

# q-deformation of the group algebra $k[\widehat{W}]$ associated to the elliptic root system $A_l^{(1,1)}$ ( $l \geq 2$ )

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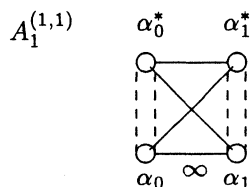
## 1 Introduction

Elliptic root systems ( extended affine root systems ), have been introduced and studied by K. Saito ( [1], [2] ). The Weyl group associated to the elliptic root system is called elliptic Weyl group, and the relations of the generators of the elliptic Weyl group have been described by the author and K. Saito from the view point of a generalization of Coxeter relations ( [5], [6] ). In the cases of finite or affine root systems, the Hecke algebras and their representation theory are well known ( [3], [4] ), and one can consider the Hecke algebra as a q-deformation of the group algebra  $k[W]$ . In this paper, in the case of the elliptic root system  $A_l^{(1,1)}$  ( $l \geq 2$ ), we give a similar q-deformation  $k[\widehat{W}](q)$  of the group algebra  $k[\widehat{W}]$ , where  $\widehat{W}$  is a hyperbolic extension ( a central extension ) of the elliptic Weyl group  $W$  ( [6] ).

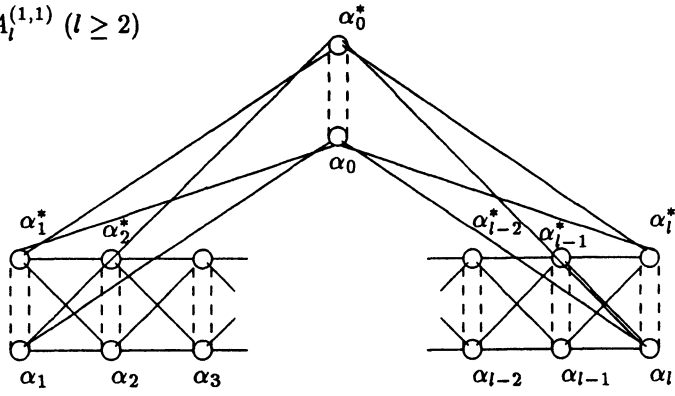
## 2 Elliptic Weyl group and elliptic root system of $A_l^{(1,1)}$

Let  $F$  be a real vector space of finite rank with a symmetric bilinear form  $I : F \times F \rightarrow \mathbf{R}$ . A subset  $\Phi$  of  $F$  is called an elliptic root system, if  $I$  is positive semi definite and the radical  $rad(I) = \{x \in F \mid I(x, y) = 0 \text{ for } \forall y \in F\}$  is of rank 2 over  $\mathbf{R}$  and satisfies a system of an axiom for a generalized root system belonging to  $I$  ([1]). In the sequel, we restrict in the case of  $A_l^{(1,1)}$ . Let  $\epsilon_1, \epsilon_2, \dots, \epsilon_{l+1}$  be orthonormal vectors belonging to the real vector space  $F$ , and  $rad(I) = \mathbf{R}a \oplus \mathbf{R}b$ . Then the root system  $\Phi$  and the Dynkin diagram of  $A_l^{(1,1)}$  are given as follows ([1]) :

$$\Phi = \{ \pm(\epsilon_i - \epsilon_j) + nb + ma \mid (1 \leq i < j \leq l+1), (n, m \in \mathbf{Z}) \},$$



$A_l^{(1,1)} \ (l \geq 2)$



where  $\alpha_0 := \epsilon_{l+1} - \epsilon_1 + b$ ,  $\alpha_i := \epsilon_i - \epsilon_{i+1} \ (1 \leq i \leq l)$ , and  $\alpha_i^* := \alpha_i + a \ (0 \leq i \leq l)$ .

Let  $w_i := w_{\alpha_i}$  and  $w_i^* := w_{\alpha_i^*}$  be the reflections corresponding to the roots  $\alpha_i$  and  $\alpha_i^*$ , respectively. Then  $w_i$  and  $w_i^* \ (0 \leq i \leq l)$  generate the elliptic Weyl group of  $A_l^{(1,1)}$ , and the following has been established.

**Proposition 2.1** ([5], [6])

(I) The relations of  $w_i$  and  $w_i^*$  are given as follows :

For any  $\alpha, \beta, \in \{\alpha_0, \alpha_1, \dots, \alpha_l, \alpha_0^*, \alpha_1^*, \dots, \alpha_l^*\}$

$$(i) \quad \bigcirc_{\alpha} \implies w_{\alpha}^2 = 1,$$

$$\bigcirc_{\alpha} \quad \bigcirc_{\beta} \implies (w_{\alpha} w_{\beta})^2 = 1,$$

$$\bigcirc_{\alpha} \text{---} \bigcirc_{\beta} \implies (w_{\alpha} w_{\beta})^3 = 1,$$

$$\begin{array}{c} \alpha^* \quad \beta^* \\ \bigcirc \quad \bigcirc \\ \diagup \quad \diagdown \\ \bigcirc \quad \bigcirc \\ \alpha \quad \beta \end{array} \implies w_{\alpha} w_{\alpha^*} w_{\beta} w_{\beta^*} = w_{\alpha^*} w_{\beta} w_{\beta^*} w_{\alpha} = w_{\beta} w_{\beta^*} w_{\alpha} w_{\alpha^*} = w_{\beta^*} w_{\alpha} w_{\alpha^*} w_{\beta}$$

$$\begin{array}{c} \alpha^* \quad \beta^* \\ \bigcirc \quad \bigcirc \\ \diagup \quad \diagdown \\ \bigcirc \quad \bigcirc \\ \alpha \quad \beta \end{array} \implies w_{\alpha} w_{\beta^*} w_{\alpha} = w_{\beta} w_{\alpha^*} w_{\beta},$$

$$(ii) \quad w_0 w_0^* w_1 w_1^* \cdots w_{l-1} w_{l-1}^* w_l w_l^* = 1.$$

(II) If we consider only the relation (i), then the corresponding group is the hyperbolic extension  $\widehat{W}$  of the elliptic Weyl group  $W$ .

(Remark) In the case of  $A_l^{(1,1)}$ , the following relations can be obtained from (i):

$$\begin{array}{c} \alpha^* \\ \circ \\ \vdots \\ \circ \\ \alpha \end{array} \begin{array}{c} \beta \\ \circ \end{array} \Rightarrow (w_\alpha w_\beta w_\alpha^* w_\beta)^3 = 1, \quad \begin{array}{c} \beta^* \\ \circ \\ \alpha \quad \circ \quad \gamma \\ \vdots \\ \circ \\ \beta \end{array} \Rightarrow (w_\alpha w_\beta^* w_\alpha w_\beta w_\gamma w_\beta)^2 = 1.$$

### 3 q-deformation algebra $k[\widehat{W}](q)$ of the group algebra $k[\widehat{W}]$

In this section, let  $k$  be a field, and we define a q-deformation algebra  $k[\widehat{W}](q)$  of the hyperbolic extension  $\widehat{W}$  of the elliptic Weyl group  $W$  of  $A_l^{(1,1)}$  ( $l \geq 2$ ), according to Kazhdan-Lusztig [4].

**Definition 3.1** For each simple root  $\alpha \in \{\alpha_0, \alpha_1, \dots, \alpha_l\}$ , we define the elements  $T_\alpha, T_\alpha^* \in \text{End}(F \otimes k(q^{\frac{1}{2}}, q^{-\frac{1}{2}}))$  by the following actions:

$$\begin{cases} T_\alpha(\alpha) = -\alpha, \\ T_\alpha(\alpha^*) = q\alpha^* - (q+1)\alpha, \\ T_\alpha(\beta) = q\beta + q^{1/2}\alpha \quad (\text{if } \begin{array}{c} \circ \text{---} \circ \\ \alpha \quad \beta \end{array}), \\ T_\alpha(\beta) = q\beta \quad (\text{if } \begin{array}{c} \circ \quad \circ \\ \alpha \quad \beta \end{array}), \end{cases}$$

$$\begin{cases} T_\alpha^*(\alpha) = q\alpha - (q+1)\alpha^*, \\ T_\alpha^*(\alpha^*) = -\alpha^*, \\ T_\alpha^*(\beta) = q\beta + q^{1/2}\alpha^* \quad (\text{if } \begin{array}{c} \circ \text{---} \circ \\ \alpha^* \quad \beta \end{array}), \\ T_\alpha^*(\beta) = q\beta \quad (\text{if } \begin{array}{c} \circ \quad \circ \\ \alpha^* \quad \beta \end{array}). \end{cases}$$

Under these actions, we examine the relations of the generators  $T_i := T_{\alpha_i}$  and  $T_i^* := T_{\alpha_i^*}$  ( $0 \leq i \leq l$ ), then we obtain the following.

#### Proposition 3.2

For any  $\alpha, \beta \in \{\alpha_0, \alpha_1, \dots, \alpha_l, \alpha_0^*, \alpha_1^*, \dots, \alpha_l^*\}$

$$(I) \quad \begin{array}{c} \circ \\ \alpha \end{array} \Rightarrow T_\alpha^2 = (q-1)T_\alpha + q,$$

$$(II) \quad \begin{array}{c} \circ \\ \alpha \end{array} \quad \begin{array}{c} \circ \\ \beta \end{array} \Rightarrow T_\alpha T_\beta = T_\beta T_\alpha,$$

$$\begin{array}{ll}
\text{(I)} & \begin{array}{c} \circ \quad \circ \\ \alpha \quad \beta \end{array} \Rightarrow T_\alpha T_\beta T_\alpha = T_\beta T_\alpha T_\beta, \\
\text{(II)} & \begin{array}{c} \circ \quad \circ \\ \alpha \quad \beta \end{array} \Rightarrow T_\alpha T_\beta = T_\beta T_\alpha, \\
\text{(III)} & \begin{array}{c} \alpha^* \quad \beta^* \\ \circ \quad \circ \\ \circ \quad \circ \\ \alpha \quad \beta \end{array} \Rightarrow \begin{aligned} & T_\alpha T_\beta^* T_\alpha - T_\beta T_\alpha^* T_\beta = Q([T_\alpha, T_\alpha^*] - [T_\beta, T_\beta^*]), \\ & T_\alpha^* T_\beta T_\alpha - T_\beta T_\alpha T_\beta^* = Q[T_\beta, T_\beta^*], \\ & T_\alpha^* T_\beta^* T_\alpha - T_\beta T_\alpha^* T_\beta^* = Q[T_\alpha, T_\alpha^*], \end{aligned} \\
& \text{where, } Q = \frac{q^{\frac{3}{2}} - q^2}{1+q}.
\end{array}$$

By this proposition, we define the  $k[\widehat{W}](q)$  as follows.

**Definition 3.3** The  $q$ -deformation algebra  $k[\widehat{W}](q)$  is the algebra presented by the following generators and relations:

**Generators:**  $T_\alpha, T_\alpha^* \quad \forall \alpha \in \{\alpha_0, \alpha_1, \dots, \alpha_l\}$ .

**Relations:**

$$\begin{array}{ll}
\text{(I)} & \begin{array}{c} \circ \\ \alpha \end{array} \Rightarrow T_\alpha^2 = (q-1)T_\alpha + q, \\
\text{(II)} & \begin{array}{c} \circ \quad \circ \\ \alpha \quad \beta \end{array} \Rightarrow T_\alpha T_\beta = T_\beta T_\alpha, \\
& \begin{array}{c} \circ \quad \circ \\ \alpha \quad \beta \end{array} \Rightarrow T_\alpha T_\beta T_\alpha = T_\beta T_\alpha T_\beta, \\
\text{(III)} & \begin{array}{c} \alpha^* \quad \beta^* \\ \circ \quad \circ \\ \circ \quad \circ \\ \alpha \quad \beta \end{array} \Rightarrow \begin{aligned} & T_\alpha T_\beta^* T_\alpha - T_\beta T_\alpha^* T_\beta = Q([T_\alpha, T_\alpha^*] - [T_\beta, T_\beta^*]), \\ & T_\alpha^* T_\beta T_\alpha - T_\beta T_\alpha T_\beta^* = Q[T_\beta, T_\beta^*], \\ & T_\alpha^* T_\beta^* T_\alpha - T_\beta T_\alpha^* T_\beta^* = Q[T_\alpha, T_\alpha^*], \end{aligned} \\
& \text{where, } Q = \frac{q^{\frac{3}{2}} - q^2}{1+q}.
\end{array}$$

(Remark) The subalgebra generated by the  $T_\alpha, \forall \alpha \in \{\alpha_0, \dots, \alpha_l\}$  ( or  $T_\alpha^*$  ) is isomorphic to the affine Hecke algebra of type  $A_l$ .

In the following, we prove Proposition 3.2. Let  $\tilde{T}_i$  be the representation matrix on the basis  $(\alpha_0, \alpha_0^*, \dots, \alpha_l, \alpha_l^*)$ , i.e.  $\tilde{T}_i (\alpha_0, \alpha_0^*, \dots, \alpha_l, \alpha_l^*) = (\alpha_0, \alpha_0^*, \dots, \alpha_l, \alpha_l^*) \tilde{T}_i$ , then from Definition 3.1, we see that

$$\begin{aligned}
\tilde{T}_0 &= q^{\frac{1}{2}}(E_{0,2} + E_{0,3} + E_{0,2l} + E_{0,2l+1}) - (q+1)(E_{0,0} + E_{0,1}) + qI, \\
\tilde{T}_i &= q^{\frac{1}{2}}(E_{2i,2i-2} + E_{2i,2i-1} + E_{2i,2i+2} + E_{2i,2i+3}) - (q+1)(E_{2i,2i} + E_{2i,2i+1}) + qI, \quad (1 \leq i \leq l-1), \\
\tilde{T}_l &= q^{\frac{1}{2}}(E_{2l,0} + E_{2l,1} + E_{2l,2l-2} + E_{2l,2l-1}) - (q+1)(E_{2l,2l} + E_{2l,2l+1}) + qI,
\end{aligned}$$

where  $E_{i,j}$  is the  $(2l+2) \times (2l+2)$  matrix whose  $(i,j)$  component is 1 and all others are 0, and  $I$  is the identity matrix. We set  $\tilde{T}_i := \tilde{T}_i' + qI$ , and identify  $\tilde{T}_i, \tilde{T}_i'$  with  $T_i, T_i'$ , respectively. In

the same way,  $\tilde{T}_i^* = \tilde{T}_i^{*'} + qI$  are given by

$$\begin{aligned}\tilde{T}_0^* &= q^{\frac{1}{2}}(E_{1,2} + E_{1,3} + E_{1,2l} + E_{1,2l+1}) - (q+1)(E_{1,0} + E_{1,1}) + qI, \\ \tilde{T}_i^* &= q^{\frac{1}{2}}(E_{2i+1,2i-2} + E_{2i+1,2i-1} + E_{2i+1,2i+2} + E_{2i+1,2i+3}) - (q+1)(E_{2i+1,2i} + E_{2i+1,2i+1}) + qI, \\ \tilde{T}_l^* &= q^{\frac{1}{2}}(E_{2l+1,0} + E_{2l+1,1} + E_{2l+1,2l-2} + E_{2l+1,2l-1}) - (q+1)(E_{2l+1,2l} + E_{2l+1,2l+1}) + qI.\end{aligned}$$

Then there hold the following relations.

**Lemma 3.4**

For any  $\alpha, \beta \in \{\alpha_0, \alpha_1, \dots, \alpha_l, \alpha_0^*, \alpha_1^*, \dots, \alpha_l^*\}$

$$(I) \quad \begin{array}{c} \bigcirc \\ \alpha \end{array} \quad \Rightarrow \quad T_\alpha'^2 = -(q+1)T_\alpha',$$

$$(II) \quad \begin{array}{c} \bigcirc \\ \alpha \end{array} \quad \begin{array}{c} \bigcirc \\ \beta \end{array} \quad \Rightarrow \quad T_\alpha' T_\beta' = T_\beta' T_\alpha',$$

$$\begin{array}{c} \bigcirc \text{---} \bigcirc \\ \alpha \quad \beta \end{array} \quad \Rightarrow \quad T_\alpha' T_\beta' T_\alpha' = qT_\beta',$$

$$\begin{array}{c} \alpha^* \\ \bigcirc \\ \vdots \\ \bigcirc \\ \alpha \end{array} \quad \Rightarrow \quad \begin{aligned} [T_\alpha', T_\alpha^{*'}] &= -(q+1)(T_\alpha' - T_\alpha^{*'}), \\ T_\alpha^{*'} T_\alpha' &= -(q+1)T_\alpha^{*'}, \\ T_\alpha' T_\alpha^{*'} &= -(q+1)T_\alpha', \end{aligned}$$

$$(III) \quad \begin{array}{c} \alpha^* \quad \beta^* \\ \bigcirc \quad \bigcirc \\ \vdots \quad \vdots \\ \bigcirc \quad \bigcirc \\ \alpha \quad \beta \end{array} \quad \Rightarrow \quad \begin{aligned} T_\alpha^{*'} T_\beta' - T_\alpha' T_\beta^{*'} &= q^{\frac{1}{2}}(T_\beta^{*'} - T_\beta'), \\ T_\alpha^{*'} T_\beta^{*'} T_\alpha' &= T_\alpha^{*'} T_\beta' T_\alpha', \\ T_\alpha' T_\beta' T_\alpha^{*'} &= T_\alpha' T_\beta^{*'} T_\alpha', \\ T_\alpha' T_\beta' T_\alpha^{*'} + T_\alpha' T_\alpha^{*'} + T_\alpha' &= 0, \\ T_\alpha^{*'} T_\beta' T_\alpha' + T_\alpha^{*'} T_\alpha' + T_\alpha^{*'} &= 0. \end{aligned}$$

(Proof) It is proved by direct calculations by using the relation, in the representation matrix  $E_{2i+1,k} - E_{2i,k} = E_{2j+1,k} - E_{2j,k}$  for any  $i, j, k$ , which is obtained from the property  $\alpha_i^* - \alpha_i = \alpha_j^* - \alpha_j$ .  $\square$

( Proof of Proposition 3.2 )

( (I),  $T_\alpha^2 = (q-1)T_\alpha + qI$  )

$$\begin{aligned} T_\alpha^2 &= (T_\alpha' + qI)^2 \\ &= T_\alpha'^2 + 2qT_\alpha' + q^2I \\ &= -(q+1)T_\alpha' + 2qT_\alpha' + q^2I \quad (\Leftarrow \text{ Lemma 3.4 (I)}) \\ &= (q-1)(T_\alpha' - qI) + q^2I \\ &= (q-1)T_\alpha + qI \end{aligned}$$

$$( \text{ (II)}, T_\alpha T_\beta = T_\beta T_\alpha )$$

$$\begin{aligned} T_\alpha T_\beta &= (T'_\alpha + qI)(T'_\beta + qI) \\ &= T'_\alpha T'_\beta + q(T'_\alpha + T'_\beta) + q^2 I \\ \text{and } T_\beta T_\alpha &= T'_\beta T'_\alpha + q(T'_\beta + T'_\alpha) + q^2 I \end{aligned}$$

so from Lemma 3.4 (II), we obtain the result.

$$( \text{ (II)}, T_\alpha T_\beta T_\alpha = T_\beta T_\alpha T_\beta )$$

$$\begin{aligned} T_\alpha T_\beta T_\alpha &= T'_\alpha T'_\beta T'_\alpha + q(T'_\alpha T'_\beta + T'_\beta T'_\alpha) + q^2(T'_\alpha + T'_\beta) + qT'^2_\alpha + q^2 T'_\alpha + q^3 I \\ &= T'_\alpha T'_\beta T'_\alpha + q(T'_\alpha T'_\beta + T'_\beta T'_\alpha) + q^2(T'_\alpha + T'_\beta) - qT'_\alpha + q^3 I \quad (\Leftarrow \text{ Lemma 3.4 (I)}) \\ &= q(T'_\alpha T'_\beta + T'_\beta T'_\alpha) + q^2(T'_\alpha + T'_\beta) + q^3 I \quad (\Leftarrow \text{ Lemma 3.4 (II)}) \end{aligned}$$

in the same way,

$$T_\beta T_\alpha T_\beta = q(T'_\beta T'_\alpha + T'_\alpha T'_\beta) + q^2(T'_\beta + T'_\alpha) + q^3 I$$

so we get the result.

$$( \text{ (III)}, T_\alpha T_\beta^* T_\alpha - T_\beta T_\alpha^* T_\beta = Q([T_\alpha, T_\alpha^*] - [T_\beta, T_\beta^*]) )$$

In the same way as the previous case,

$$\begin{aligned} T_\alpha T_\beta^* T_\alpha &= q(T'_\alpha T'^{*'}_\beta + T'^{*'}_\beta T'_\alpha + q^2(T'_\alpha + T'^{*'}_\beta) + q^3 I, \\ T_\beta T_\alpha^* T_\beta &= q(T'_\beta T'^{*'}_\alpha + T'^{*'}_\alpha T'_\beta) + q^2(T'_\beta + T'^{*'}_\alpha) + q^3 I, \\ \text{so, } T_\alpha T_\beta^* T_\alpha - T_\beta T_\alpha^* T_\beta &= q(T'_\alpha T'^{*'}_\beta + T'^{*'}_\beta T'_\alpha - T'_\beta T'^{*'}_\alpha - T'^{*'}_\alpha T'_\beta) + q^2(T'_\alpha + T'^{*'}_\beta - T'_\beta - T'^{*'}_\alpha) \\ &= q^{\frac{3}{2}}(T'_\beta - T'^{*'}_\beta + T'^{*'}_\alpha - T'_\alpha) + q^2(T'_\alpha + T'^{*'}_\beta - T'_\beta - T'^{*'}_\alpha) \quad (\Leftarrow \text{ Lemma 3.4 (III)}) \\ &= (q^2 - q^{\frac{3}{2}})((T'_\alpha - T'^{*'}_\alpha) - (T'_\beta - T'^{*'}_\beta)) \\ &= Q([T_\alpha, T_\alpha^*] - [T_\beta, T_\beta^*]) \quad (\Leftarrow \text{ Lemma 3.4 (II)}) \end{aligned}$$

$$( \text{ (III)}, T_\alpha^* T_\beta T_\alpha - T_\beta T_\alpha^* T_\beta = Q[T_\beta, T_\beta^*] )$$

$$\begin{aligned} T_\alpha^* T_\beta T_\alpha &= T'^{*'}_\alpha T'_\beta T'_\alpha + q(T'^{*'}_\alpha T'_\alpha + T'_\beta T'_\alpha + T'^{*'}_\alpha T'_\beta) + q^2(T'_\alpha + T'^{*'}_\alpha + T'_\beta) + q^3 I \\ &= (q-1)T'^{*'}_\alpha T'_\alpha + q(T'_\beta T'_\alpha + T'^{*'}_\alpha T'_\beta) + (q^2-1)T'^{*'}_\alpha + q^2(T'_\alpha + T'_\beta) + q^3 I \quad (\Leftarrow \text{ Lemma 3.4 (III)}) \\ &= q(T'_\beta T'_\alpha + T'^{*'}_\alpha T'_\beta) + q^2(T'_\alpha + T'_\beta) + q^3 I \quad (\Leftarrow \text{ Lemma 3.4 (II)}) \end{aligned}$$

$$\text{and similarly, } T_\beta T_\alpha^* T_\beta = q(T'_\alpha T'^{*'}_\beta + T'_\beta T'_\alpha) + q^2(T'^{*'}_\beta + T'_\alpha) + q^3 I$$

$$\begin{aligned} \text{so, } T_\alpha^* T_\beta T_\alpha - T_\beta T_\alpha^* T_\beta &= q(T'^{*'}_\alpha T'_\beta - T'_\alpha T'^{*'}_\beta) + q^2(T'_\beta - T'^{*'}_\beta) \\ &= q^{\frac{3}{2}}(T'^{*'}_\beta - T'_\beta) + q^2(T'_\beta - T'^{*'}_\beta) \quad (\Leftarrow \text{ Lemma 3.4 (III)}) \\ &= (q^{\frac{3}{2}} - q^2)(T'^{*'}_\beta - T'_\beta) \\ &= Q[T_\beta, T_\beta^*] \quad (\Leftarrow \text{ Lemma 3.4 (II)}) \end{aligned}$$

The relation  $T_\alpha^* T_\beta^* T_\alpha - T_\beta T_\alpha^* T_\beta = Q[T_\alpha, T_\alpha^*]$ , is similarly proved as the previous .  $\square$

## References

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