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# On the coefficients of Okamoto polynomials *

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**Abstract**. 岡本多項式は Painlevé IV 方程式の有理解を与える整数係数の多項式の族である。この多項式の係数の性質をいくつか論じたい。そもそも Yablonskii-Vorob'ev 多項式 (= Painlevé II 方程式の場合) に対しての類似の性質は金子昌信との共同研究で得られたものであるが、それも合わせて説明する。

## 1 Okamoto polynomials

### 1.1 Introduction

The Okamoto polynomial $Q_n = Q_n(x) \in \mathbf{Z}[x]$ $(n = 0, 1, 2, \dots)$, is defined by the recursion

$$Q_{n+1}Q_{n-1} = (x^2 + 2n - 1)Q_n^2 + Q_n''Q_n - {Q_n'}^2 \tag{1}$$

with the initial condition $Q_0 = Q_1 = 1$. The first few is

$Q_2 = x^2 + 1,$
$Q_3 = x^6 + 5x^4 + 5x^2 + 5,$
$Q_4 = x^{12} + 14x^{10} + 65x^8 + 140x^6 + 175x^4 + 350x^2 + 175,$
$Q_6 = x^{30} + 55x^{28} + 1295x^{26} + 17325x^{24} + 147525x^{22} + 854315x^{20} + 3570875x^{18}$
$\quad +11836825x^{16} + 36295875x^{14} + 118243125x^{12} + 371896525x^{10} + 838212375x^8$
$\quad +1146520375x^6 + 1011635625x^4 + 1011635625x^2 + 337211875,$
$Q_7 = x^{42} + 91x^{40} + \cdots + 86075017153125x^2 + 28691672384375.$

The non-trivial fact that the right-hand side of the recursion can be divisible by $Q_{n-1}$ in $\mathbf{Q}[x]$ is proved by Okamoto [9]. It is easy to see that these are

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monic polynomials with integral coefficients. Okamoto polynomials arises from the $\tau$-functions for rational solutions of Painlevé IV,

$$y'' = \frac{1}{2y}(y')^2 + \frac{3}{2}y^3 + 4xy^2 + 2(x^2 - a)y + \frac{b}{y}$$

with $a = -n$ and $b = -2(n + \frac{1}{3})^2$. For further information, e.g., the explicit form of the rational solutions in terms of Okamoto polynomials, see [9].

## 1.2 The constant term

The constant term of the Okamoto polynomials are given by the following formula.

**Theorem 1**

$$Q_{2n}(0) = \prod_{k=1}^{n-1}(6k+1)^{2(n-k)-1} \times \prod_{k=1}^{n-1}(6k-1)^{2(n-k)},$$
$$Q_{2n+1}(0) = \prod_{k=1}^{n}(6k-1)^{2(n-k)+1} \times \prod_{k=1}^{n-1}(6k+1)^{2(n-k)}.$$

**Remark 2** *This formula is also derived from the hook-type formula [10].*

## 1.3 Higher coefficients

It is easy to see that Okamoto polynomials are even; we denote

$$Q_n(x) = \sum_{j=0}^{\infty} q_j(n)x^{2j}.$$

**Theorem 3** *For each $j$, the function*

$$\mathbf{N} \ni m \mapsto q_j(2m)/q_0(2m) \in \mathbf{Q}$$

*extends to a polynomial function in $m$. Also for*

$$\mathbf{N} \ni m \mapsto q_j(2m+1)/q_0(2m+1) \in \mathbf{Q}.$$

**Example 4**

$q_1(2m)/q_0(2m) = m,$
$q_2(2m)/q_0(2m) = m(m-1)/2,$
$q_3(2m)/q_0(2m) = m(m-1)(5m+2)/30,$
$q_4(2m)/q_0(2m) = m(m-1)(m+1)(35m-18)/840,$
$q_5(2m)/q_0(2m) = m(m-1)(m+1)(35m^2-38m-8)/4200,$
$q_6(2m)/q_0(2m) = m(m-1)(m+1)(385m^3-687m^2-94m+120)/277200,$
$q_7(2m)/q_0(2m) = m(m-1)(m+1)(m-2)(1001m^3-167m^2-942m-72)/5045040,$

$q_1(2m+1)/q_0(2m+1) = m,$
$q_2(2m+1)/q_0(2m+1) = m(m+1)/2,$
$q_3(2m+1)/q_0(2m+1) = m(m+1)(5m-2)/30.$

**Remark 5** *The polynomial giving* $q_j(2m+1)/q_0(2m+1)$ *is obtained from that of* $(-1)^j q_j(2m)/q_0(2m)$ *by the substitution* $m \mapsto -m$.

## 1.4 Interpolation

We denote the polynomial $a_j(t), b_j(t) \in \mathbf{Q}[t]$ for $j = 0, 1, \ldots$ such that

$$a_j(m) = q_j(2m)/q_0(2m), \qquad b_j(m) = q_j(2m+1)/q_0(2m+1).$$

Note that $b_j(t) = a_j(-t)$ for an even $j$, and $b_j(t) = -a_j(-t)$ for an odd $j$. We define

$$A(x,t) = \sum_{j=0}^{\infty} a_j(t)x^{2j}, \qquad B(x,t) = \sum_{j=0}^{\infty} b_j(t)x^{2j}.$$

Note that $B(\sqrt{-1}x, -t) = A(x,t)$. Okamoto polynomials are obtained by the substitution $t = m$ up to a constant multiple:

$$Q_{2m}(x) = Q_{2m}(0)A(x,m), \qquad Q_{2m+1}(x) = Q_{2m+1}(0)B(x,m).$$

We divide the recursion (1) by the constant

$$Q_{2m+1}(0)Q_{2m-1}(0) = (6m-1)Q_{2m}(0)^2, \text{ or}$$
$$Q_{2m+2}(0)Q_{2m}(0) = (6m+1)Q_{2m+1}(0)^2, \text{ respectively.}$$

Then we obtain

**Theorem 6**

$$(6t-1)B(x,t)B(x,t-1) = (x^2+4t-1)A(x,t)^2 + A''(x,t)A(x,t) - A'(x,t)^2,$$
$$(6t+1)A(x,t+1)A(x,t) = (x^2+4t+1)B(x,t)^2 + B''(x,t)B(x,t) - B'(x,t)^2,$$

*where* $' = \frac{\partial}{\partial x}$, *as before.*

The element $A$ (also $B$) here is considered as an element in the ring $\mathbf{Q}[t][[x]]$ of formal power series with coefficients in $\mathbf{Q}[t]$. It, moreover, is contained in a smaller subalgebra.

Let $R$ be a commutative ring and $R = I_0 \supset I_1 \supset I_2 \supset \cdots$ be a decreasing sequence of ideals of $R$ such that the intersection $\cap I_k = \{0\}$. We define the valuation $\nu(a)$ of an element $0 \neq a \in R$ by the largest $k$ such that $a \in I_k$. The norm of $a$ is defined to be $|a| = e^{-\nu(a)}$. Then $R$ is a normed ring. We denote by $\bar{R}$ its completion. The ring

$$\bar{R}\langle\langle u\rangle\rangle = \{\sum a_n u^n \in \bar{R}[[u]] \mid \lim_{n\to\infty} \nu(a_n) = \infty\}$$

is of our interest. The image of the natural ring homomorphism

$$\bar{R}\langle\langle u\rangle\rangle \to \bar{R}[[u]] \to (R/I_k)[[u]]$$

is contained in $(R/I_k)[u] \subset (R/I_k)[[u]]$.

**Example:**

(i) Take $R = \mathbf{Z}$ and $I_k = (p^k)$, then the completion $\bar{R}$ is $\mathbf{Z}_p$, the ring of $p$-adic integers.

$$\mathbf{Z}_p\langle\langle u\rangle\rangle = \{\sum_{n=0}^{\infty} a_n u^n \in \mathbf{Z}_p[[u]] \mid \lim_{n\to\infty} \nu(a_n) = \infty\}$$

is called the Tate algebra. This ring naturally arises in the function theory of $p$-adic analytic functions. For example, the Weierstrass preparation theorem holds in this ring as in the ring of germs of (usual) holomorphic functions [1].

(ii) Let $F$ be a field, and take $R = F[t]$ and $I_k = (t^k)$, then the completion $\bar{R}$ is the ring of formal power series $F[[t]]$.

(iii) In our case, we take $R = \mathbf{Q}[t]$ and the ideals are $I_k = (\prod_{j=-k}^{k}(t-j))$, which is a shifted analogue of the case (ii). We see that our functions $A$ and $B$ arising above are contained in $\bar{R}\langle\langle x\rangle\rangle$. The equations in Theorem 6 are considered as an interpolation (or an analytic continuation) of the Toda equation (1).

**Remark 7** *We also have a different type of description of coefficients. We denote by $d_n$ the degree of $Q_n$; that is, $d_n = n(n-1)$. Then, for example,*

$$q_{d_n-2}(n) = (n-1)n(2n-1)/6,$$
$$q_{d_n-4}(n) = (n-2)(n-1)n(n+1)(4n^2-4n-9)/72,$$
$$q_{d_n-6}(n) = (n-2)(n-1)n(n+1)(2n-1)(4n^4-8n^3-41n^2+45n+180)/1296.$$

## 2 Another family of Okamoto polynomials

There is another family of Okamoto polynomials, denoted by $R_n = R_n(x) \in \mathbf{Z}[x]$, with the recursion

$$R_{n+1}R_{n-1} = (x^2+2n)R_n^2 + R_n''R_n - R_n'^{\,2} \tag{2}$$

with the initial condition $R_0 = 1$, $R_1 = x$. The first few is

$$\begin{aligned}
R_2 &= x^4 + 2x^2 - 1,\\
R_3 &= x^9 + 8x^7 + 14x^5 - 35x,\\
R_4 &= x^{16} + 20x^{14} + 140x^{12} + 420x^{10} + 350x^8 - 980x^6 - 4900x^4 - 4900x^2 + 1225,\\
R_5 &= x^{25} + 40x^{23} + 650x^{21} + 5600x^{19} + 27475x^{17} + 72240x^{15} + 34300x^{13}\\
&\quad -509600x^{11} - 2627625x^9 - 7007000x^7 - 7357350x^5 + 6131125,\\
R_6 &= x^{36} + 70x^{34} + \cdots + 184117683750x^2 - 30686280625.
\end{aligned}$$

It is easy to see that $R_{2m}$ is even and that $R_{2m+1}$ is odd for a non-negative integer $m$. The constant terms (resp. the lowest order terms) of these Okamoto polynomials are given by

**Theorem 8**

$$\begin{aligned}
R_{2n}(0) &= (-1)^n \prod_{k=1}^{n-1}\{(6k+1)(6k-1)\}^{2(n-k)},\\
R_{2n+1}(0) &= 0,\\
R'_{2n+1}(0) &= (-1)^n \prod_{k=1}^{n}\{(6k+1)(6k-1)\}^{2(n-k)+1}.
\end{aligned}$$

We denote

$$R_{2m}(x) = \sum_{j=0}^{\infty} r_j(2m)x^{2j}, \qquad R_{2m+1}(x) = \sum_{j=0}^{\infty} r_j(2m+1)x^{2j+1}.$$

Then the same statement as Theorem 3 holds. First several examples are

$$\begin{aligned}
r_1(2m)/r_0(2m) &= -2m,\\
r_2(2m)/r_0(2m) &= -m^2,\\
r_3(2m)/r_0(2m) &= -2(m-1)m(m+1)/15,\\
r_4(2m)/r_0(2m) &= (m-1)m^2(m+1)/42,\\
r_5(2m)/r_0(2m) &= (m-1)m(m+1)(7m^2+2)/525,\\
r_6(2m)/r_0(2m) &= (m-1)m^2(m+1)(101m^2-74)/34650,\\
r_7(2m)/r_0(2m) &= (m-1)m(m+1)(649m^2-1501m^2-90)/1576575,
\end{aligned}$$

$r_1(2m+1) = 0,$
$r_2(2m+1)/r_0(2m+1) = -m(m+1)/5,$
$r_3(2m+1)/r_0(2m+1) = -4m(m+1)(2m+1)/105,$
$r_4(2m+1)/r_0(2m+1) = -m(m+1)(m^2+m-1)/70,$
$r_5(2m+1)/r_0(2m+1) = -4(m-1)m(m+1)(m+2)(2m+1)/5775,$
$r_6(2m+1)/r_0(2m+1) = (m-1)m(m+1)(m+2)(m^2+m+15)/90090,$
$r_7(2m+1)/r_0(2m+1) = 2(m-1)m(m+1)(m+2)(2m+1)(19m^2+19m+15)/2627625.$

**Question 9** *The fact that $r_0(2m)$ has a product-type formula as in Theorem 8 is explained by the hook-type formula, Yamada [10], see also Mizukawa-Yamada [6]. There is no hook-type formula applicable for $r_j(n)$ with $j \geq 1$, as well as $r_0(2m+1)$, but still the above expression suggests us to have some product-type formula.*

# 3 Proof

## 3.1 Schur functions

Okamoto polynomials are expressible by a special kind of the Schur functions. We recall standard notations briefly. Let $\lambda$ be the partition of $N = |\lambda|$. We denote by $\chi_\lambda$ the irreducible character of the symmetric group $S_N$ of $N$ letters, and the value at the cycle type $1^{m_1}2^{m_2}\cdots$ by $\chi_\lambda(1^{m_1}2^{m_2}\cdots)$. A definition of the Schur function is

$$s_\lambda = \sum_{m_1,m_2,\cdots\geq 0} \chi_\lambda(1^{m_1}2^{m_2}\cdots)\frac{t_1^{m_1}}{m_1!}\frac{t_2^{m_2}}{m_2!}\cdots.$$

The coefficient of $t_1^N$ in $s_\lambda$ is given by the hook formula;

$$\chi_\lambda(1^N)/N! = (\prod_{s\in\lambda} h(s))^{-1},$$

where $h(s)$ is the hook length of $\lambda$ at the box $s$ in the Young diagram $\lambda$.

In terms of symmetric functions of variables $\{x_1, x_2, \dots\}$, the variable $t_k = \sum_{i\geq 1} x_i^k/k$ is essentially a power sum of degree $k$. The generating function of the complete symmetric functions $h_n$ in $\{x_1, x_2, \dots\}$ is

$$\sum_{n=0}^{\infty} h_n u^n = \prod_{i\geq 1}(1-ux_i)^{-1} = \exp\left(\sum_{k=1}^{\infty} t_k u^k\right).$$

The Schur function in variables $(x_1, \ldots, x_N)$ is given by the Weyl character formula

$$s_\lambda = \det(x_j^{\lambda_i+N-i}) / \det(x_j^{N-i}).$$

The Jacobi-Trudi formula is

$$s_\lambda = \det(h_{\lambda_i-i+j}),$$

which is used in the proof of our theorem.

The Okamoto polynomials are expressible by the Schur function [4] [7] with the special type of partition $(2n-2, \ldots, 6, 4, 2)$;

$$Q_n(x) = c_n s_{(2n-2,\ldots,6,4,2)},$$

after the substitution $t_1 = x$, $t_2 = \frac{1}{2}$, $t_3 = t_4 = \cdots = 0$ in the right-hand side of the expression above. Since $Q_n(x)$ is monic, the constant $c_n$ above is explicitly described by the relation $c_n \chi_{(2n-2,\ldots,6,4,2)}(1^N) = N!$, where the number of boxes is $N = (n-1)n$. Also, $R_n$ corresponds to the partition $(2n-1, \ldots, 5, 3, 1)$. A partition $\lambda$ is called <u>$k$-reduced</u> if no hook with length of a multiple of $k$. The partition $(2n-2, \ldots, 6, 4, 2)$ and $(2n-1, \ldots, 5, 3, 1)$ are examples of 3-reduced partitions.

# 4 Yablonskii-Vorob'ev

Yablonskii-Vorob'ev polynomial corresponds to Painlevé II, and has been discussed in the joint work with Kaneko [5]. Here we only summarize the data.

| | Okamoto | Yablonskii-Vorob'ev | |
|---|---|---|---|
| Painlevé | IV | II | ... |
| symmetry | $A_2^{(1)}$ | $A_1^{(1)}$ | |
| Schur | 3-reduced | 2-reduced | |
| parition | $(2n-2, \ldots, 4, 2)$<br>$(2n-1, \ldots, 5, 3, 1)$ | $(n, \ldots, 3, 2, 1)$ | |
| specialization | $t_1 = x, t_2 = 1/2,$<br>$t_3 = t_4 = \cdots = 0$ | $t_1 = x, t_3 = 1/3,$<br>$t_2 = t_4 = \cdots = 0$ | |
| cycle type | $1^*2^j$ | $1^*3^j$ | |
| period | 2 | 3 | |

The recursion

$$T_{n+1}T_{n-1} = xT_n^2 + T_n''T_n - (T_n')^2$$

with $T_0 = 1$, $T_1 = x$ $(T_{-1} = 1)$. Painlevé II

$$P_{II}: \qquad y'' = 2y^3 - 4xy + 4n.$$

The expression of the rational solutions of $P_{II}$ in terms of Yablonskii-Vorob'ev;

$$y = (\log(T_n/T_{n-1}))' = T'_n/T_n - T'_{n-1}/T_{n-1}.$$

First few examples of Yablonskii-Vorob'ev polynomials are

$T_2 = x^3 - 1,$
$T_3 = x^6 - 5x^3 - 5,$
$T_4 = x^{10} - 15x^7 - 175x,$
$T_5 = x^{15} - 35x^{12} + 175x^9 - 1225x^6 - 12250x^3 + 6125,$
$T_6 = x^{21} - 70x^{18} + 1155x^{15} - 9800x^{12} - 67375x^9 - 1414875x^6 + 4716250x^3 + 2358125,$
$T_7 = x^{28} - 216x^{25} + 4725x^{22} - 80850x^{19} + 242550x^{16} - 12733875x^{13} - 202327125x^{10}$
$\quad +3034906875x^7 + 11802415625,$
$T_8 = x^{36} - 210x^{33} + \cdots + 177213270609375x^3 - 59071090203125.$

Let us denote

$$T_n(x) = \sum_{j=0}^{\infty} t_j(n) x^{3j+\delta},$$

where $\delta = \delta_n$ is the remainder of the degree of $T_n$ by 3, i.e., $\delta = 0$ for $n = 3m - 1$ or $n = 3m$ and $\delta = 1$ for $n = 3m + 1$. We have

$$t_0(n) = (-1)^{[(n+1)/3]} 3^{-((n+1)n-2\delta)/6} \mu_n / (\mu_{[(n-1)/3]} \mu_{[n/3]} \mu_{[(n+1)/3]}).$$

Note that $t_0(3m+1)$ is not a constant term but the coefficient of $x^1$. Finally, the same statement as Theorem 3 holds. Examples are

$t_1(3m)/t_0(3m) = m,$
$t_2(3m)/t_0(3m) = -m(m+1)/10,$
$t_3(3m)/t_0(3m) = -(m-1)m(m+1)/210,$
$t_4(3m)/t_0(3m) = -(m-1)m(m+1)(19m-6)/46200,$
$t_5(3m)/t_0(3m) = -(m-1)m(m+1)(155m^2 + 572m - 48)/21021000,$

$t_1(3m+1) = 0,$
$t_2(3m+1)/t_0(3m+1) = 3m(m+1)/70,$
$t_3(3m+1)/t_0(3m+1) = -m(m+1)/350,$
$t_4(3m+1)/t_0(3m+1) = -9(m-1)m(m+1)(m+2)/200200,$
$t_5(3m+1)/t_0(3m+1) = 3(m-1)m(m+1)(m+2)/3503500,$
$t_6(3m+1)/t_0(3m+1) = -(m-1)m(m+1)(m+2)(207m^2 + 207m + 50)/4526522000,$
$t_7(3m+1)/t_0(3m+1) = 9(m-1)m(m+1)(m+2)(107m^2 + 1 - 7m + 4)/348542194000.$

The polynomial giving $t_j(3m-1)/t_0(3m-1)$ is obtained from that of $t_j(3m)/t_0(3m)$ by the substitution $m \mapsto -m$.

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