表現論シンポジウム講演集, 2001
pp.112-118

AUTOMORPHIC FORMS AND HARMONIC ANALYSIS ON GROUP MANIFOLDS

SHIGERU SANO* MASATOSHI IIDA**

Introduction

In this note, we first recall harmonic analysis on reductive groups which is studied deeply by Harish-Chandra. From the philosophy of automorphic forms, his theory becomes clear and is completed grandly. We look back the history of the influence of automorphic forms and study some problems after the works.

Let G be a reductive Lie group of H-C class with Lie algebra $\mathfrak{g}$. Let θ be a Cartan involution of $\mathfrak{g}$ and let $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ be the corresponding Cartan decomposition. Let K denote the maximal compact subgroup of G with Lie algebra $\mathfrak{k}$. Let $\mathfrak{h}_1, \mathfrak{h}_2, ..., \mathfrak{h}_n$ be a maximal set of θ-invariant Cartan subalgebras of $\mathfrak{g}$ which are not conjugate each other (cf. [S]). The Cartan subgroup corresponding to $\mathfrak{h}_j$ is denoted by H_j $(j = 1, 2, ..., n)$.

The Cartan subgroups are considered as cusps for a fundamental domain under some cofinite discrete subgroup Γ in the theory of automorphic forms. In fact, the parabolic subgroups P_j $(j = 1, 2, ..., n)$ corresponding to H_j are called cuspidal parabolic subgroups. Let $P_j = M_jA_jN_j$ be the Langlands decomposition of P_j.

Let $\{\tau, V\}$ be a double representation of K. For a parabolic subgroup $P = MAN$, we take $\phi \in C^\infty(M, V)$ satisfying that $\phi(m_1xm_2) = \tau(m_1)\phi(x)\tau(m_2)$ $(x \in M, m_1, m_2 \in M \cap K)$ and extend it to a function on G by

$$\phi(kman) = \tau(k)\phi(m) \qquad (k \in K, m \in M, a \in A, n \in N).$$

Let $\mathfrak{a}$ be the Lie algebra of A and $\mathfrak{a}^*$ its dual. For $\nu \in \mathfrak{a}^*$, an Eisenstein integral is defined by

$$E(P:\phi:\nu:x) = \int_K \phi(xk)\tau(k^{-1})\exp\{(\sqrt{-1}\nu - \rho_P)H_P(xk)\}dk.$$

For $P = P_j$ and matrix coefficients ϕ of discrete series representations of M_j, principal series representations of G are characterized by $E(P_j : \phi : \nu : x)$ $(j = 1, 2, ..., n)$. Harish-Chandra discussed harmonic analysis on G using the integrals as Eisenstein series of the theory of automorphic forms. He also characterized discrete series representations by the method of cusp forms and constant terms.

If G is not compact, $\Gamma = \{e\}$ is a coinfinite subgroup of G. Then we find that the cofinite condition of discrete subgroups can be removed in the theory of automorphic forms. Many achievements of harmonic analysis on $G = \Gamma \setminus G$ express that abundant results will be expected from the research direction.

$SL(2, \mathbb{Z})$ is a coinfinite subgroup of $SL(2, \mathbb{C})$. So we study structures of the space $SL(2, \mathbb{Z}) \setminus SL(2, \mathbb{C})/SU(2)$ and Eisenstein series on the space.

*Polytechnic University, Sagamihara,Kanagawa, Japan, e-mail address: ssano@uitec.ac.jp
**Polytechnic University, Sagamihara, Kanagawa, Japan, e-mail address: iida@cs.uitec.ac.jp

§1. Structure of $SL(2,\mathbb{Z}) \setminus SL(2,\mathbb{C})/SU(2)$

Let G be $SL(2,\mathbb{C})$ and K its maximal compact subgroup $SU(2)$, Γ its discrete subgroup $SL(2,\mathbb{Z})$. Let $\mathbf{X}$ denote the Riemannian symmetric space G/K. Put $G_{\mathbb{R}} = SL(2,\mathbb{R})$. From the Oshima compactification, there exists a compact manifold $\widetilde{\mathbf{X}}$, which is divided into three G-orbits. Two of them are diffeomorphic to $\mathbf{X}$ and the rest to $G/P \simeq \mathbb{P}^1\mathbb{C}$ (one dimensional complex projective space), where P is the minimal parabolic subgroup of G. The boundary $\partial\mathbf{X}$ of $\mathbf{X}$ in $\widetilde{\mathbf{X}}$ is the third G-orbit G/P.

From Iwasawa decomposition, $G/K \simeq \left\{\begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix} : z \in \mathbb{C}\right\}\left\{\begin{pmatrix} r^{1/2} & 0 \\ 0 & r^{-1/2} \end{pmatrix} : r > 0\right\}$ holds and we choose (z,r) as the coordinate of $\mathbf{X}$. $\mathbf{X}$ is considered as the subset $\{w = z + jr : z = x + iy \in \mathbb{C}, r > 0\}$ of the quaternion field $\mathbb{H}$. The space $\mathbf{X}$ is also given as the upper half-space in Euclidean three-space

$$\mathbf{X} = \{(z,r) : z \in \mathbb{C}, r > 0\} = \{(x,y,r) : x,y \in \mathbb{R}, r > 0\}.$$

G acts on $\mathbf{X}$ as

$$g\cdot(z,r) = \left(\frac{(az+b)(\bar{c}\bar{z}+\bar{d}) + a\bar{c}r^2}{|cz+d|^2+|c|^2r^2}, \frac{r}{|cz+d|^2+|c|^2r^2}\right), \quad g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G. \tag{1.1}$$

Note that $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ is the parallel transformation along $x : (z,r) \to (z+1,r)$. Note also that the $G_{\mathbb{R}}$-action on the subspace $\{(x,0,r) \in \mathbf{X}\}$ of $\mathbf{X}$ is the same as the $SL(2,\mathbb{R})$-action on the upper half-plane $SL(2,\mathbb{R})/SO(2)$. It is easily proved that the isotropy group of G for $(0,1) \in \mathbf{X}$ is K. The boundary of $\mathbf{X}$ is $\partial\mathbf{X} \simeq \mathbb{P}^1\mathbb{C} = \{[z_1,z_2] : z_1,z_2 \in \mathbb{C}, (z_1,z_2) \neq (0,0)\}$.

Actions of G on $\partial\mathbf{X}$ are given as

$$g\cdot[z_1,z_2] = [az_1+bz_2, cz_1+dz_2], \quad g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G. \tag{1.2}$$

For $\kappa \in \partial\mathbf{X}$, put $\Gamma_\kappa = \{g \in \Gamma : g\cdot\kappa = \kappa\}$. $\kappa \in \partial\mathbf{X}$ is called a cusp of Γ if Γ_κ is generated by parabolic elements. For example, $\infty = [1,0]$ is a cusp as $\Gamma_\infty = \left\{\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} : a \in \mathbb{Z}\right\}$.

Proposition 1.1. *An element κ on $\partial\mathbf{X}$ is a cusp if and only if*

$$\kappa = [m,n], \quad m,n \in \mathbb{Z}, (m,n) \neq (0,0).$$

Now we find a fundamental domain of Γ in $\mathbf{X}$. The closed subset $\mathcal{D}$ is a fundamental domain if it satisfies following conditions:

(C1) $\mathcal{D}$ meets each Γ-orbit at least once,
(C2) if $w, w' \in \mathcal{D}$ are Γ-equivalent, then $w, w' \in \partial\mathcal{D}$ (the boundary of $\mathcal{D}$ in $\mathbf{X}$),
(C3) $\partial\mathcal{D}$ has Lebesgue measure 0.

The result is the following.

Proposition 1.2. $\mathcal{D} = \{(x,y,r) \in \mathbf{X} : -1/2 \le x \le 1/2, y \in \mathbb{R}, r > 0, x^2+y^2+r^2 \ge 1\}$ *is a fundamental domain of* Γ.

§2. Eisenstein series

From a Riemannian metric equipped on $\mathbf{X}$, a volume element on $\mathbf{X}$ is defined by

$$dw = \frac{dxdydr}{r^3},$$

and the Laplace-Beltrami operator on $\mathbf{X}$ is given by

$$\Delta = r^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial r^2} \right) - r\frac{\partial}{\partial r}.$$

Let $\mathcal{D}$ be the fundamental domain of Γ in $\mathbf{X}$ as in Proposition 1.2. The volume of $\mathcal{D}$ is infinite. Let $\partial\widetilde{\mathcal{D}}$ denote the boundary of $\mathcal{D}$ in $\widetilde{\mathbf{X}}$, then $\partial\mathcal{D} \subset \partial\widetilde{\mathcal{D}}$. There is only a cusp ∞ on $\partial\widetilde{\mathcal{D}}$. The Eisenstein series $E(w,s)$ for Γ at ∞ is then defined by

$$E(w,s) = \sum_{g \in \Gamma_\infty \backslash \Gamma} r(g \cdot w)^{1+s}. \tag{2.1}$$

It converges absolutely for Re$s > 0$ (cf. [EGM] Ch.3 Proposition 2.1) and satisfies $\Delta E(w,s) = (s^2-1)E(w,s)$ as Δ is G-invariant. $E(w,s)$ is a periodic function of x with period 1 as $E(w,s)$ is Γ-invariant. We consider its Fourier expansion:

$$\begin{cases} a_m(y, r : s) = \displaystyle\int_0^1 E((x+yi, r), s) e^{-2m\pi i x} dx, \\ E(w,s) = \displaystyle\sum_{m \in \mathbb{Z}} a_m(y, r : s) e^{2m\pi i x}. \end{cases}$$

Put $\Gamma_0 = \Gamma_\infty$. If

$$\Gamma_0 g \Gamma_0 = \Gamma_0 g' \Gamma_0, \; g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \; g' = \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix},$$

then $c = c', \; d \equiv d' (\text{mod } c)$. We have an orbit decomposition of Γ

$$\Gamma_0 \backslash \Gamma / \Gamma_0 = \Gamma_0 \bigcup (\bigcup_{c,d} \Gamma_0 g \Gamma_0) \; \left(g = \begin{pmatrix} * & * \\ c & d \end{pmatrix} \in \Gamma, c, d \, (\text{mod } c) \right).$$

It follows that

$$\begin{aligned} a_m(y, r : s) &= \int_0^1 \sum_{g \in \Gamma_0 \backslash \Gamma} r(g \cdot w)^{1+s} e^{-2m\pi i x} dx \\ &= \delta_{0m} r^{1+s} + \sum_{c,d} \int_{-\infty}^{\infty} \left(\frac{r}{|cz+d|^2 + |c|^2 r^2} \right)^{1+s} e^{-2m\pi i x} dx \\ &= \delta_{0m} r^{1+s} + \sum_{c,d} \frac{e^{2m\pi i d/c}}{|c|^{2(1+s)}} \int_{-\infty}^{\infty} \left(\frac{r}{|z|^2 + r^2} \right)^{1+s} e^{-2m\pi i x} dx. \end{aligned} \tag{2.2}$$

We set

$$\phi_m(s) = \sum_{c,d} \frac{e^{2m\pi i d/c}}{|c|^{2(1+s)}} \qquad \left(g = \begin{pmatrix} * & * \\ c & d \end{pmatrix} \in \Gamma, c, d \,(\mathrm{mod}\; c)\right).$$

Using the modified Bessel function K_s, we get, in the case $m \neq 0$,

$$\begin{aligned}
&\int_{-\infty}^{\infty} \left(\frac{1}{|z|^2 + r^2}\right)^{1+s} e^{-2m\pi i x} dx \\
&= \int_{-\infty}^{\infty} \left(\frac{1}{x^2 + y^2 + r^2}\right)^{1+s} e^{-2m\pi i x} dx \\
&= \frac{1}{(y^2 + r^2)^{1/2+s}} \int_{-\infty}^{\infty} \frac{e^{-2m\pi i x\sqrt{y^2+r^2}}}{(1+x^2)^{1+s}} dx \\
&= \frac{2\pi^{1+s}|m|^{1/2+s}}{(\sqrt{y^2+r^2})^{1/2+s}} K_{s+1/2}(2|m|\pi\sqrt{y^2+r^2})\, \Gamma(1+s)^{-1},
\end{aligned}$$

in the case $m = 0$,

$$\begin{aligned}
&\int_{-\infty}^{\infty} \left(\frac{1}{|z|^2 + r^2}\right)^{1+s} dx \\
&= \frac{1}{(y^2 + r^2)^{1/2+s}} \int_{-\infty}^{\infty} \frac{1}{(1+x^2)^{1+s}} dx \\
&= \frac{\pi^{1/2}}{(y^2 + r^2)^{1/2+s}} \frac{\Gamma(1/2+s)}{\Gamma(1+s)}.
\end{aligned}$$

Then we have the following assertion.

Proposition 2.1. *Fourier coefficients of the Eisenstein series are given by*

$$a_m(y, r : s) = \frac{2\pi^{1+s}|m|^{1/2+s} r^{1+s}}{(\sqrt{y^2+r^2})^{1/2+s}} K_{s+1/2}(2|m|\pi\sqrt{y^2+r^2})\Gamma(1+s)^{-1}\phi_m(s),$$

$$(m \neq 0)$$

and

$$a_0(y, r : s) = r^{1+s} + \frac{\pi^{1/2} r^{1+s}}{(y^2+r^2)^{1/2+s}} \frac{\Gamma(1/2+s)}{\Gamma(1+s)} \phi_0(s).$$

We take an another coordinate (x, u, θ) of $\mathbf{X}$ as

$$x = x,\ y = u \sin\theta,\ r = u\cos\theta \ \left(-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2},\ u > 0\right).$$

For any $(\xi, \theta) = (x + iu, \theta)$, actions of $G_{\mathbb{R}}$ are given by

$$g \cdot (\xi, \theta) = \left(\frac{a\xi + b}{c\xi + d}, \theta\right) \text{ for } \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G_{\mathbb{R}}.$$

The Laplace-Beltrami operator and the volume element become

$$\Delta = u^2 \cos^2\theta \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial u^2}\right) + \cos^2\theta \frac{\partial^2}{\partial \theta^2} + \cos\theta \sin\theta \frac{\partial}{\partial \theta}$$

and

$$dw = \frac{1}{u^2 \cos^3 \theta} dx du d\theta.$$

Let f be a Γ-invariant function on $\mathbf{X}$ with the Fourier expansion

$$f(w) = \sum_{m \in \mathbf{Z}} a_m(u, \theta) e^{2m\pi i x}.$$

For $Y > 0$, we put

$$f^Y(w) = \begin{cases} f(w) - a_0(u(w), \theta(w)) & u(w) > Y \\ f(w) & otherwise. \end{cases}$$

The u-part of $E(w, s)$ is defined by

(2.3) $$E^0(w, s) = \sum_{g \in \Gamma_0 \backslash \Gamma} u(g \cdot w)^{1+s}.$$

As $\theta(w)$ is Γ-invariant, we have

$$E(w, s) = E^0((x, u), s) \cos^{1+s} \theta \qquad \text{for } w = (x, u, \theta).$$

The constant term of Fourier coefficients is given by

$$a_0(u, \theta : s) = (u^{1+s} + \phi(s) u^{-s}) \cos^{1+s} \theta$$

where $\phi(s) = \pi^{1/2} \frac{\Gamma(1/2+s)}{\Gamma(1+s)} \phi_0(s)$. We also define the u-part of Δ by

$$\Delta^0 = u^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial u^2} \right).$$

Then $\Delta^0 E^0(w, s) = s(s+1) E^0(w, s)$ as Δ^0 is $G_{\mathbf{R}}$-invariant.

We have the following analogous relation of Maass-Selberg.

Theorem 2.2. *Let $E(w, s)$ and $E(w, t)$ be Eisenstein series which satisfies respectively*

$$\Delta E(w, s) = (s^2 - 1) E(w, s) \text{ and } \Delta E(w, t) = (t^2 - 1) E(w, t).$$

Let $E^0(w, s)$ (resp. $E^0(w, t)$) be the u-part of $E(w, s)$ (resp. $E(w, t)$) with the constant term of its Fourier expansion

$$u^{1+s} + \phi(s) u^{-s} \qquad (resp.\ u^{1+t} + \phi(t) u^{-t}).$$

If $(s + t + 1)(s - t) \neq 0$, we have the inner product formula

(2.4) $$\begin{aligned} &(E^Y(w, s), \overline{E^Y(w, t)}) \\ &= c \left(\frac{Y^{s+t+1} - \phi(s)\phi(t) Y^{-s-t-1}}{s + t + 1} + \frac{\phi(t) Y^{s-t} - \phi(s) Y^{-s+t}}{s - t} \right) \end{aligned}$$

where $c = \int_{-\pi/2}^{\pi/2} \cos^{s+t-1} \theta \, d\theta$.

$E^Y(w, s)$ and $E^Y(w, t)$ are then square integrable on $\mathcal{D}$.

§3. Cusp forms

From the property of $\phi(s)$ in the constant term of Fourier coefficients of $E(w,s)$, we have the following assertion.

Lemma 3.1. *The Eisenstein series $E(w,s)$ has a holomorphic continuation to the domain*

$$\left\{s \in \mathbb{C} : \operatorname{Re} s \geq -\frac{1}{2}, s \neq 0\right\},$$

and $E(w,s)$ has a meromorphic continuation to the whole s-plane.

For the Eisenstein series, its constant term which is invariant in general under the action of N is important in the theory of automorphic forms. In the symmetric space $SL(2,\mathbb{C})/SU(2)$, there are two variables x, y in $\left\{\begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix} : z \in \mathbb{C}\right\}$. The fundamental domain $\mathcal{D}$ is bounded along the variable x and unbounded along the variable y. Furthermore the cusp ∞ is the limit of y in the domain $\mathcal{D}$. Then constant terms and cusp forms are considered under the variable x.

Let $\mathcal{D}_0$ be the fundamental domain of $SL(2,\mathbb{R})/SO(2)$ with the coordinate (x,u). With the coordinate, the Laplace-Beltrami operator is $\Delta^0 = u^2(\partial^2/\partial x^2 + \partial^2/\partial u^2)$.

Proposition 3.2. *Let $f \in C^2(\mathcal{D}) \cap L^2(\mathcal{D}, d\omega)$ satisfy $\Delta f = (s^2-1)f$ and assume the separation of variables as $f(w) = g(x,u)\, h(\theta)$.*

Then, functions g, h are of the form

$$g(x,u) = u^{1/2} \sum_{n \neq 0} \rho(n) K_{i\nu}(2\pi|n|u)e(nx)$$

and

$$h(\theta) = C_1 \cos^{1-s}\theta\, {}_2F_1(\alpha,\beta;1-s;\cos^2\theta) + C_2 \cos^{1+s}\theta\, {}_2F_1(\alpha+s,\beta+s;s+1;\cos^2\theta),$$

where α, β and ν are constants with conditions

$$\alpha+\beta = \frac{1}{2} - s, \quad \alpha\beta = -\frac{1}{4}(\nu^2 + \frac{1}{4} - s(s-1)), \quad \operatorname{Im}\nu > 0, \nu \neq \frac{1}{2}.$$

C_1 and C_2 are also constants which satisfy,

$$C_1 = 0 \text{ if } \operatorname{Re} s \geq \frac{3}{2} \text{ and } C_2 = 0 \text{ if } \operatorname{Re} s \leq -\frac{3}{2}.$$

Proof. Functions $g(x,u)$ and $h(\theta)$ satisfy following differential equations.

$$u^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial u^2}\right) g(x,u) = \left(\nu^2 + \frac{1}{4}\right) g(x,u),$$

$$\left[\cos^2\theta \frac{\partial^2}{\partial\theta^2} + \cos\theta\sin\theta\frac{\partial}{\partial\theta} + \lambda(\lambda-1)\cos^2\theta - (s^2-1)\right] h(\theta) = 0.$$

The former differential equation was solved in Lemma1.4 in [M]. So we treat only the latter here.

If we put $h(\theta) = \cos^{1-s}\theta\, l(\theta)$, $l(\theta)$ satisfies the differential equation:

$$\cos\theta\, l''(\theta) + (2s-1)\sin\theta\, l'(\theta) + (\nu^2 + \frac{1}{4} - s(s-1))\cos\theta\, l(\theta) = 0.$$

Then, putting $x = \cos^2\theta$ and $p(x) = l(\theta)$, we have the Gaussian hypergeometric equation:

$$x(1-x)p''(x) + \left\{(1-s) - (\frac{3}{2} - s)x\right\} p'(x) + \frac{1}{4}\left\{\nu^2 + \frac{1}{4} - s(s-1)\right\} p(x) = 0.$$

A base of the solution space is

$$\cos^{1-s}\theta\, {}_2F_1(\alpha,\beta;1-s;\cos^2\theta), \quad \cos^{1+s}\theta\, {}_2F_1(\alpha+s,\beta+s;s+1;\cos^2\theta).$$

The conditions on C_1, C_2 are led from square integrability of $h(\theta)$. □

REFERENCES

[EGM] Elstrodt, J., Grunewald, F. and Mennicke, J., *Groups acting on hyperbolic space*, Springer (1998).

[I] Iwanic, H., *Introduction to the spectral theory of automorphic forms*, Revista Matematica Iberoamerinana (1995).

[K] Kubota, T., *Elementary theory of Eisenstein series*, Kodansha (1973).

[M] Motohashi, Y., *Spectral Theory of the Riemann Zeta-Function*, Cambridge University Press (1997).

[O] Oshima, T., *A realization of Riemannian symmetric spaces*, J. Math. Soc. Japan **30** (1978), 117–132.

[SI] Sano, I., IIDA, M., *Eisenstein Series on the space* $SL(2,\mathbb{Z})\backslash SL(2,\mathbb{C})/SU(K)$, Bull. Polytechnic University **30-A** (2001), 187–193.

[S] Sugiura, M., *Conjugate classes of Cartan subalgebras in real semi-simple Lie algebras*, J. Math. Soc. Japan **11** (1959), 374–434.