}_3+(1,0,0)}$ is given as follows:

(i-a) $P_{(1,0,0)}\left(f(M) \otimes \left(\begin{smallmatrix}1\,0\,0\\1\,0\\1\end{smallmatrix}\right)\right) = f'\left(M\left(\begin{smallmatrix}1\,0\,0\\1\,0\\1\end{smallmatrix}\right)\right).$

(i-b) $P_{(1,0,0)}\left(f(M) \otimes \left(\begin{smallmatrix}1\,0\,0\\1\,0\\0\end{smallmatrix}\right)\right) = f'\left(M\left(\begin{smallmatrix}1\,0\,0\\1\,0\\0\end{smallmatrix}\right)\right) + \chi_-(M)f'\left(M\left(\begin{smallmatrix}1\,0\,0\\0\,1\\0\end{smallmatrix}\right)\right).$

(i-c) $P_{(1,0,0)}\left(f(M) \otimes \left(\begin{smallmatrix}1\,0\,0\\0\,0\\0\end{smallmatrix}\right)\right) = f'\left(M\left(\begin{smallmatrix}1\,0\,0\\0\,0\\0\end{smallmatrix}\right)\right).$

(ii) The projector $P_{(0,1,0)} : V_{\mathbf{m}_3} \otimes V_{(1,0,0)} \to V_{\mathbf{m}_3+(0,1,0)}$ is given as follows:

(ii-a) $P_{(0,1,0)}\left(f(M) \otimes \left(\begin{smallmatrix}1\,0\,0\\1\,0\\1\end{smallmatrix}\right)\right)$
$= -(m_{13} - m_{12})f'\left(M\left(\begin{smallmatrix}0\,1\,0\\1\,0\\1\end{smallmatrix}\right)\right) + \chi_+(M)D(M)f'\left(M\left(\begin{smallmatrix}0\,1\,0\\0\,1\\1\end{smallmatrix}\right)\right).$

(ii-b) $P_{(0,1,0)}\left(f(M) \otimes \left(\begin{smallmatrix}1\,0\,0\\1\,0\\0\end{smallmatrix}\right)\right)$
$= -(m_{13} - m_{12})f'\left(M\left(\begin{smallmatrix}0\,1\,0\\1\,0\\0\end{smallmatrix}\right)\right) + C_1(M)f'\left(M\left(\begin{smallmatrix}0\,1\,0\\0\,1\\0\end{smallmatrix}\right)\right).$

(ii-c) $P_{(0,1,0)}\left(f(M) \otimes \left(\begin{smallmatrix}1\,0\,0\\0\,0\\0\end{smallmatrix}\right)\right)$

$= (m_{12} - m_{23}) f'\left(M\left(\begin{smallmatrix} 0\,1\,0 \\ 0\,0 \\ 0 \end{smallmatrix}\right)\right) + \chi_+(M) C_1(M) f'\left(M\left(\begin{smallmatrix} 0\,1\,0 \\ -1\,1 \\ 0 \end{smallmatrix}\right)\right).$

Here $D(M) = -m_{13} + m_{12} - \delta(M)$.

(iii) The projector $P_{(0,0,1)} : V_{\mathbf{m}_3} \otimes V_{(1,0,0)} \to V_{\mathbf{m}_3+(0,0,1)}$ is given as follows:

(iii-a) $P_{(0,0,1)}\left(f(M) \otimes \left(\begin{smallmatrix} 1\,0\,0 \\ 1\,0 \\ 1 \end{smallmatrix}\right)\right)$

$= -(m_{13} - m_{12})(m_{22} - m_{33}) f'\left(M\left(\begin{smallmatrix} 0\,0\,1 \\ 1\,0 \\ 1 \end{smallmatrix}\right)\right) + E(M) f'\left(M\left(\begin{smallmatrix} 0\,0\,1 \\ 0\,1 \\ 1 \end{smallmatrix}\right)\right).$

(iii-b) $P_{(0,0,1)}\left(f(M) \otimes \left(\begin{smallmatrix} 1\,0\,0 \\ 1\,0 \\ 0 \end{smallmatrix}\right)\right)$

$= -(m_{13} - m_{12})(m_{22} - m_{33}) f'\left(M\left(\begin{smallmatrix} 0\,0\,1 \\ 1\,0 \\ 0 \end{smallmatrix}\right)\right) + F(M) f'\left(M\left(\begin{smallmatrix} 0\,0\,1 \\ 0\,1 \\ 0 \end{smallmatrix}\right)\right)$

$- \chi_-(M) C_2(M) f'\left(M\left(\begin{smallmatrix} 0\,0\,1 \\ -1\,2 \\ 0 \end{smallmatrix}\right)\right).$

(iii-c) $P_{(0,0,1)}\left(f(M) \otimes \left(\begin{smallmatrix} 1\,0\,0 \\ 0\,0 \\ 0 \end{smallmatrix}\right)\right)$

$= (m_{12} - m_{33} + 1)(m_{22} - m_{33}) f'\left(M\left(\begin{smallmatrix} 0\,0\,1 \\ 0\,0 \\ 0 \end{smallmatrix}\right)\right) - C_2(M) f'\left(M\left(\begin{smallmatrix} 0\,0\,1 \\ -1\,1 \\ 0 \end{smallmatrix}\right)\right).$

Here

$$\begin{aligned} E(M) &= \bar{C}_1(M)(m_{13} - m_{33} + 1 - C_1(M)), \\ F(M) &= -C_2(M) - \chi_-(M)\{(m_{13} - m_{12})(m_{22} - m_{33}) - (m_{13} - m_{33} + 1)\delta(M)\}. \end{aligned}$$

Remark As we can see in the next subsection, in order to prove Theorem 1, it suffices to show that any of three projector given above is a $\mathfrak{gl}_3$-homomorphism. But the actual method to find these formula is to use the relation between the canonical basis with the Gelfand-Tsetlin basis found by Gelfand-Zelvinsky [?]. To write this computation seems to take more space than the proof below.

2.2 Proof of Theorem 1

The proof is direct compuation to check that either of three projectors is a $\mathfrak{gl}_3$-modules. The action of the Cartan subgroup is diagonal. Therefore the essential computation is those of simple root vetors $E_{i,i+1}$, $E_{i+1,i}$.

2.3 The symmetric tensor product $V_{(2,0,0)}$ of the standard representation $V_{(1,0,0)}$

If we apply Theorem 1 for the special case $V_{\mathbf{m_3}} = V_{(1,0,0)}$, we have only two irreducible constituents $V_{(2,0,0)}$ and $V_{(1,1,0)}$ which occur with multiplicities one:

$$V_{(1,0,0)} \otimes V_{(1,0,0)} \cong V_{(2,0,0)} \oplus V_{(1,1,0)}.$$

The first factor $V_{(2,0,0)}$ is the symmetric tensor product of $V_{(1,0,0)}$, and the second the anti-symmetric tensor product. Here we write the correspondence between canonical basis of $V_{(1,0,0)}$ and $V_{(2,0,0)}$ explicitly.

Lemma 2 Via identification $V_{(2,0,0)}$ with $\mathrm{Sym}^2(V_{(1,0,0)})$ wich is unique up to a scalar multiple, we have identifications :

$$\begin{aligned}
\left(\begin{smallmatrix}2\,0\,0\\0\,0\\0\end{smallmatrix}\right) &= \left(\begin{smallmatrix}1\,0\,0\\0\,0\\0\end{smallmatrix}\right)\otimes\left(\begin{smallmatrix}1\,0\,0\\0\,0\\0\end{smallmatrix}\right)\\
\left(\begin{smallmatrix}2\,0\,0\\1\,0\\0\end{smallmatrix}\right) &= \frac{1}{2}\left\{\left(\begin{smallmatrix}1\,0\,0\\0\,0\\0\end{smallmatrix}\right)\otimes\left(\begin{smallmatrix}1\,0\,0\\1\,0\\0\end{smallmatrix}\right)+\left(\begin{smallmatrix}1\,0\,0\\1\,0\\0\end{smallmatrix}\right)\otimes\left(\begin{smallmatrix}1\,0\,0\\0\,0\\0\end{smallmatrix}\right)\right\}\\
\left(\begin{smallmatrix}2\,0\,0\\1\,0\\1\end{smallmatrix}\right) &= \frac{1}{2}\left\{\left(\begin{smallmatrix}1\,0\,0\\0\,0\\0\end{smallmatrix}\right)\otimes\left(\begin{smallmatrix}1\,0\,0\\1\,0\\1\end{smallmatrix}\right)+\left(\begin{smallmatrix}1\,0\,0\\1\,0\\1\end{smallmatrix}\right)\otimes\left(\begin{smallmatrix}1\,0\,0\\0\,0\\0\end{smallmatrix}\right)\right\}\\
\left(\begin{smallmatrix}2\,0\,0\\2\,0\\0\end{smallmatrix}\right) &= \left(\begin{smallmatrix}1\,0\,0\\1\,0\\0\end{smallmatrix}\right)\otimes\left(\begin{smallmatrix}1\,0\,0\\1\,0\\0\end{smallmatrix}\right)\\
\left(\begin{smallmatrix}2\,0\,0\\2\,0\\1\end{smallmatrix}\right) &= \frac{1}{2}\left\{\left(\begin{smallmatrix}1\,0\,0\\1\,0\\0\end{smallmatrix}\right)\otimes\left(\begin{smallmatrix}1\,0\,0\\1\,0\\1\end{smallmatrix}\right)+\left(\begin{smallmatrix}1\,0\,0\\1\,0\\1\end{smallmatrix}\right)\otimes\left(\begin{smallmatrix}1\,0\,0\\1\,0\\0\end{smallmatrix}\right)\right\}\\
\left(\begin{smallmatrix}2\,0\,0\\2\,0\\2\end{smallmatrix}\right) &= \left(\begin{smallmatrix}1\,0\,0\\1\,0\\1\end{smallmatrix}\right)\otimes\left(\begin{smallmatrix}1\,0\,0\\1\,0\\1\end{smallmatrix}\right)
\end{aligned}$$

Remark. Similarly to the case of the standard representation, to denote the canonical basis of $V_{(2,0,0)}$ we do not write the letter 'f' before its G-pattern.

3 Tensor product with $V_{(2,0,0)}$

In this section, we want to have the irreducible decomposition of the tensor product $V_{\mathbf{m_3}} \otimes V_{(2,0,0)}$ and an explicit formula of the projectors to its irreducible components. Generically this tensor product has six irreducible components $V_{\mathbf{m_3}+\mathbf{e}_i+\mathbf{e}_j}$ $(1 \le i, j \le 3)$. Here $\mathbf{e}_i$ is the unit vector with unity at the i-th entry and zero at the remaining entries. Each component occurs with multiplicity one, if the weight vector $\mathbf{m_3} + \mathbf{e}_i + \mathbf{e}_j$ is dominant.

3.1 The projectors for $V_{\mathbf{m}_3} \otimes V_{(2,0,0)}$.

In this subsection, we give an explicit formula for the projectors from the tencor product $V_{\mathbf{m}_3} \otimes V_{(2,0,0)}$ to its six irreducible components in terms of canonical basis. The proof of this is given in the next subsection. Let $\{f(M)\}$ and $\{f'(M)\}$ be the canonical basis of the representation $V_{\mathbf{m}_3}$ and of the target representation, respectively. Similarly for the standard representation, we suppress the letter 'f' before G-patterns to denote the canonical basis of $V_{(2,0,0)}$.

Formula 1: The projector $P_{(2,0,0)} : V_{\mathbf{m}_3} \otimes V_{(2,0,0)} \to V_{\mathbf{m}_3+(2,0,0)}$.

1. $P_{(2,0,0)}\left(f(M) \otimes \left(\substack{2\,0\,0 \\ 0\,0 \\ 0}\right)\right) = f'\left(M\left(\substack{2\,0\,0 \\ 0\,0 \\ 0}\right)\right).$
2. $P_{(2,0,0)}\left(f(M) \otimes \left(\substack{2\,0\,0 \\ 1\,0 \\ 0}\right)\right) = f'\left(M\left(\substack{2\,0\,0 \\ 1\,0 \\ 0}\right)\right) + \chi_-(M) f'\left(M\left(\substack{2\,0\,0 \\ 0\,1 \\ 0}\right)\right).$
3. $P_{(2,0,0)}\left(f(M) \otimes \left(\substack{2\,0\,0 \\ 1\,0 \\ 1}\right)\right) = f'\left(M\left(\substack{2\,0\,0 \\ 1\,0 \\ 1}\right)\right).$
4. $P_{(2,0,0)}\left(f(M) \otimes \left(\substack{2\,0\,0 \\ 2\,0 \\ 0}\right)\right) = f'\left(M\left(\substack{2\,0\,0 \\ 2\,0 \\ 0}\right)\right) + \left\{\chi_-(M) + \chi_-^{(1)}(M)\right\} f'\left(M\left(\substack{2\,0\,0 \\ 1\,1 \\ 0}\right)\right) +$
   $\chi_-^{(1)}(M) f'\left(M\left(\substack{2\,0\,0 \\ 0\,2 \\ 0}\right)\right).$
5. $P_{(2,0,0)}\left(f(M) \otimes \left(\substack{2\,0\,0 \\ 2\,0 \\ 1}\right)\right) = f'\left(M\left(\substack{2\,0\,0 \\ 2\,0 \\ 1}\right)\right) + \chi_-(M) f'\left(M\left(\substack{2\,0\,0 \\ 1\,1 \\ 1}\right)\right).$
6. $P_{(2,0,0)}\left(f(M) \otimes \left(\substack{2\,0\,0 \\ 2\,0 \\ 2}\right)\right) = f'\left(M\left(\substack{2\,0\,0 \\ 2\,0 \\ 2}\right)\right).$

Formula 2: The projector $P_{(1,1,0)} : V_{\mathbf{m}_3} \otimes V_{(2,0,0)} \to V_{\mathbf{m}_3+(1,1,0)}$.

1. $P_{(1,1,0)}\left(f(M) \otimes \left(\substack{2\,0\,0 \\ 0\,0 \\ 0}\right)\right) = (m_{12}-m_{23}) f'\left(M\left(\substack{1\,1\,0 \\ 0\,0 \\ 0}\right)\right) + C_1(M)\chi_+(M) f'\left(M\left(\substack{1\,1\,0 \\ -1\,1 \\ 0}\right)\right).$
2. $P_{(1,1,0)}\left(f(M) \otimes \left(\substack{2\,0\,0 \\ 1\,0 \\ 0}\right)\right) = \frac{1}{2}(2m_{12} - m_{13} - m_{23}) f'\left(M\left(\substack{1\,1\,0 \\ 1\,0 \\ 0}\right)\right) +$
   $C_1(M) f'\left(M\left(\substack{1\,1\,0 \\ 0\,1 \\ 0}\right)\right).$
3. $P_{(1,1,0)}\left(f(M) \otimes \left(\substack{2\,0\,0 \\ 1\,0 \\ 1}\right)\right) = \frac{1}{2}(2m_{12} - m_{13} - m_{23}) f'\left(M\left(\substack{1\,1\,0 \\ 1\,0 \\ 1}\right)\right)$
   $+\frac{1}{2}\chi_+(M)\left\{C_1(M) + D(M)\right\} f'\left(M\left(\substack{1\,1\,0 \\ 0\,1 \\ 1}\right)\right).$
4. $P_{(1,1,0)}\left(f(M) \otimes \left(\substack{2\,0\,0 \\ 2\,0 \\ 0}\right)\right) = -(m_{13} - m_{12}) f'\left(M\left(\substack{1\,1\,0 \\ 2\,0 \\ 0}\right)\right)$
   $+\left\{C_1(M) - \chi_-(M)(m_{13} - m_{12})\right\} f'\left(M\left(\substack{1\,1\,0 \\ 1\,1 \\ 0}\right)\right) + C_1(M)\chi_-(M) f'\left(M\left(\substack{1\,1\,0 \\ 0\,2 \\ 0}\right)\right).$

5. $P_{(1,1,0)}\left(f(M)\otimes\left(\begin{smallmatrix}2\;0\;0\\2\;0\\1\end{smallmatrix}\right)\right)=-(m_{13}-m_{12})f'\left(M\left(\begin{smallmatrix}1\;1\;0\\2\;0\\1\end{smallmatrix}\right)\right)$
$+\frac{1}{2}\left\{C_1(M)-(m_{13}-m_{12})-\delta(M)\chi_+(M)\right\}f'\left(M\left(\begin{smallmatrix}1\;1\;0\\1\;1\\1\end{smallmatrix}\right)\right).$

6. $P_{(1,1,0)}\left(f(M)\otimes\left(\begin{smallmatrix}2\;0\;0\\2\;0\\2\end{smallmatrix}\right)\right)=-(m_{13}-m_{12})f'\left(M\left(\begin{smallmatrix}1\;1\;0\\2\;0\\2\end{smallmatrix}\right)\right)$
$+\chi_+(M)D(M)f'\left(M\left(\begin{smallmatrix}1\;1\;0\\1\;1\\2\end{smallmatrix}\right)\right).$

Formula 3: The projector $P_{(1,0,1)}:V_{\mathbf{m}_3}\otimes V_{(2,0,0)}\to V_{\mathbf{m}_3+(1,0,1)}$.

1. $P_{(1,0,1)}\left(f(M)\otimes\left(\begin{smallmatrix}2\;0\;0\\0\;0\\0\end{smallmatrix}\right)\right)=(m_{12}-m_{33}+1)(m_{22}-m_{33})f'\left(M\left(\begin{smallmatrix}1\;0\;1\\0\;0\\0\end{smallmatrix}\right)\right)$
$-C_2(M)f'\left(M\left(\begin{smallmatrix}1\;0\;1\\-1\;1\\0\end{smallmatrix}\right)\right).$

2. $P_{(1,0,1)}\left(f(M)\otimes\left(\begin{smallmatrix}2\;0\;0\\1\;0\\0\end{smallmatrix}\right)\right)=\frac{1}{2}(m_{22}-m_{33})(2m_{12}-m_{13}-m_{33}+1)f'\left(M\left(\begin{smallmatrix}1\;0\;1\\1\;0\\0\end{smallmatrix}\right)\right)$
$+\frac{1}{2}\left\{F(M)-C_2(M)+\chi_-(M)(m_{22}-m_{33})(m_{12}-m_{33}+1)\right\}f'\left(M\left(\begin{smallmatrix}1\;0\;1\\0\;1\\0\end{smallmatrix}\right)\right)-$
$C_2(M)\chi_-(M)f'\left(M\left(\begin{smallmatrix}1\;0\;1\\-1\;2\\0\end{smallmatrix}\right)\right).$

3. $P_{(1,0,1)}\left(f(M)\otimes\left(\begin{smallmatrix}2\;0\;0\\1\;0\\1\end{smallmatrix}\right)\right)=\frac{1}{2}(m_{22}-m_{33})(2m_{12}-m_{13}-m_{33}+1)f'\left(M\left(\begin{smallmatrix}1\;0\;1\\1\;0\\1\end{smallmatrix}\right)\right)$
$+\frac{1}{2}\left\{E(M)-C_2(M)\right\}f'\left(M\left(\begin{smallmatrix}1\;0\;1\\0\;1\\1\end{smallmatrix}\right)\right).$

4. $P_{(1,0,1)}\left(f(M)\otimes\left(\begin{smallmatrix}2\;0\;0\\2\;0\\0\end{smallmatrix}\right)\right)=-(m_{13}-m_{12})(m_{22}-m_{33})f'\left(M\left(\begin{smallmatrix}1\;0\;1\\2\;0\\0\end{smallmatrix}\right)\right)$
$+\left\{F(M)-\chi_-^{(1)}(M)(m_{13}-m_{12})(m_{22}-m_{33})\right\}f'\left(M\left(\begin{smallmatrix}1\;0\;1\\1\;1\\0\end{smallmatrix}\right)\right)$
$+\left\{-\chi_-(M)C_2(M)+\chi_-^{(1)}(M)F(M)\right\}f'\left(M\left(\begin{smallmatrix}1\;0\;1\\0\;2\\0\end{smallmatrix}\right)\right)$
$-\chi_-^{(1)}(M)C_2(M)f'\left(M\left(\begin{smallmatrix}1\;0\;1\\-1\;3\\0\end{smallmatrix}\right)\right).$

5. $P_{(1,0,1)}\left(f(M)\otimes\left(\begin{smallmatrix}2\;0\;0\\2\;0\\1\end{smallmatrix}\right)\right)=-(m_{13}-m_{12})(m_{22}-m_{33})f'\left(M\left(\begin{smallmatrix}1\;0\;1\\2\;0\\1\end{smallmatrix}\right)\right)$
$+\frac{1}{2}\left\{E(M)+F(M)-\chi_-(M)(m_{13}-m_{12})(m_{22}-m_{33})\right\}f'\left(M\left(\begin{smallmatrix}1\;0\;1\\1\;1\\1\end{smallmatrix}\right)\right)+$
$\frac{1}{2}\chi_-(M)\left\{E(M)-C_2(M)\right\}f'\left(M\left(\begin{smallmatrix}1\;0\;1\\0\;2\\1\end{smallmatrix}\right)\right).$

6. $P_{(1,0,1)}\left(f(M)\otimes\left(\begin{smallmatrix}2\;0\;0\\2\;0\\2\end{smallmatrix}\right)\right)=-(m_{13}-m_{12})(m_{22}-m_{33})f'\left(M\left(\begin{smallmatrix}1\;0\;1\\2\;0\\2\end{smallmatrix}\right)\right)$
$+E(M)f'\left(M\left(\begin{smallmatrix}1\;0\;1\\1\;1\\2\end{smallmatrix}\right)\right).$

Formula 4: The projector $P_{(0,2,0)}:V_{\mathbf{m}_3}\otimes V_{(2,0,0)}\to V_{\mathbf{m}_3+(0,2,0)}$.

1. $P_{(0,2,0)}\left(f(M)\otimes\left(\begin{smallmatrix}2\;0\;0\\0\;0\\0\end{smallmatrix}\right)\right)=\sum_{i=0}^{2}c_i f'\left(M\left(\begin{smallmatrix}0\;2\;0\\0\;0\\0\end{smallmatrix}\right)[-i]\right)$ with

$$
\begin{aligned}
c_0 &= (m_{12}-m_{23})(m_{12}-m_{23}-1),\\
c_1 &= C_1(M)\left\{(m_{12}-m_{23})\chi_+^{(1)}(M)+(m_{12}-m_{23}-2)\chi_+(M)\right\},\\
c_2 &= C_1(M)\left\{C_1(M)-1\right\}\chi_+^{(1)}(M).
\end{aligned}
$$

2. $P_{(0,2,0)}\left(f(M)\otimes\left(\begin{smallmatrix}2\;0\;0\\1\;0\\0\end{smallmatrix}\right)\right)=\sum_{i=0}^{2}c_i f'\left(M\left(\begin{smallmatrix}0\;2\;0\\1\;0\\0\end{smallmatrix}\right)[-i]\right)$ with

$$
\begin{aligned}
c_0 &= -(m_{12}-m_{23})(m_{13}-m_{12}),\\
c_1 &= C_1(M)\left\{(m_{12}-m_{23}-1)-\chi_+(M)(m_{13}-m_{12})\right\},\\
c_2 &= C_1(M)\left\{C_1(M)-1\right\}\chi_+(M).
\end{aligned}
$$

3. $P_{(0,2,0)}\left(f(M)\otimes\left(\begin{smallmatrix}2\;0\;0\\1\;0\\1\end{smallmatrix}\right)\right)=\sum_{i=0}^{2}c_i f'\left(M\left(\begin{smallmatrix}0\;2\;0\\1\;0\\1\end{smallmatrix}\right)[-i]\right)$ with

$$
\begin{aligned}
c_0 &= -(m_{12}-m_{23})(m_{13}-m_{12}),\\
c_1 &= \chi_+(M)D(M)(m_{12}-m_{23}-1)-\chi_+^{(1)}(M)\left\{C_1(M)+1\right\}(m_{13}-m_{12}),\\
c_2 &= C_1(M)D(M)\chi_+^{(1)}(M).
\end{aligned}
$$

4. $P_{(0,2,0)}\left(f(M)\otimes\left(\begin{smallmatrix}2\;0\;0\\2\;0\\0\end{smallmatrix}\right)\right)=\sum_{i=0}^{2}f'\left(M\left(\begin{smallmatrix}0\;2\;0\\2\;0\\0\end{smallmatrix}\right)[-i]\right)$ with

$$
\begin{aligned}
c_0 &= (m_{13}-m_{12})(m_{13}-m_{12}-1),\\
c_1 &= -2C_1(M)(m_{13}-m_{12}),\\
c_2 &= C_1(M)\left\{C_1(M)-1\right\}.
\end{aligned}
$$

5. $P_{(0,2,0)}\left(f(M)\otimes\left(\begin{smallmatrix}2\;0\;0\\2\;0\\1\end{smallmatrix}\right)\right)=\sum_{i=0}^{2}c_i f'\left(M\left(\begin{smallmatrix}0\;2\;0\\2\;0\\1\end{smallmatrix}\right)[-i]\right)$ with

$$
\begin{aligned}
c_0 &= (m_{13}-m_{12})(m_{13}-m_{12}-1),\\
c_1 &= -(m_{13}-m_{12})\left[\chi_I(M)\left\{D(M)+1\right\}+C_1(M)\right],\\
c_2 &= \chi_+(M)C_1(M)D(M).
\end{aligned}
$$

6. $P_{(0,2,0)}\left(f(M)\otimes\left(\begin{smallmatrix}2\;0\;0\\2\;0\\2\end{smallmatrix}\right)\right)=\sum_{i=0}^{2}c_i f'\left(M\left(\begin{smallmatrix}0\;2\;0\\2\;0\\2\end{smallmatrix}\right)[-i]\right)$ with

$$
\begin{aligned}
c_0 &= (m_{13}-m_{12})(m_{13}-m_{12}-1),\\
c_1 &= -(m_{13}-m_{12})\left[\chi_+(M)D(M)+\chi_+^{(1)}(M)\left\{D(M)+2\right\}\right],\\
c_2 &= \chi_+^{(1)}(M)D(M)\left\{D(M)+1\right\}.
\end{aligned}
$$

Formula 5: The projector $P_{(0,1,1)} : V_{\mathbf{m}_3} \otimes V_{(2,0,0)} \to V_{\mathbf{m}_3+(0,1,1)}$.

1. $P_{(0,1,1)}\left(f(M) \otimes \left(\begin{smallmatrix} 2\,0\,0 \\ 0\,0 \\ 0 \end{smallmatrix}\right)\right) = \sum_{i=0}^{2} c_i f''\left(M\left(\begin{smallmatrix} 0\,1\,1 \\ 0\,0 \\ 0 \end{smallmatrix}\right)[-i]\right)$ with

$$
\begin{aligned}
c_0 &= (m_{12} - m_{23})(m_{12} - m_{33} + 1)(m_{22} - m_{33}), \\
c_1 &= \chi_+(M)C_1(M)(m_{12} - m_{33} + 1)(m_{22} - m_{33}) - (m_{12} - m_{23} - 1)C_2(M), \\
c_2 &= -\chi_+(M)C_2(M)\left\{C_1(M) - 1\right\}.
\end{aligned}
$$

2. $P_{(0,1,1)}\left(f(M) \otimes \left(\begin{smallmatrix} 2\,0\,0 \\ 1\,0 \\ 0 \end{smallmatrix}\right)\right) = \sum_{i=0}^{2} c_i f''\left(M\left(\begin{smallmatrix} 0\,1\,1 \\ 1\,0 \\ 0 \end{smallmatrix}\right)[-i]\right)$ with

$$
\begin{aligned}
c_0 &= -\frac{1}{2}(2m_{12} - m_{23} - m_{33} + 2)(m_{13} - m_{12})(m_{22} - m_{33}), \\
c_1 &= \frac{1}{2}(m_{12} - m_{23})F(M) - \frac{1}{2}(m_{13} - m_{12})(m_{22} - m_{33})C_1(M)\left\{1 - \chi_-(M)\right\} \\
&\quad + \frac{1}{2}(m_{13} - m_{12} + 1)C_2(M) + \frac{1}{2}(m_{12} - m_{33} + 1)C_1(M)(m_{22} - m_{33}), \\
c_2 &= -C_2(M)\left\{C_1(M) - 1\right\}.
\end{aligned}
$$

3. $P_{(0,1,1)}\left(f(M) \otimes \left(\begin{smallmatrix} 2\,0\,0 \\ 1\,0 \\ 1 \end{smallmatrix}\right)\right) = \sum_{i=0}^{2} c_i f'\left(M\left(\begin{smallmatrix} 0\,1\,1 \\ 1\,0 \\ 1 \end{smallmatrix}\right)[-i]\right)$ with

$$
\begin{aligned}
c_0 &= -\frac{1}{2}(2m_{12} - m_{23} - m_{33} + 2)(m_{13} - m_{12})(m_{22} - m_{33}), \\
c_1 &= \frac{1}{2}(m_{12} - m_{23})E(M) + \frac{1}{2}(m_{13} - m_{12} + 1)C_2(M) \\
&\quad + \frac{1}{2}\chi_+(M)(m_{22} - m_{33})\left[-\left\{C_1(M) + 1\right\}(m_{13} - m_{12}) + D(M)(m_{12} - m_{33} + 1)\right] \\
c_2 &= \frac{1}{2}\chi_+(M)C_2(M)\left\{m_{13} - m_{33} + 2 - C_1(M) - D(M)\right\}.
\end{aligned}
$$

4. $P_{(0,1,1)}\left(f(M) \otimes \left(\begin{smallmatrix} 2\,0\,0 \\ 2\,0 \\ 0 \end{smallmatrix}\right)\right) = \sum_{i=0}^{3} c_i f''\left(M\left(\begin{smallmatrix} 0\,1\,1 \\ 2\,0 \\ 0 \end{smallmatrix}\right[-i])\right)$ with

$$
\begin{aligned}
c_0 &= (m_{13} - m_{12})(m_{13} - m_{12} - 1)(m_{22} - m_{33}), \\
c_1 &= -(m_{13} - m_{12})\left[F(M) + (m_{22} - m_{33})\left\{C_1(M) + \chi_-(M)\right\}\right], \\
c_2 &= \left\{C_1(M) - 1 + \chi_-(M)\right\}F(M) + \chi_-(M)(m_{13} - m_{12} + 1)C_2(M), \\
c_3 &= -\chi_-(M)C_2(M)\left\{C_1(M) - 1\right\}.
\end{aligned}
$$

5. $P_{(0,1,1)}\left(f(M)\otimes\left(\begin{smallmatrix}2\,0\,0\\2\,0\\1\end{smallmatrix}\right)\right)=\sum_{i=0}^{2}c_if''\left(M\left(\begin{smallmatrix}0\,1\,1\\2\,0\\1\end{smallmatrix}\right)[-i]\right)$ with

$$\begin{aligned}
c_0 &= (m_{13}-m_{12})(m_{13}-m_{12}-1)(m_{22}-m_{33}),\\
c_1 &= -\frac{1}{2}(m_{13}-m_{12})\left[F(M)+E(M)+(m_{22}-m_{33})\left\{C_1(M)+1\right\}\right]\\
&\quad -\frac{1}{2}(m_{13}-m_{12})D(M)\left\{1-\chi_-(M)\right\}(m_{22}-m_{33}),\\
c_2 &= \frac{1}{2}C_1(M)E(M)+\frac{1}{2}C_2(M)\left\{1-D(M)-\chi_-(M)\delta(M)\right\}.
\end{aligned}$$

6. $P_{(0,1,1)}\left(f(M)\otimes\left(\begin{smallmatrix}2\,0\,0\\2\,0\\2\end{smallmatrix}\right)\right)=\sum_{i=0}^{2}c_if'\left(M\left(\begin{smallmatrix}0\,1\,1\\2\,0\\2\end{smallmatrix}\right)[-i]\right)$ with

$$\begin{aligned}
c_0 &= (m_{13}-m_{12})(m_{13}-m_{12}-1)(m_{22}-m_{33}),\\
c_1 &= -(m_{13}-m_{12})\left[E(M)+\chi_+(M)\left\{D(M)+1\right\}(m_{22}-m_{33})\right],\\
c_2 &= \chi_+(M)D(M)E(M).
\end{aligned}$$

Formula 6: The projector $P_{(0,0,2)}:V_{\mathbf{m}_3}\otimes V_{(2,0,0)}\rightarrow V_{\mathbf{m}_3+(0,0,2)}$.

1. $P_{(0,0,2)}\left(f(M)\otimes\left(\begin{smallmatrix}2\,0\,0\\0\,0\\0\end{smallmatrix}\right)\right)=\sum_{i=0}^{2}c_if'\left(M\left(\begin{smallmatrix}0\,0\,2\\0\,0\\0\end{smallmatrix}\right)[-i]\right)$ with

$$\begin{aligned}
c_0 &= (m_{12}-m_{33}+1)(m_{12}-m_{33})(m_{22}-m_{33})(m_{22}-m_{33}-1),\\
c_1 &= -2(m_{12}-m_{33})(m_{22}-m_{33})C_2(M),\\
c_2 &= C_2(M)\left\{C_1(M)-1\right\}\left\{\bar{C}_1(M)-1\right\}.
\end{aligned}$$

2. $P_{(0,0,2)}\left(f(M)\otimes\left(\begin{smallmatrix}2\,0\,0\\1\,0\\0\end{smallmatrix}\right)\right)=\sum_{i=0}^{3}c_if''\left(M\left(\begin{smallmatrix}0\,0\,2\\1\,0\\0\end{smallmatrix}\right)[-i]\right)$ with

$$\begin{aligned}
c_0 &= -(m_{13}-m_{12})(m_{12}-m_{33}+1)(m_{22}-m_{33})(m_{22}-m_{33}-1),\\
c_1 &= (m_{22}-m_{33})\left\{(m_{12}-m_{33})F(M)+(m_{13}-m_{12})C_2\left(M\left(\begin{smallmatrix}0\,0\,1\\1\,0\\0\end{smallmatrix}\right)\right)\right\},\\
c_2 &= -F(M)C_2\left(M\left(\begin{smallmatrix}0\,0\,1\\0\,1\\0\end{smallmatrix}\right)\right)-\chi_-(M)C_2(M)(m_{12}-m_{33}-1)(m_{22}-m_{33}+1),\\
c_3 &= \chi_-(M)C_2(M)\left\{C_1(M)-1\right\}\left\{\bar{C}_1(M)-1\right\}.
\end{aligned}$$

3. $P_{(0,0,2)}\left(f(M)\otimes\left(\begin{smallmatrix}2\,0\,0\\1\,0\\1\end{smallmatrix}\right)\right)=\sum_{i=0}^{2}c_if'\left(M\left(\begin{smallmatrix}0\,0\,2\\1\,0\\1\end{smallmatrix}\right)[-i]\right)$ with

$$\begin{aligned}
c_0 &= -(m_{12}-m_{33}+1)(m_{13}-m_{12})(m_{22}-m_{33})(m_{22}-m_{33}-1),\\
c_1 &= (m_{22}-m_{33})\left[(m_{12}-m_{33})E(M)+(m_{13}-m_{12})\left\{C_1(M)+1\right\}\bar{C}_1(M)\right],\\
c_2 &= -E(M)C_1(M)\left\{\bar{C}_1(M)-1\right\}.
\end{aligned}$$

4. $P_{(0,0,2)}\left(f(M)\otimes\left(\begin{smallmatrix}2\,0\,0\\2\,0\\0\end{smallmatrix}\right)\right)=\sum_{i=0}^{4}c_i f''\left(M\left(\begin{smallmatrix}0\,0\,2\\2\,0\\0\end{smallmatrix}\right)[-i]\right)$ with

$$
\begin{aligned}
c_0 &= (m_{13}-m_{12})(m_{13}-m_{12}-1)(m_{22}-m_{33})(m_{22}-m_{33}-1),\\
c_1 &= -(m_{13}-m_{12})(m_{22}-m_{33})\left\{F(M)+F\left(M\left(\begin{smallmatrix}0\,0\,1\\1\,0\\0\end{smallmatrix}\right)\right)\right\},\\
c_2 &= \chi_-(M)C_2(M)(m_{13}-m_{12}+1)(m_{22}-m_{33}+1)\\
&\quad +\chi_-^{(1)}(M)\{C_1(M)+1\}\{\bar{C}_1(M)+1\}(m_{13}-m_{12})(m_{22}-m_{33})\\
&\quad +F(M)F\left(M\left(\begin{smallmatrix}0\,0\,1\\0\,1\\0\end{smallmatrix}\right)\right),\\
c_3 &= -C_2(M)\left\{\chi_-^{(1)}(M)F(M)+\chi_-(M)F\left(M\left(\begin{smallmatrix}0\,0\,1\\-1\,2\\0\end{smallmatrix}\right)\right)\right\},\\
c_4 &= \chi_-^{(1)}(M)C_2(M)\{C_1(M)-1\}\{\bar{C}_1(M)-1\}.
\end{aligned}
$$

5. $P_{(0,0,2)}\left(f(M)\otimes\left(\begin{smallmatrix}2\,0\,0\\2\,0\\1\end{smallmatrix}\right)\right)=\sum_{i=0}^{3}c_i f''\left(M\left(\begin{smallmatrix}0\,0\,2\\2\,0\\1\end{smallmatrix}\right)[-i]\right)$ with

$$
\begin{aligned}
c_0 &= (m_{13}-m_{12})(m_{13}-m_{12}-1)(m_{22}-m_{33})(m_{22}-m_{33}-1),\\
c_1 &= -(m_{13}-m_{12})(m_{22}-m_{33})\left\{F\left(M\left(\begin{smallmatrix}0\,0\,1\\1\,0\\1\end{smallmatrix}\right)\right)+E(M)\right\},\\
c_2 &= E(M)F\left(M\left(\begin{smallmatrix}0\,0\,1\\0\,1\\1\end{smallmatrix}\right)\right)+\chi_-(M)(m_{13}-m_{12})(m_{22}-m_{33})\{C_1(M)+1\}\bar{C}_1(M),\\
c_3 &= -\chi_-(M)E(M)C_1(M)\{\bar{C}_1(M)-1\}.
\end{aligned}
$$

6. $P_{(0,0,2)}\left(f(M)\otimes\left(\begin{smallmatrix}2\,0\,0\\2\,0\\2\end{smallmatrix}\right)\right)=\sum_{i=0}^{2}c_i f'\left(M\left(\begin{smallmatrix}0\,0\,2\\2\,0\\2\end{smallmatrix}\right)[-i]\right)$ with

$$
\begin{aligned}
c_0 &= (m_{13}-m_{12})(m_{13}-m_{12}-1)(m_{22}-m_{33})(m_{22}-m_{33}-1),\\
c_1 &= -2(m_{13}-m_{12})(m_{22}-m_{33})\{E(M)-\bar{C}_1(M)\},\\
c_2 &= E(M)\{\bar{C}_1(M)-1\}\{m_{13}-m_{33}-C_1(M)\}.
\end{aligned}
$$