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WAVELET ANALYSIS WITH LATTICE BASIS

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§1. Notations and Preliminaries. Let $A \in GL(n;\mathbb{R})$ and $A^* = (A^t)^{-1}$.
$A = [\vec{a_1}, \dots, \vec{a_n}]$, $\vec{a_j} = ((a_{jk}); k = 1, \dots, n)$, $j = 1, \dots, n$.
We call $\vec{a_j}$'s the basis of the lattice Γ_A.
In the following, we define the Fourier transform of $f \in L^2(\mathbb{R}^n)$ as

$$(1)\ \widehat{f(\xi)} := \int_{\mathbb{R}^n} \exp(-\sqrt{-1}\xi \cdot x) f(x) dx,$$

and the inverse Fourier transform as

$$(2)\ f(x) := \left(\frac{1}{2\pi}\right)^n \int_{\mathbb{R}^n} \exp(-\sqrt{-1}\xi \cdot x) \widehat{f(\xi)} d\xi.$$

Define

$$\Gamma = \Gamma_A = \{Ak; k \in \mathbb{Z}_n\},$$

$$\Gamma^* = \Gamma_{A^*} = \{A^*k; k \in \mathbb{Z}_n\},$$

$$\Omega = \Omega_A = \{Ax; x \in Q_n\},\ where\ Q_n = [0,1]^n.$$

We call Ω "the fundamental domain" of the lattice Γ.

Definition 1. *(Multiresolution Analysis with Lattice Basis)*
A Mutiresolution Analysis with Lattice Basis(MRALB) is a family of closed subspaces, V_j, $j \in \mathbb{Z}$, of $L^2(\mathbb{R}^n)$ such that :
(1) V_j, $j \in \mathbb{Z}$ is an increasing sequence such that

$$(3)\ \bigcap_{j\in\mathbb{Z}} V_j = \{0\}, \overline{\bigcup_{j\in\mathbb{Z}} V_j} = L^2(\mathbb{R}^n);$$

(2) $f \in V_j$ if and only if $f(2x) \in V_{j+1}$ for $j \in \mathbb{Z}$;
(3) $f \in L^2(\mathbb{R}^n)$ belongs V_0 if and only in $f(x-\gamma) \in V_0$ for all $\gamma \in \Gamma$;
(4) There exists $g \in V_0$ such that $\{g(x-\gamma); \gamma \in \Gamma\}$ is a Riesz basis of V_0.

The above condition(4) means that
there exists constants C_1, C_2 , $0 < C_1 \le C_2$ such that
for any sequence $a(\gamma), \gamma \in \Gamma$, we have an inequality

$$(*)\ C_1 \sum_{\gamma\in\Gamma} |a(\gamma)|^2 \le |\sum_{\gamma\in\Gamma} a(\gamma) g(x-\gamma)|^2 \le C_2 \sum_{\gamma\in\Gamma} |a(\gamma)|^2.$$

Assume that $\{\varphi(x-\gamma); \gamma \in \Gamma\}$ be an orthonormal basis of V_0 and let

$$\left(\frac{1}{2}\right)^n \varphi\left(\frac{x}{2}\right) = \sum_{\gamma\in\Gamma} a(\gamma)\varphi(x-\gamma).$$

Then $\widehat{\varphi(2\xi)} = m_0(\xi)\widehat{\varphi(\xi)}$, where $m_0(\xi) = \sum_{\gamma\in\Gamma} a(\gamma)\exp(-\sqrt{-1}\xi\cdot\gamma)$, which is $2\pi\Gamma^* - periodic$ for $\xi \in \mathbb{R}^n$ and we call it a filtering function.

For $f, g \in L^2(\mathbb{R}^n)$, define

$$(4)\ C(f, g)(\xi) = \sum_{\gamma\in\Gamma} \widehat{f(\xi + 2\pi\gamma^*)}\overline{\widehat{g(\xi + 2\pi\gamma^*)}},$$

which is $2\pi\Gamma^*$ – *periodic.*

Theorem 1. *Suppose $\varphi(x) \in L^2(\mathbb{R}^n)$. Then, a system*

$$\{2^{\frac{nj}{2}}\varphi(2^j x - \gamma); \gamma \in \Gamma,\ j \in \mathbb{Z}\}$$

is an orthonormal basis of $V_j (j \in \mathbb{Z})$ if and only if

$$(5)\ C(\varphi, \varphi)(\xi) = |\det(A)|,\ a.a.\ \xi \in \mathbb{R}^n.$$

Corollary 1. *Let a system $\{\varphi(x-\gamma); \gamma \in \Gamma\}$ be an orthonormal basis of V_0. Then we have an identity*

$$(6)\ \sum_{\eta\in E} |m_0(\xi + \pi A^*\eta|^2 = 1\ a.a.\ \xi \in \mathbb{R}^n,$$

where $E - \{0,1\}^n$.

To prove the Theorem1, it is sufficient prove it for V_0,
and use the $2\pi\Gamma$ – *periodicity* of the function $C(\varphi, \varphi)(\xi)$ with its Fourier series expansion, with the Planchrel formula.
Note the fact that

$$(7)\ \textit{Assume}\ |\widehat{\varphi(0)}| = \sqrt{|\det(A)|}.\ \textit{Then}\ \widehat{\varphi(2\pi\gamma^*)}\ \textit{are}\ 0,\ \gamma^* \in \Gamma^*,$$

$$(8)\ \textit{Assume}\ |m_0(0)| = 1.\ \textit{Then}\ m_0(\pi A^*\eta) = 0,\ \eta \in \tilde{E} = E - \{0\}.$$

Assume that $\{g(x-\gamma); \gamma \in \Gamma\}$ be a Riesz basis of V_0 and define
the function $\varphi(x)$ as

$$\widehat{\varphi(\xi)} = \sqrt{|\det(A)|}\frac{\widehat{g(\xi)}}{\sqrt{C(g, g)(\xi)}}. \tag{9}$$

Then, $\{2^{\frac{nj}{2}}\varphi(2^j x - \gamma); \gamma \in \Gamma\}$ is an orthonormal basis of V_j.
We intend to make decomposition

$$V_{j+1} = V_j \bigoplus W_j (j \in \mathbb{Z}). \tag{10}$$

Suppose $\psi_\varepsilon(x) \in V_1, \varepsilon \in E$ and let

$$\widehat{\psi_\varepsilon(2\xi)} = m_\varepsilon(\xi)\widehat{\varphi(\xi)},$$

where $\psi_0(x) = \varphi(x)$, *and $m_\varepsilon(\xi)$ is $2\pi\Gamma^*$ – periodic.*
Then we have

Theorem 2. *The system $\{\varphi_\varepsilon(x-\gamma); \gamma \in \Gamma, \varepsilon \in E\}$ is an orthonormal basis of V_1 if and only if the matrix*

$$U(\xi) = ((m_\varepsilon(\xi + \pi A^*\eta))_{(\varepsilon,\ \eta)\in E^2},$$

is unitary for almost all $\xi \in \mathbb{R}^n$.

Let

$$P_0 = \sum_{\gamma\in\Gamma} \varphi(x-\gamma)\overline{\varphi(x-\gamma)},$$

and

$$P_j = 2^{nj}\sum_{\gamma\in\Gamma} \varphi(2^j x - \gamma)\overline{\varphi(2^j x - \gamma)}.$$

The operators $P_j (j \in \mathbb{Z})$ are orthogonal projections of $L^2(\mathbb{R}^n)$ onto V_j, and

define $Q_j = P_{j+1} - P_j$.
Define $W_j = Image(Q_j)$. Then $V_{j+1} = V_j \bigoplus W_j$.
For $\varepsilon \in \tilde{E}$,

$$W_{(0,\eta)} = \overline{\langle \psi_\eta(x-\gamma); \gamma \in \Gamma \rangle},$$

and

$$W_0 = \bigoplus_{\eta \in \tilde{E}} W_{(0,\eta)}.$$

Then, we have a decomposition $V_1 = V_0 \bigoplus W_0$ under the conditions of Theorem2.
In this case, for $j \in \mathbb{Z}$, $\eta \in \tilde{E}$, and $\gamma \in \Gamma$, let $\psi_{(j,\eta,\gamma)}(x) = 2^{\frac{nj}{2}} \psi_\eta(2^j x - \gamma)$.
Define $W_{(j,\eta)} = \overline{\langle \psi_{(j,\eta,\gamma)}(x); \gamma \in \Gamma \rangle}$, and $W_j = \bigoplus_{\eta \in \tilde{E}} W_{(j,\eta)}$, then

$$V_{j+1} = V_j \bigoplus W_j = V_0 \bigoplus \bigoplus_{k=0}^{j} W_k,$$

$$L^2(\mathbb{R}^n) = V_0 \bigoplus \bigoplus_{k=0}^{\infty} W_k.$$

Definition 2. *We call the system* $(\psi_{(j,\eta,\gamma)}(x); j \in \mathbb{Z},\ \eta \in \tilde{E},\ \gamma \in \Gamma)$ *an wavelet basis of* $L^2(\mathbb{R}^n)$.

§2. Simple example(n = 2). Let $A = (\overrightarrow{a_1}\ \overrightarrow{a_2})$ and $g(x) = \chi_\Omega(x)$,
, $\overrightarrow{a_j} = (a_{kj})_{k=1}^2 (j = 1, 2)$.
Put *a point* $p \in \mathbb{R}^2$ *as* $s\overrightarrow{a_1} + t\overrightarrow{a_2}$.
Then we identify Ω with Q_2.
In the folowing, we put $\eta_0^t = (0,0)$, $\eta_1^t = (0,1)$, $\eta_2^t = (1,0)$, *and* $\eta_3^t = (1,1)$.
In this case, we get

$$\widehat{g(\xi)} = |\det(A)| \exp(-\sqrt{-1}\frac{\xi}{2} \cdot A\eta_3) \prod_{j=1}^{2} sinc(\frac{\xi \cdot A\eta_j}{2\pi}),$$

where $sinc(z) = \frac{\sin(\pi z)}{\pi z}$.

$$\varphi(x) = |\det(A)|^{-\frac{1}{2}} g(x).$$

Let $u(s,t)$ be a function such that

$$\widehat{u(\xi)} = |\det(A)|^{\frac{1}{2}} \prod_{j=1}^{2} sinc(\frac{\xi \cdot A\eta_j}{2\pi}).$$

Then, we have the following formula's,

$$\varphi(s,t) = u(s - \frac{A\eta_2}{2}, t - \frac{A\eta_1}{2}),$$
$$\widehat{\varphi(2\xi)} = \exp(-\sqrt{-1}\frac{\xi \cdot A\eta_3}{2}) m(\xi) \widehat{\varphi(\xi)},$$
$$m_0(\xi) = \exp(-\sqrt{-1}\frac{\xi \cdot A\eta_3}{2}) m(\xi)$$

,where $\widehat{u(2\xi)} = m(\xi)\widehat{u(\xi)}$, and

$$m(\xi) = \prod_{j=1}^{2} \cos(\frac{\xi \cdot A\eta_j}{2}).$$

Thus, we have the following formula's

$$
(1)\quad
\begin{aligned}
m(\xi+\pi A^*\eta_0)=m(\xi) &= \cos(\frac{\xi\cdot A\eta_1}{2})\cos(\frac{\xi\cdot A\eta_2}{2})\\
m(\xi+\pi A^*\eta_1) &= -\sin(\frac{\xi\cdot A\eta_1}{2})\cos(\frac{\xi\cdot A\eta_2}{2})\\
m(\xi+\pi A^*\eta_2) &= -\cos(\frac{\xi\cdot A\eta_1}{2})\sin(\frac{\xi\cdot A\eta_2}{2})\\
m(\xi+\pi A^*\eta_3) &= \sin(\frac{\xi\cdot A\eta_1}{2})\sin(\frac{\xi\cdot A\eta_2}{2})
\end{aligned}
$$

Now, define

$$
\begin{aligned}
m(\xi;\varepsilon_0) &= m(\xi)\\
m(\xi;\varepsilon_1) &= m(\xi+\pi A^*\eta_1)\\
m(\xi;\varepsilon_2) &= m(\xi+\pi A^*\eta_2)\\
m(\xi;\varepsilon_3) &= m(\xi+\pi A^*\eta_3)
\end{aligned}
$$

Then , the matrix

$$U(\xi) = (m(\xi+\pi A^*\eta_j;\varepsilon_k))_{j,k} \ is\ unitary$$

where $\varepsilon_k = \eta_k$, $k = 0,1,2,3$.
Thus we could apply Thorem2 in §1 to get wavelet basis ψ_{ε_k}'s.
In the following, we will give these wavelet basis. Put $P = (s,t)^t$.

$[\psi_{\varepsilon_1}(x)\ :\ ]\quad m_{\varepsilon_1}(\xi) = \sqrt{-1}\exp(-\sqrt{-1}\frac{\xi\cdot A\eta_3}{2})m(\xi;\varepsilon_1)$

$$\widehat{\psi_{\varepsilon_1}(\xi\,;u)} \stackrel{def}{=} \sqrt{-1}m(\frac{\xi}{2};\varepsilon_1)\widehat{u(\frac{\xi}{2})}$$

$$\psi_{\varepsilon_1}(P\,;u) = \frac{1}{\sqrt{|det(A)|}}\chi_{[-1/2,1/2]}(\eta_2^tA^{-1}P)[-\chi_{[-1/2,0]}(\eta_1^tA^{-1}P)+\chi_{[0,1/2]}(\eta_1^tA^{-1}P)]$$

$$\psi_{\varepsilon_1}(P) = \frac{1}{\sqrt{|det(A)|}}\chi_{[0,1]}(\eta_2^tA^{-1}P)[-\chi_{[0,1/2]}(\eta_1^tA^{-1}P)+\chi_{[1/2,1]}(\eta_1^tA^{-1}P)]$$

$[\psi_{\varepsilon_2}(x)\ :\ ]\quad m_{\varepsilon_2}(\xi) = \sqrt{-1}\exp(-\sqrt{-1}\frac{\xi\cdot A\eta_3}{2})m(\xi;\varepsilon_2)$

$$\widehat{\psi_{\varepsilon_2}(\xi\,;u)} \stackrel{def}{=} \sqrt{-1}m(\frac{\xi}{2};\varepsilon_2)\widehat{u(\frac{\xi}{2})}$$

$$\psi_{\varepsilon_2}(P\,;u) = \frac{1}{\sqrt{|det(A)|}}\chi_{[-1/2,1/2]}(\eta_1^tA^{-1}P)[-\chi_{[-1/2,0]}(\eta_2^tA^{-1}P)+\chi_{[0,1/2]}(\eta_2^tA^{-1}P)]$$

$$\psi_{\varepsilon_2}(P) = \frac{1}{\sqrt{|det(A)|}}\chi_{[0,1]}(\eta_1^tA^{-1}P)[-\chi_{[0,1/2]}(\eta_2^tA^{-1}P)+\chi_{[1/2,1]}(\eta_2^tA^{-1}P)]$$

$[\psi_{\varepsilon_3}(x)\ :\ ]\quad m_{\varepsilon_3}(\xi) = -\exp(-\sqrt{-1}\frac{\xi\cdot A\eta_3}{2})m(\xi;\varepsilon_3)$

$$\widehat{\psi_{\varepsilon_3}(\xi\,;u)} \stackrel{def}{=} -m(\frac{\xi}{2};\varepsilon_3)\widehat{u(\frac{\xi}{2})}$$

$$\psi_{\varepsilon_3}(P\,;u) = \frac{1}{\sqrt{|det(A)|}}[-\chi_{[-1/2,0]}(\eta_2^tA^{-1}P)+\chi_{[0,1/2]}(\eta_2^tA^{-1}P)][-\chi_{[-1/2,0]}(\eta_1^tA^{-1}P)+\chi_{[0,1/2]}(\eta_1^tA^{-1}P)$$

$$\psi_{\varepsilon_3}(P) = \frac{1}{\sqrt{|det(A)|}}[-\chi_{[0,1/2]}(\eta_2^tA^{-1}P)+\chi_{[1/2,1]}(\eta_2^tA^{-1}P)][-\chi_{[0,1/2]}(\eta_1^tA^{-1}P)+\chi_{[1/2,1]}(\eta_1^tA^{-1}P)]$$

In summary, we get

Proposition 1. *The functions $\varphi(s,t)$, $\psi_{\varepsilon_1}(s,t)$, $\psi_{\varepsilon_2}(s,t)$, and $\psi_{\varepsilon_3}(s,t)$ generate an othonormal basis of the space V_1, and $\psi_{\varepsilon_1}(s,t)$, $\psi_{\varepsilon_2}(s,t)$, and $\psi_{\varepsilon_3}(s,t)$ generate an orthonormal basis of the space W_0.*

Note that

$$\int \varphi(s,t)dsdt = \sqrt{|\det(A)|}; \int \psi_{\varepsilon_j}(s,t)dsdt = 0 \ (j = 1,2,3).$$

In the next paragraph, we will describe "2-dimensional splines" .

§3. Spline system of order r(n = 2). Let $\varphi(s,t) = \frac{1}{\sqrt{|\det(A)|}}\chi_{\Omega_A}(s,t)$.
For $r = 0,1,2,\ldots$, define

$$F_r(s,t) = \varphi^{*(r+1)}(s,t).$$

The Fourier transform of the function $F_r(s,t)$ is

$$\begin{aligned}\widehat{F_r(\xi)} &= [\widehat{\varphi(\xi)}]^{r+1} \\ &= |\det(A)|^{\frac{r+1}{2}}\exp(-\mathrm{i}\frac{r+1}{2}\xi\cdot A\eta_3)[\widehat{u(\xi)}]^{r+1}\end{aligned} \tag{2}$$

We have the correlation function $C(F_r,F_r)(\xi)$

$$\begin{aligned}C(F_r,F_r)(\xi) &= |\det(A)|^{r+1}\sum_{\gamma^{*}\in\Gamma^*}|\widehat{u(\xi+2\pi\gamma^*)}|^2 \\ &= \prod_{j=1}^{2}(\sin^{2(r+1)}(\frac{\xi\cdot A\eta_j}{2})\sum_{k_j\in\mathbf{Z}}\frac{1}{[\frac{\xi\cdot A\eta_j}{2}+\pi k_j]^{2(r+1)}})\end{aligned} \tag{3}$$

Let

$$for\ q\in\mathbb{N},\ q\geq 2,\ \sum_{k\in\mathbf{Z}}\frac{1}{(t+\pi k)^q} = \frac{P_{q-2}(\cos(t))}{\sin^q(t)}$$

, where P_q is a polynomial of degree q . Then, we have an equation

$$C(F_r,F_r)(\xi) = |\det(A)|^{r+1}\prod_{j=1}^{2}P_{2r}(\cos(\frac{\xi\cdot A\eta_j}{2})) \tag{4}$$

Define a function $G_r(s,t)$ having its Fourier transform

$$\widehat{G_r(\xi)} = \sqrt{|\det(A)|}\frac{\widehat{F_r(\xi)}}{\sqrt{C(F_r,F_r)(\xi)}}.$$

Note that $\widehat{G_r(\xi)} = O(|\xi|^{-2r})\ as\ |\xi|\to\infty$.

Definition 3. *We call the system $\{G_r(x-\gamma);\gamma\in\Gamma\}$ a spline system of order r on the lattice basis.*

Now let $\widehat{U_r(\xi)} = \sqrt{|\det(A)|}\prod_{j=1}^{2}(sinc^{r+1}(\frac{\xi\cdot A\eta_j}{2\pi})P_{2r}^{-\frac{1}{2}}(\cos(\frac{\xi\cdot A\eta_j}{2})))$.
Then, we have an expression

$$G_r(s,t) = U_r(s-\frac{(r+1)(a_{11}+a_{12})}{2}, t-\frac{(r+1)(a_{21}+a_{22})}{2}).$$

In the following, we treat cases of r is odd,and even, separately.

Case of $r+1 = 2\sigma(\sigma = 1,2,\ldots)$; (r is odd)]

In this case, we have expressions

$$\widehat{U_r(\xi)} = \sqrt{|\det(A)|}\prod_{j=1}^{2}(sinc^{2\sigma}(\frac{\xi\cdot A\eta_j}{2\pi})P_{4\sigma-2}^{-\frac{1}{2}}(\cos(\frac{\xi\cdot A\eta_j}{2}))$$

(5)
$$m_r(\xi;\eta_0) = \prod_{j=1}^{2}(\cos^{2\sigma}(\frac{\xi\cdot A\eta_j}{2})\sqrt{\frac{P_{4\sigma-2}(\cos(\frac{\xi\cdot A\eta_j}{2}))}{P_{4\sigma-2}(\cos(\xi\cdot A\eta_j))}})$$

$$\widehat{U_r(2\xi)} = m_r(\xi;\eta_0)\widehat{U_r(\xi)}$$

We have also expressions

$$m_r(\xi+\pi A^*\eta_1;\eta_0) = \sin^{2\sigma}(\frac{\xi\cdot A\eta_1}{2})\sqrt{\frac{P_{4\sigma-2}(\sin(\frac{\xi\cdot A\eta_1}{2}))}{P_{4\sigma-2}(\cos(\xi\cdot A\eta_1)}}\cos^{2\sigma}(\frac{\xi\cdot A\eta_2}{2})\sqrt{\frac{P_{4\sigma-2}(\cos(\frac{\xi\cdot A\eta_2}{2}))}{P_{4\sigma-2}(\cos(\xi\cdot A\eta_2)}}$$

$$m_r(\xi+\pi A^*\eta_2;\eta_0) = \cos^{2\sigma}(\frac{\xi\cdot A\eta_1}{2})\sqrt{\frac{P_{4\sigma-2}(\cos(\frac{\xi\cdot A\eta_1}{2}))}{P_{4\sigma-2}(\cos(\xi\cdot A\eta_1)}}\sin^{2\sigma}(\frac{\xi\cdot A\eta_2}{2})\sqrt{\frac{P_{4\sigma-2}(\sin(\frac{\xi\cdot A\eta_2}{2}))}{P_{4\sigma-2}(\cos(\xi\cdot A\eta_2)}}$$

$$m_r(\xi+\pi A^*\eta_3;\eta_0) = \sin^{2\sigma}(\frac{\xi\cdot A\eta_1}{2})\sqrt{\frac{P_{4\sigma-2}(\sin(\frac{\xi\cdot A\eta_1}{2}))}{P_{4\sigma-2}(\cos(\xi\cdot A\eta_1)}}\sin^{2\sigma}(\frac{\xi\cdot A\eta_2}{2})\sqrt{\frac{P_{4\sigma-2}(\sin(\frac{\xi\cdot A\eta_2}{2}))}{P_{4\sigma-2}(\cos(\xi\cdot A\eta_2)}}$$

Define

$$m_r(\xi;\eta_j) = m_r(\xi+\pi A^*\eta_j;\eta_0)$$

(6)
$$m_(\xi;\eta_j) = \exp(-\sqrt{-1}\frac{r+1}{2}\xi\cdot A\eta_3)m_r(\xi;\eta_j)(j=0,1,2,3)$$

Then the matrix $(m_(\xi+\pi A^{ast}\varepsilon_j;\eta_k))_{j,k}$ is unitary.
To calculate wavelet functions "$\psi(s,t;\eta_j)(j=1,2,3)$" , we introduce two auxiliary functions

$$\psi_r^{(1)}(z) = \frac{1}{2\pi}\int\exp(\sqrt{-1}\xi z)\frac{sinc^{r+1}(\frac{\xi}{2\pi})}{\sqrt{P_{2r}(\cos(\frac{\xi}{2})}}d\xi$$

$$\psi_r^{(2)}(z) = \frac{1}{2\pi}\int\exp(\sqrt{-1}\xi z)[\sin(\frac{\xi}{4})sinc(\frac{\xi}{4\pi})]^{r+1}\sqrt{\frac{P_{2r}(\sin(\frac{\xi}{4}))}{P_{2r}(\cos(\frac{\xi}{4}))P_{2r}(\cos(\frac{\xi}{2}))}}d\xi$$

The function $\psi_r^1(z)$ is even, real-valued function for $r=0,1,2,\dots$,
however the function $\psi_r^2(z)$ is even, real-valued for $r: odd$ and
is odd, pure-imaginary valued for $r: even$. The attached are graphs of some of integrand functions of these functions.
For cases of $r: odd(i.e. r+1: even)$, we have the following results :

Theorem 3. *Let* $r: odd(r=1,3,5,\dots)$. *Then we have the following expressions for the scaling function* $G_r(s,t)$ *and the wavelet functions* $\psi_r(s,t;\varepsilon_j)(j=1,2,3)))$:

$$\psi_r(s,t;\varepsilon_0) = \frac{1}{\sqrt{|\det(A)|}}\psi_r^{(1)}(\frac{a_{22}s-a_{12}t}{\det(A)}-\frac{r+1}{2})\psi_r^{(1)}(\frac{-a_{21}s+a_{11}t}{\det(A)}-\frac{r+1}{2})$$

$$\psi_r(s,t;\varepsilon_1) = \frac{(1)}{\sqrt{|\det(A)|}}\psi_r^{(1)}(\frac{a_{22}s-a_{12}t}{\det(A)}-\frac{r+1}{2})\psi_r^{(2)}(\frac{-a_{21}s+a_{11}t}{\det(A)}-\frac{r+1}{2})$$

$$\psi_r(s,t;\varepsilon_0) = \frac{1}{\sqrt{|\det(A)|}}\psi_r^{(2)}(\frac{a_{22}s-a_{12}t}{\det(A)}-\frac{r+1}{2})\psi_r^{(1)}(\frac{-a_{21}s+a_{11}t}{\det(A)}-\frac{r+1}{2})$$

$$\psi_r(s,t;\varepsilon_0)=\frac{1}{\sqrt{|\det(A)|}}\psi_r^{(2)}\left(\frac{a_{22}s-a_{12}t}{\det(A)}-\frac{r+1}{2}\right)\psi_r^{(2)}\left(\frac{-a_{21}s+a_{11}t}{\det(A)}-\frac{r+1}{2}\right)$$

, *where* $\psi_r(s,t;\varepsilon_0)=G_r(s,t)$.

Case of $r+1=2\sigma+1(\sigma=1,2,\ldots)$; (r is even)]

In this case, we define

$$\begin{aligned}m_r(\xi;\varepsilon_1)&=(-\sqrt{-1})^{r+1}m_r(\xi+\pi A^*\eta_1;\varepsilon_0)\\ m_r(\xi;\varepsilon_2)&=(-\sqrt{-1})^{r+1}m_r(\xi+\pi A^*\eta_2;\varepsilon_0)\\ m_r(\xi;\varepsilon_3)&=(-1)^{r+1}m_r(\xi+\pi A^*\eta_3;\varepsilon_0)\end{aligned}\tag{7}$$

and

$\psi_r^{(2)}(z)=\frac{1}{2\pi}\int\exp(\sqrt{-1}\xi z)[-\sqrt{-1}\sin(\frac{\xi}{4})sinc(\frac{\xi}{4\pi})]^{r+1}\sqrt{\frac{P_{2r}(\sin(\frac{\xi}{4}))}{P_{2r}(\cos(\frac{\xi}{4}))P_{2r}(\cos(\frac{\xi}{2}))}}d\xi$

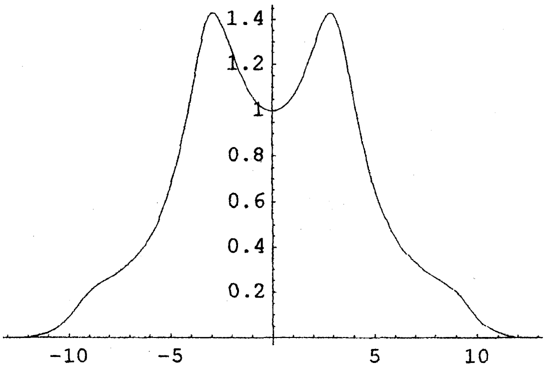
Then, the Theorem 3 is valid.

$$\mathbf{sinc[x\_]\ :=\ Sin[\pi\, x]\ /\ (\pi\, x)}$$

$$\mathbf{P[t\_]\ :=\ \frac{1 + 2\, t^2}{3}}$$

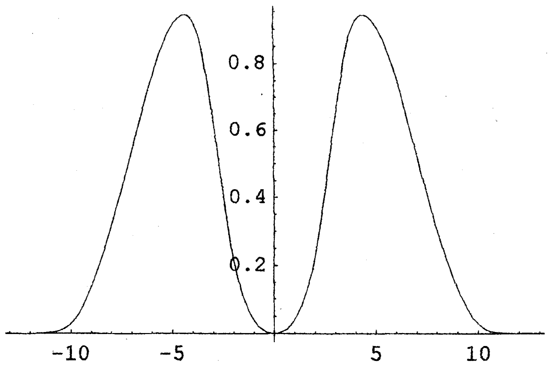
$$\mathbf{f[x\_]\ :=\ \frac{sinc[\frac{x}{4\pi}]^2}{\sqrt{P[Cos[\frac{x}{2}]]}}}$$

{ Fourier Tansform of function U(2;1)}



$$\mathbf{g[x\_]\ :=\ \left(Sin\left[\frac{x}{4}\right]\ sinc\left[\frac{x}{4\,\pi}\right]\right)^2 \sqrt{\frac{P[Sin[\frac{x}{4}]]}{P[Cos[\frac{x}{4}]]\ P[Cos[\frac{x}{2}]]}}}$$

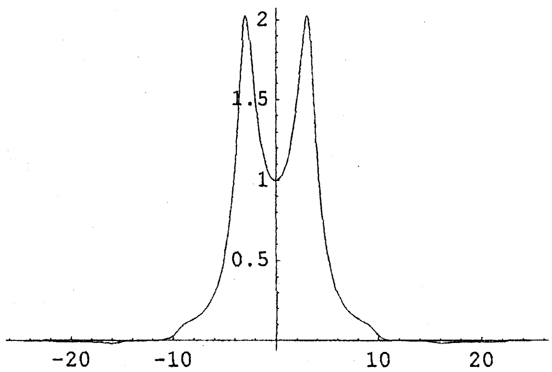
{Fourier Transform of function U(2;2)}



$$\mathbf{P[t\_]\ :=\ \frac{2 + 11\, t^2 + 2\, t^4}{15}}$$

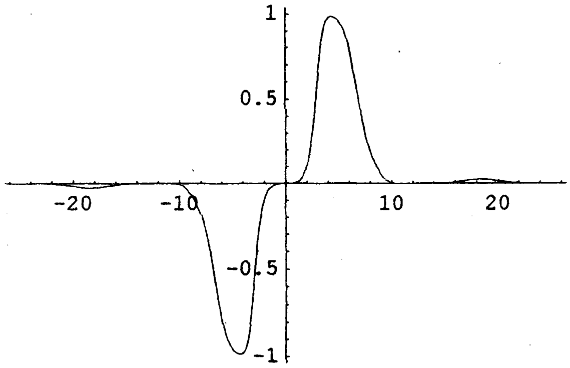
$$\mathbf{f[x\_]\ :=\ \frac{sinc[\frac{x}{4\pi}]^3}{\sqrt{P[Cos[\frac{x}{2}]]}}}$$

{Fourier Transform of function U(2;1)}



$$g[x\_] := \left(\mathrm{Sin}\left[\frac{x}{4}\right]\,\mathrm{sinc}\left[\frac{x}{4\pi}\right]\right)^3 \sqrt{\frac{P[\mathrm{Sin}[\frac{x}{4}]]}{P[\mathrm{Cos}[\frac{x}{4}]]\;P[\mathrm{Cos}[\frac{x}{2}]]}}$$

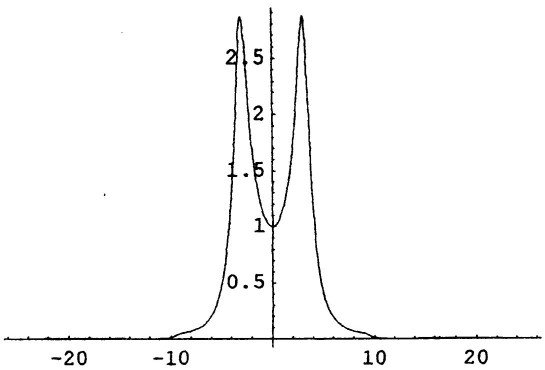
{Fourier Transform of function U(2;2)}



$$P[t\_] := \frac{17 + 180\ t^2 + 114\ t^4 + 4\ t^6}{315}$$

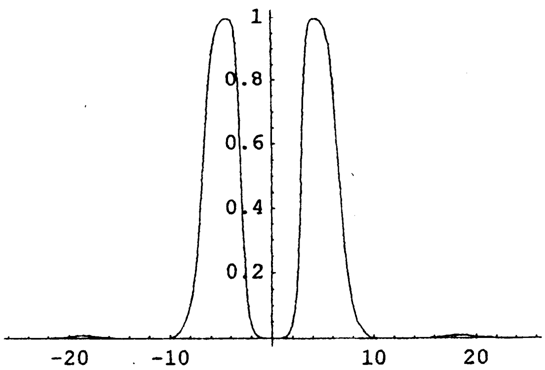
$$f[x\_] := \frac{\mathrm{sinc}[\frac{x}{4\pi}]^4}{\sqrt{P[\mathrm{Cos}[\frac{x}{2}]]}}$$

{Fourier Transform of function U(3;1)}



$$g[x\_] := \left(\mathrm{Sin}\left[\frac{x}{4}\right]\,\mathrm{sinc}\left[\frac{x}{4\pi}\right]\right)^4 \sqrt{\frac{P[\mathrm{Sin}[\frac{x}{4}]]}{P[\mathrm{Cos}[\frac{x}{4}]]\;P[\mathrm{Cos}[\frac{x}{2}]]}}$$

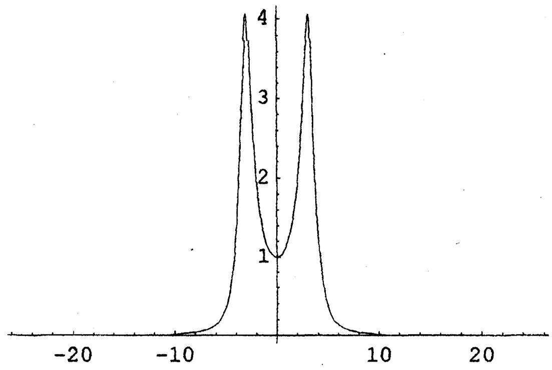
{Fourier Transform of function U(3;2)}



$$P[t\_] := \frac{62 + 1072\ t^2 + 1452\ t^4 + 247\ t^6 + 2\ t^8}{2835}$$

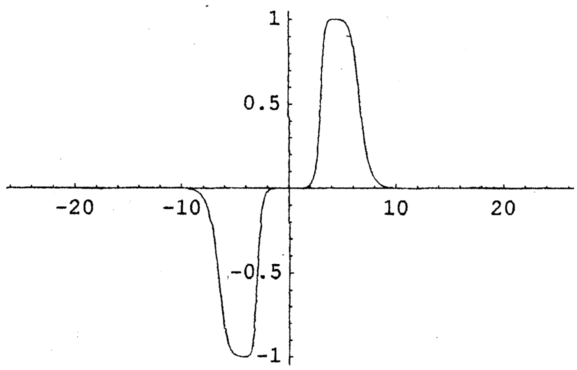
$$\mathbf{f[x\_]\ :=\ \frac{sinc[\frac{x}{4\pi}]^{5}}{\sqrt{P[Cos[\frac{x}{2}]]}}}$$

{Fourier Transform of function U(4;1)}



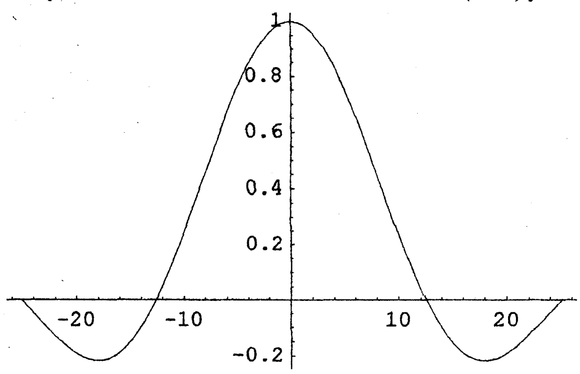
$$\mathbf{g[x\_]\ :=\ \left(Sin\left[\frac{x}{4}\right]\ sinc\left[\frac{x}{4\pi}\right]\right)^{5}\sqrt{\frac{P[Sin[\frac{x}{4}]]}{P[Cos[\frac{x}{4}]]\ P[Cos[\frac{x}{2}]]}}}$$

{Fourier Transform of function U(4;2)}



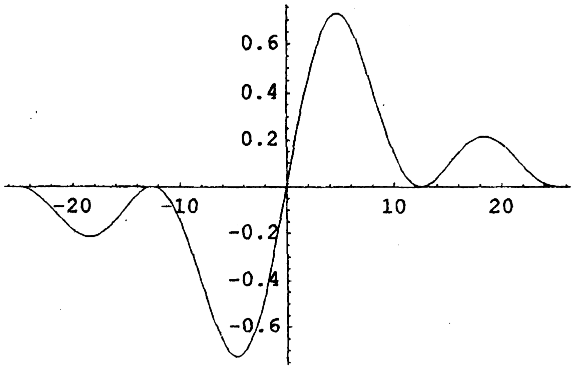
$$\mathbf{f[x\_]\ :=\ \frac{sinc[\frac{x}{4\pi}]^{1}}{1}}$$

{Fourier Transform of function U(0;1)}



$$g[x\_] := \left(\sin\left[\frac{x}{4}\right] \,\mathrm{sinc}\left[\frac{x}{4\pi}\right]\right)^{1}$$

{Fourier Transform of function U(0;2)}



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