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CANONICAL REPRESENTATIONS AND BEREZIN KERNELS*

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The theory of so-called canonical representations is about to become a very interesting part of representation theory because it combines so many areas of mathematics: complex analysis, theory of Jordan algebras, harmonic analysis on symmetric spaces, etc. The terminology originates from a paper on SL(2,$\mathbf{R}$), where $\mathbf{R}$ is a ring of functions, by Vershik, Gel'fand and Graev [39]. But actually these representations were introduced by Berezin, around 1975. Our lecture presents an introduction to canonical representations, where we restrict as far as the group is concerned to $G = \mathrm{SU}(1,n)$. At certain points in the exposition we give an outlook to the general context. Among the main recent contributors to the subject are (in alphabetical order): Arazy, van Dijk,Engliš, Hille, Molchanov, Neretin, Olafsson, Ørsted, Pasquale, Peetre, Upmeier, Unterberger, Zhang (see References).

1 Outline of the exposition

We will discuss the following items:

1. Definition of the canonical representation π_λ, λ a real number
2. Asymptotic behaviour for $\lambda \to \infty$
3. Spectral decomposition of π_λ
4. Connection with tensor products and restrictions of holomorphic and anti-holomorphic representations

*Notes based on a lecture in Tabarka (Tunesia) 1999

2 Definition of π_λ

Let $G = \mathrm{SU}(1,n)$ be the group of $(n+1)\times(n+1)$ complex matrices of determinant 1, which leaves the following Hermitian form invariant

$$[x, y] = \overline{y}_0 x_0 - \overline{y}_1 x_1 - \ldots - \overline{y}_n x_n. \tag{2.1}$$

Let $K = \mathrm{S}(\mathrm{U}(1)\times \mathrm{U}(n))$ be the standard maximal compact subgroup of G and set $X = G/K$. X has several realizations; a *bounded* realization is given by

$$B = \{y \in \mathbf{C}^n : \|y\|^2 = |y_1|^2 + \ldots + |y_n|^2 < 1\}. \tag{2.2}$$

The group G acts transitively on B: let $g \in G$, $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with a : 1×1, b : $1\times n$, c : $n \times 1$, d : $n\times n$ matrices, then

$$g \cdot y = \frac{dy + c}{\langle b, y\rangle + a} \tag{2.3}$$

where $\langle b, y\rangle = b_1 y_1 + \ldots b_n y_n$. Observe that K is the stabilizer of $z = 0$. A G-invariant measure on B is given by

$$d\mu(y) = (1 - \|y\|^2)^{-(n+1)}\, dy. \tag{2.4}$$

Define for $\lambda \in \mathbf{R}$:

$$\psi_\lambda(g) = (1 - \|y\|^2)^\lambda \tag{2.5}$$

if $y = g \cdot 0$.

The function ψ_λ is left and right K-invariant and thus gives in a standard way rise to a G-invariant kernel on $X \times X$ by

$$\psi_\lambda(g_1^{-1} g_2) = \left\{ \frac{(1 - \|y\|^2)(1 - \|z\|^2)}{[1 - (y|z)]\,[1 - (z|y)]} \right\}^\lambda = B_\lambda(y, z) \tag{2.6}$$

if $z = g_1 \cdot 0$, $y = g_2 \cdot 0$, which is called the *Berezin* kernel of X. It is not difficult to see that B_λ is a positive-definite kernel for $\lambda \geq 0$, which means that for every finite set of complex numbers $\mu_1, \ldots, \mu_N$ and points $z_1, \ldots, z_N \in B$ the expression

$$\sum_{i,j}^{N} \mu_i \overline{\mu}_j\, B_\lambda(z_j, z_i) \tag{2.7}$$

is positive. There are several ways to associate to a G-invariant positive-definite kernel on $X \times X$ a unitary representation of G. The following method is appropriate. Let V be the complex linear space of all functions f of the form

$$f(z) = \sum_i \mu_i B_\lambda(z, z_i) \quad \text{(finite sum)}. \tag{2.8}$$

If $g(z) = \sum_j \nu_j B_\lambda(z, y_j)$, we define a scalar product by

$$(f|g) = \sum_{i,j} \mu_i \bar{\nu}_j B_\lambda(y_j, z_i). \tag{2.9}$$

This is really an inner product. Set $\mathcal{H}_\lambda$ for the Hilbert space completion of V with respect to this inner product and let G act by

$$\pi_\lambda(g)f(z) = f(g^{-1} \cdot z). \tag{2.10}$$

The representation π_λ is unitary and is called a *canonical* representation of G. The kernel B_λ or the function ψ_λ is called the reproducing kernel of π_λ.

3 Asymptotic behaviour of π_λ for $\lambda \to \infty$

The function

$$Q(z) = (1 - \|z\|^2)^{-1} \tag{3.1}$$

gives a parametrization of the K orbits on B (spheres), which is suitable for our purposes. Observe that $\psi_\lambda(z) = Q(z)^{-\lambda}$. We will consider the distribution

$$f \to \int_B Q(z)^{-\lambda} f(z) d\mu(z) \quad (f \in \mathcal{D}(B)) \tag{3.2}$$

for $\lambda \to \infty$. In order to get some finite answer, let T_λ be the normalized distribution

$$\frac{\Gamma(\lambda)}{\pi^n \, \Gamma(\lambda - n)} Q(z)^{-\lambda}. \tag{3.3}$$

Then one has

$$\langle T_\lambda, f \rangle = f(0) + \frac{1}{4\lambda} \Delta f(0) + \mathcal{O}(\frac{1}{\lambda^2}) \tag{3.4}$$

for $\lambda \to \infty$, where Δ is the G-invariant Laplace operator of B:

$$\Delta = 4(1 - \|z\|^2) \sum_{k,l}^{n} (\delta_{kl} - z_k \bar{z}_l) \frac{\partial^2}{\partial z_k \partial \bar{z}_l}. \tag{3.5}$$

So $T_\lambda \to \delta$, or π_λ tends to the regular representation of G on $L^2(B, d\mu)$. In terms of "Berezin quantization" this implies, putting $\lambda = \frac{1}{h}$, with h Planck's constant, that the correspondence principle holds [2].

4 Spectral decomposition of π_λ

In order to decompose π_λ into irreducible unitary representations, we expand ψ_λ into zonal spherical functions of positive type. This is an equivalent setting. For $\lambda > \rho$ $(\rho = n)$ this is easily done, since then ψ_λ belongs to $L^1(G) \cap L^2(G)$.

The formula reads as follows:

$$\psi_\lambda(g) = \frac{1}{2\pi K} \int_0^\infty a_\lambda(\mu)\, \varphi_{i\mu}(g)\, \frac{d\mu}{|c(i\mu)|^2} \tag{4.1}$$

with

$$a_\lambda(\mu) = \int_G \psi_\lambda(g)\, \varphi_{-i\mu}(g)\, dg. \tag{4.2}$$

Furthermore, K a constant depending on the normalization of measures and

$$c(s) = \Gamma(n) 2^{n-s} \frac{\Gamma(s)}{\Gamma((s+n)/2)^2}. \tag{4.3}$$

If A is the subgroup of G consisting of the matrices:

$$a_t = \begin{pmatrix} \cosh t & 0 & \sinh t \\ 0 & I_{n-1} & 0 \\ \sinh t & 0 & \cosh t \end{pmatrix} \tag{4.4}$$

one has the decomposition: $G = KA_+K$ or $D = KA_+ \cdot 0$ (the plus sign means $t \geq 0$), and

$$\psi_\lambda(a_t) = \cosh t^{-2\lambda}. \tag{4.5}$$

The spherical function $\varphi_{-i\mu}$ has an explicit form:

$$\varphi_{-i\mu}(a_t) = F(\frac{i\mu + n}{2}, \frac{-i\mu + n}{2}; n; -\sinh^2 t), \tag{4.6}$$

and

$$dg = \frac{2\pi^n}{\Gamma(n)} (\sinh t)^{2n-2} (\frac{\sinh 2t}{2})\, dk dt dk'. \tag{4.7}$$

We thus get to compute (setting $x = \sinh^2 t$):

$$a_\lambda(\mu) = \frac{\pi^n}{\Gamma(n)} \int_0^\infty F(\frac{i\mu + n}{2}, \frac{-i\mu + n}{2}; n; -x).$$

$$x^{n-1}(1+x)^{-\lambda}\, dx. \tag{4.8}$$

By [18], 20.2 (9) this is equal to

$$a_\lambda(\mu) = \pi^n \frac{\Gamma(\lambda + \frac{i\mu - n}{2})\,\Gamma(\lambda + \frac{-i\mu - n}{2})}{\Gamma(\lambda)^2}. \tag{4.9}$$

The theory, developed so far for SU$(1, n)$, has been extended to all *Hermitian symmetric spaces*. The decomposition of ψ_λ has been obtained by Berezin for the classical spaces and by Upmeier and Unterberger [37] for all spaces, but only for "large" λ (as by us, up to now).

We now consider the case $0 < \lambda \leq \rho$, where ψ_λ still is a positive-definite function. By analytic continuation in λ we have obtained, applying the Paley Wiener theorem for $D = G/K$, see [8]:

$$\begin{aligned}(\psi_\lambda, f) &= 2\pi \sum_{l, s_l > 0} r_l(\lambda)\,(\varphi_{s_l}, f) + \\ & \frac{1}{2\pi K} \int_0^\infty a_\lambda(i\mu)\,(\varphi_{-i\mu}, f)\,\frac{d\mu}{|c(i\mu)|^2}\end{aligned} \tag{4.10}$$

with

$$s_l(\lambda) = \rho - 2\lambda - 2l, \; l = 0, 1, 2, \ldots \tag{4.11}$$

and

$$r_l(\lambda) = \frac{2^{2\lambda + 2l - 2\rho}\pi^{n-1}}{\Gamma(n)} \frac{(\frac{-2\lambda - 2l + 2}{2})_l\,(1 - \lambda - l)_l}{(\rho - 2\lambda - 2l + 1)_l\, l!}. \tag{4.12}$$

So we pick up finitely many *complementary series* representations. The case SU(p, q) has recently been treated by Hille (thesis) [24]. How to proceed for a general Hermitian symmetric space is an open problem. For reference we give the list of irreducible Hermitian symmetric spaces below in Table 1, upper part.

5 Tensor products and restrictions of holomorphic and anti-holomorphic representations

There is a very nice application of canonical representations: their reproducing kernel can be identified with the reproducing kernel of tensor products of holomorphic and anti-holomorphic representations (note: not of two holomorphic representations)

The space $L^2(G/K, l)$

Denote by χ_l (l an integer) the character of K given by

$$\chi_l : \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} \to a^l \tag{5.1}$$

where $|a| = 1$, $d \in \mathrm{U}(n)$, $a \det d = 1$.

Let $\rho_l = \mathrm{Ind}_{K\uparrow G}\, \chi_l$ and V_l the space of ρ_l.

So $f \in V_l$ if

(i) $f : G \to \mathbf{C}$ is measurable,

(ii) $f(gk) = \chi_l(k^{-1}) f(g)$,

(ii) $\|f\|^2 = \int_{G/K} |f(g)|^2 \, d\mu(\overline{g}) < \infty$, where $\overline{g} = gK$.

Here $d\mu(\overline{g})$ is the invariant measure on $G/K \simeq B$. Instead of V_l one also uses the notation $L^2(G/K, l)$. G acts (via ρ_l) by left translations.

We shall identify V_l with a space of functions on the unit ball B in $\mathbf{C}^n$. Therefore, define

$$Af(g) = a^l f(g) \tag{5.2}$$

for $f \in L^2(G/K, l)$, $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Then $Af(gk) = Af(g)$ for all $k \in K$. So Af is defined on B and one has

$$\|f\|^2 = \int_B |Af(z)|^2 \, (1 - \|z\|^2)^l \, d\mu(z). \tag{5.3}$$

Let $\mathcal{H}_l$ denote the Hilbert space of all measurable functions φ on B such that

$$\int_B |\varphi(z)|^2 \, (1 - \|z\|^2)^l \, d\mu(z) < \infty. \tag{5.4}$$

$\mathcal{H}_l$ is a G-space; G acts unitarily in $\mathcal{H}_l$ by σ_l, given by

$$\sigma_l(g)\varphi(z) = \varphi(g^{-1} \cdot z) \, (\langle b, z \rangle + a)^{-l} \tag{5.5}$$

if $g^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

A is a unitary intertwining operator between ρ_l and σ_l.

The holomorphic discrete series; Fock spaces

For $\lambda \in \mathbf{R}$ consider the Fock space $\mathcal{F}_\lambda$ of holomorphic functions on B satisfying

$$\|f\|_\lambda^2 := \int_B |f(z)|^2 (1 - \|z\|^2)^\lambda \, d\mu(z) < \infty. \tag{5.6}$$

This space is non-trivial for $\lambda > \rho$ ($\rho = n$), since $\mathcal{F}_\lambda$ contains the function which is identically 1 in this case. One has

$$\|1\|_\lambda^2 = \frac{\pi^n}{2(\lambda - 1)\cdots(\lambda - n)}. \tag{5.7}$$

Moreover, $\mathcal{F}_\lambda$ is a closed subspace of $L^2(B, d\mu_\lambda)$, hence a Hilbert space, where

$$d\mu_\lambda(z) = (1 - \|z\|^2)^\lambda \, d\mu(z). \tag{5.8}$$

It also has a reproducing kernel, namely

$$E_\lambda(z, w) = \frac{(\lambda - 1)\cdots(\lambda - n)}{\pi^n} [1 - (w, z)]^{-\lambda}. \tag{5.9}$$

It is also a unitary module for the action of the universal covering group $\widetilde{G}$ of G; for integer λ ($\lambda > \rho$) it is even a G-module: a holomorphic discrete series representation of scalar type. The group G acts by

$$\tau_\lambda(g)f(z) = f\left(\frac{d \cdot z + c}{\langle b, z\rangle + a}\right)(\langle b, z\rangle + a)^{-\lambda}, \tag{5.10}$$

$g^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. τ_λ is an irreducible unitary representation.

Let us denote by $\overline{\mathcal{F}}_\lambda$ the space of complex conjugates of elements in $\mathcal{F}_\lambda$. It consists of anti-holomorphic functions and gives rise to an obvious unitary action $\overline{\tau}_\lambda$ of $\widetilde{G}$ as well. So

$$\overline{\tau}_\lambda(g)f(z) = f\left(\frac{d \cdot z + c}{\langle b, z\rangle + a}\right)\overline{(\langle b, z\rangle + a)}^{-\lambda} \tag{5.11}$$

if $g^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $f \in \overline{\mathcal{F}}_\lambda$, $\lambda \in \mathbf{Z}$.

For $\lambda \in \mathbf{N}$ ($\lambda > \rho$) we get part of the anti-holomorphic discrete series of scalar type.

Tensor products

Consider the Hilbert space tensor product

$$\mathcal{F}_\lambda \widehat{\otimes}_2 \overline{\mathcal{F}}_\lambda \tag{5.12}$$

with $\lambda > \rho$. It consists of functions $F \in L^2(B \times B, d\mu_\lambda \otimes d\mu_\lambda)$ such that $F(z,w)$ is holomorphic in z and anti-holomorphic in w. The group $\widetilde{G}$ acts diagonally. It turns out that we actually have a G-action, which for integer λ is given by

$$g_0 \cdot F(z,w) = F(g_0^{-1} \cdot z,\, g_0^{-1} \cdot w).(a + \langle b, z\rangle)^{-\lambda}\, \overline{(a + \langle b, w\rangle)}^{-\lambda} \tag{5.13}$$

if $g_0^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

Let A_λ denote the linear map $\mathcal{F}_\lambda \widehat{\otimes}_2 \overline{\mathcal{F}}_\lambda \to \mathcal{H}_0 = L^2(B, d\mu)$ given by

$$F(z,w) \to F(z,z)\,(1 - \|z\|^2)^\lambda \tag{5.14}$$

One has (see [6]):

- A_λ is a bounded, $\|A_\lambda\|^2 \leq 1/c_\lambda$, where $c_\lambda = \|1\|_\lambda^2$.
- A_λ is an intertwining operator: it intertwines the G-actions (5.13) on $\mathcal{F}_\lambda \widehat{\otimes}_2 \overline{\mathcal{F}}_\lambda$ and σ_0 on $\mathcal{H}_0$,
- A_λ has trivial kernel and dense image,

The adjoint of A_λ. The Berezin kernel

One can easily determine an explicit expression for A_λ^* and then for $A_\lambda\, A_\lambda^*$, which maps $L^2(B, d\mu)$ in itself.

One gets:

$$\begin{aligned} &A_\lambda\, A_\lambda^* f(z) = \\ &\int_B E_\lambda(z,y)\, E_\lambda(y,z)\, f(y)\, d\mu_\lambda(y)\, (1 - \|z\|^2)^\lambda. \end{aligned} \tag{5.15}$$

So $A_\lambda\, A_\lambda^*$ is a kernel operator with kernel

$$B_\lambda(z,y) = c_\lambda^{-2} \left\{ \frac{(1 - \|z\|^2)\,(1 - \|y\|^2)}{[1 - (z,y)]\,[1 - (y,z)]} \right\}^\lambda. \tag{5.16}$$

This is again the Berezin kernel (up to a factor); it is G-invariant, positive-definite, and defines a bounded Hermitian form on $L^2(G/K)$ for $\lambda > \rho$.

Restrictions

Consider the group $H = \mathrm{SO}_0(1,n)$ inside $G = \mathrm{SU}(1,n)$. It has a special property: $\mathrm{SO}_0(1,n)\cdot 0 = B(\mathbf{R}^n)$, the real unit ball, which is a *fully restrictive submanifold* of B. This means the following: if f is an entire holomorphic function on B and $f(x) = 0$ on $B(\mathbf{R}^n)$, then f is identically zero. Completely similar to the above one can restrict $f \in \mathcal{F}_\lambda$ to the real ball:

$$f(z) \to f(x)\,(1 - \|x\|^2)^{\lambda/2}. \tag{5.17}$$

Call this map A_λ again. Clearly A_λ is one-to-one.

Consider A_λ as a map $\mathcal{F}_\lambda \to \mathcal{D}'(B(\mathbf{R}^n)) \simeq \mathcal{D}'(H/H\cap K)$, the space of distributions on $B(\mathbf{R}^n)$ or $H/H\cap K$. A_λ is an intertwining operator for the H-actions, at least for $\lambda \in 2\mathbf{N}, \lambda > \rho$. It is easy to determine $A_\lambda^* : \mathcal{D}(B(\mathbf{R}^n)) \to \mathcal{F}_\lambda$ and then $A_\lambda A_\lambda^*$, which is again a kernel operator, with kernel

$$c_\lambda^{-1}\,\frac{(1-\|x\|^2)^{\lambda/2}\,(1-\|y\|^2)^{\lambda/2}}{[1-(x,y)]^\lambda}. \tag{5.18}$$

This kernel is H-invariant and positive-definite for $\lambda \geq 0$, and is thus given by a positive-definite bi-$K\cap H$-invariant function ψ_λ. One gets $\psi_\lambda(a_t) = \cosh t^{-\lambda}$ (up to a constant). One may call this the *Berezin kernel for* $H = \mathrm{SO}_0(1,n)$. The function ψ_λ is clearly in $L^1(H)$ for $\lambda > \rho$. Since $A_\lambda A_\lambda^*$ can be seen as a convolution operator on H: $A_\lambda A_\lambda^*\varphi = \varphi * \psi_\lambda$, φ right $K\cap H$-invariant, it is clear that $A_\lambda A_\lambda^*$ is a bounded map from $L^2(B(\mathbf{R}^n))$ into itself. Then it is obvious that $A_\lambda(\mathcal{F}_\lambda) \subset L^2(B(\mathbf{R}^n))$ and that A_λ is bounded for $\lambda > \rho$; indeed

$$\begin{aligned}|\langle A_\lambda f,\, \varphi\rangle|^2 &= |\langle f,\, A_\lambda^*\varphi\rangle|^2 \leq \|f\|_\lambda^2\,\|A_\lambda^*\varphi\|^2 \\ &= \|f\|_\lambda^2\,(A_\lambda A_\lambda^*\varphi\,|\varphi) \leq \|f\|_\lambda^2\,\|A_\lambda A_\lambda^*\|\,\|\varphi\|^2\end{aligned} \tag{5.19}$$

($f \in \mathcal{F}_\lambda$, $\varphi \in \mathcal{D}(B(\mathbf{R}^n))$).

This observation is due to B. Ørsted (unpublished).

In order to decompose the restriction of τ_λ ($\lambda \in 2\mathbf{N}, \lambda > \rho$) to $H = \mathrm{SO}_0(1,n)$, it is sufficient to decompose ψ_λ. This is done in [8], even for all $\lambda \geq 0$.

Observe that the tensor product case can be regarded as a similar restriction problem, namely from $G\times G$ to the diagonal $\{(g,g) : g \in G\} \simeq G$. It leads to the fully restrictive submanifold B in $B\times B$ (diagonally embedded).

This construction can be generalized to compactly causal symmetric pairs (see Table 1; $\mathfrak{g}$ is the Lie algebra of G, $\mathfrak{h}$ is the Lie algebra of the subgroup H where we restrict our holomorphic representation of G to). These pairs are studied by several people from a different point of view. The c-dual of G also has a meaning: it can be used for an alternative introduction of the Berezin kernel (see [9],[24, section 3.4]). So it is quite surprising that Table 1 has such an impact on our theory of canonical representations. The decomposition of the canonical representation (for λ large) for the subgroups H of Table 1 has recently been given by Neretin [30] for almost all classical groups and by van Dijk [6] for all tube type cases (by a different method). Here the case of small λ is a very interesting open problem too. The upper part of Table 1 has already been discussed in section 4.

Table 1: **Irreducible compactly causal pairs**

$\mathfrak{g}$ compactly causal	$\mathfrak{g}^c$ non-compactly causal	$\mathfrak{h}$
$\mathfrak{su}(p,q) \oplus \mathfrak{su}(p,q)$	$\mathfrak{sl}(p+q;\mathbf{C})$	$\mathfrak{su}(p,q)$
$\mathfrak{so}^*(2n) \oplus \mathfrak{so}^*(2n)$	$\mathfrak{so}(2n;\mathbf{C})$	$\mathfrak{so}^*(2n)$
$\mathfrak{so}(2,n) \oplus \mathfrak{so}(2,n)$	$\mathfrak{so}(2+n;\mathbf{C})$	$\mathfrak{so}(2,n)$
$\mathfrak{sp}(n,\mathbf{R}) \oplus \mathfrak{sp}(\mathbf{n},\mathbf{R})$	$\mathfrak{sp}(n,\mathbf{C})$	$\mathfrak{sp}(n,\mathbf{R})$
$\mathfrak{e}_{6(-14)} \oplus \mathfrak{e}_{6(-14)}$	$\mathfrak{e}_6$	$\mathfrak{e}_{6(-14)}$
$\mathfrak{e}_{7(-25)} \oplus \mathfrak{e}_{7(-25)}$	$\mathfrak{e}_7$	$\mathfrak{e}_{7(-25)}$
$\mathfrak{su}(p,q)$	$\mathfrak{sl}(p+q;\mathbf{R})$	$\mathfrak{so}(p,q)$
$\mathfrak{su}(n,n)$	$\mathfrak{su}(n,n)$	$\mathfrak{sl}(n;\mathbf{C}) \oplus \mathbf{R}$
$\mathfrak{su}(2p,2q)$	$\mathfrak{su}^*(2(p+q))$	$\mathfrak{sp}(p,q)$
$\mathfrak{so}^*(2n)$	$\mathfrak{so}(n,n)$	$\mathfrak{so}(n;\mathbf{C})$
$\mathfrak{so}^*(4n)$	$\mathfrak{so}^*(4n)$	$\mathfrak{su}^*(2n) \oplus \mathbf{R}$
$\mathfrak{so}(2,p+q)$	$\mathfrak{so}(p+1,q+1)$	$\mathfrak{so}(p,1) \times \mathfrak{so}(1,q)$
$\mathfrak{sp}(n,\mathbf{R})$	$\mathfrak{sp}(n,\mathbf{R})$	$\mathfrak{sl}(n;\mathbf{R}) \oplus \mathbf{R}$
$\mathfrak{sp}(2n,\mathbf{R})$	$\mathfrak{sp}(n,n)$	$\mathfrak{sp}(n,\mathbf{C})$
$\mathfrak{e}_{6(-14)}$	$\mathfrak{e}_{6(6)}$	$\mathfrak{sp}(2,2)$
$\mathfrak{e}_{6(-14)}$	$\mathfrak{e}_{6(-26)}$	$\mathfrak{f}_{4(-20)}$
$\mathfrak{e}_{7(-25)}$	$\mathfrak{e}_{7(-25)}$	$\mathfrak{e}_{6(-26)} \oplus \mathbf{R}$
$\mathfrak{e}_{7(-25)}$	$\mathfrak{e}_{7(7)}$	$\mathfrak{su}^*(8)$

(Faraut and Olafsson [20]).

6 Some notes

Here are some notes and remarks.

• Canonical representations have been introduced for classical Hermitian symmetric spaces by Berezin [2,3,4,5] and later, in a different context, by Vershik, Gel'fand and Graev for SL(2, **R**) [39,40].

• A more conceptual treatment for Hermitian symmetric spaces in the context of Jordan algebras has recently been given by Upmeier and Unterberger (1994) [37].

• An extension to hyperbolic spaces, also for small values of the parameters, and for line bundles over these spaces, is due to Hille and van Dijk (1995,1996) [8, 23]. For $G = \mathrm{SU}(p, q)$ see Hille's thesis [24].

• Canonical representations for para-Hermitian spaces were proposed and introduced by Molchanov (1996). He follows the alternative introduction, mentioned above [26].

• A thorough treatment of the rank one para-Hermitian symmetric space $\mathrm{SL}(n, \mathbf{R})/\mathrm{GL}(\mathbf{n-1}, \mathbf{R})$ is due to van Dijk and Molchanov (1998, 1999) [6,11].

References

[1] J. Arazy and B. Ørsted, Asymptotic expansions of Berezin forms. Preprint 1999.

[2] F.A. Berezin, Quantization. *Math. USSR Izv. 8*, 1974, pp. 1109–1165.

[3] F.A. Berezin, Quantization in complex symmetric space. *Math. USSR Izv. 9*, 1975, pp. 341–379.

[4] F.A. Berezin, Connection between co- and contravariant symbols of operators on the classical complex symmetric spaces. *Dokl. Akad. Nauk. USSR 241*, 1978, pp. 15–17. (In Russian; English translation in Soviet Math. Dokl. 19, 1978, pp. 786–789).

[5] F.A. Berezin and F.I. Karpelevic, Zonal spherical functions and Laplace operators on some symmetric spaces. *Dokl. Akad. Nauk USSR (N.S.) 118*, 1958, pp. 9–12 (In Russian).

[6] G. van Dijk, Berezin kernels on tube domains. *Report W97-07*, University of Leiden, 1997; *extended version* 1999.

[7] G. van Dijk, Canonical representations associated to hyperbolic spaces II. *Report W98-01*, University of Leiden, 1998. (To appear in Indag. Math.).

[8] G. van Dijk and S.C. Hille, Canonical representations related to hyperbolic spaces. *J. Funct. Anal. 147*, 1997, pp. 109–139.

[9] G. van Dijk and S.C. Hille, Maximal degenerate representations, Berezin kernels and canonical representations. *In Lie Groups and Lie Algebras, Their Representations, Generalizations and Applications.* Kluwer Academic, Dordrecht, 1997.

[10] G. van Dijk and V.F. Molchanov, Tensor products of maximal degenerate series representations of the group SL(n, **R**). *J. Math. Pures Appl. 78*, 1999, pp. 99–119.

[11] G. van Dijk and V.F. Molchanov, The Berezin form for rank one para-Hermitian symmetric spaces. *J. Math. Pures Appl. 77*, 1998, pp. 747–799.

[12] G. van Dijk and A.A. Pasquale, Canonical representations of Sp(1,n) associated with representations of Sp(1). To appear in Comm. Math. Phys. 1999.

[13] M. Engliš, Functions invariant under the Berezin transform. *J. Funct. Anal. 121*, 1994, pp. 223–254.

[14] M. Engliš, Asymptotics of the Berezin transform and quantization on planar domains. *Duke Math. J. 79*, 1995, pp. 57–76.

[15] M. Engliš and J. Peetre, On the correspondence principle for the quantized annulus. *Math. Scand. 78*, 1996, pp. 183–206.

[16] M. Engliš, Berezin quantization and reproducing kernels on complex domains. *Trans. Amer. Math. Soc. 348*, 1996, pp. 411–479.

[17] M. Engliš, Invariant operators and the Berezin transform on Cartan domains. *Math. Nachr. 195*, 1998, 61–75.

[18] A. Erdelyi, W. Magnus, F. Oberhettinger and F.G. Tricomi, Tables of integral transforms, vol. II. McGraw-Hill, New York, 1954.

[19] J. Faraut and A. Koranyi, Function spaces and reproducing kernels on bounded symmetric domains. *J. Funct. Anal. 88*, 1990, pp. 64–89.

[20] J. Faraut and A. Koranyi, Aanlysis on symmetric cones. *Oxford University Press*, 1994.

[21] J. Faraut and G. Olafsson, Causal semisimple symmetric spaces, the geometry and harmonic analysis. In: *Hofmann, Lawson and Vinberg, eds., Semigroups in Algebra, Geometry and Analysis.* De Gruyter, Berlin, 1995.

[22] L.I. Grosheva, Representations on distributions on the Lobachevsky plane concentrated at the boundary. *Tambov University Reports 3(1)*, 1998, pp. 46–49.

[23] S.C. Hille, Decomposition of tensor products of scalar unitary holomorphic and anti-holomorphic representations of the universal covering group of SU(1,n). *Report W98-12*, University of Leiden, 1998.

[24] S.C. Hille, Thesis, Leiden University, 1999.

[25] P.E.T. Jorgensen and G. Olafsson, Unitary representations of Lie groups with reflection symmetry. *J. Funct. Anal. 158*, 1998, pp. 26–88.

[26] V.F. Molchanov, Quantization on para-Hermitian symmetric spaces. *Adv. in Math. Sci., AMS Transl. Ser. 2, 175*, 1996, pp. 81–96.

[27] Yu. A. Neretin and G.I. Ol'shanskii, Boundary values of holomorphic functions, singular unitary representations of the group O(p,q) and their limits as $q \to \infty$. *Zap. Nauchn. Semin., POMIRAN, 233*, 1995, pp. 9–91 (Russian). Engl. transl.: J. Math. Sci. 1996.

[28] Yu. A. Neretin, Boundary values of holomorphic functions and some special problems for unitary representations. In: J. Hilgert, J.D. Lawson, K.-H. Neeb and E.B. Vinberg, eds., *Positivity in Lie Theory: Open Problems*, De Gruyter Expos. Math 26, pp. 221–248. De Gruyter, Berlin, 1998.

[29] Yu. A. Neretin, Pseudoriemannian symmetric spaces: one-type realizations and open embeddings to grassmannians. Preprint 1999.

[30] Yu. A. Neretin, Matrix analogs of the integral $\int_0^\infty \frac{x^{\alpha-1}\,dx}{(1+x)^\rho} = B(\alpha, \rho - \alpha)$ and Plancherel formula for Berezin kernel representations. Preprint 1999.

[31] G. Olafsson and B. Ørsted, Bargmann transform for symmetric spaces. Preprint 1999.

[32] B. Ørsted and G. Zhang, Weyl quantization and tensor products of Fock and Bergman spaces. *Indiana Math. J. 43*, 1994, pp. 551–583.

[33] B. Ørsted and G. Zhang, L^2-versions of the Howe correspondence I. *Math. Scand. 80*, 1997, pp. 115–160.

[34] B. Ørsted and G. Zhang, Tensor products of analytic continuations of holomorphic discrete series. *Can. J. Math. 49*, 1997, pp. 1224–1241.

[35] J. Peetre, The Berezin transform and Ha-plitz operators. *J. Operator Theory 24*, 1990, pp. 165–186.

[36] J. Repka, Tensor products of holomorphic discrete series representations. *Can, J. Math. 31*, 1979, pp. 836–844.

[37] A. Unterberger and H. Upmeier, Berezin transform and invariant differential operators. *Comm. Math. Phys. 164*, 1994, pp. 563–598.

[38] H. Upmeier, Jordan algebras in analysis, operator theory, and quantum mechanics. *CBMS-AMS Reg. Conf. Ser. in Math. 67*, 1987.

[39] A.M. Vershik, I.M. Gel'fand and M.I. Graev, Representations of the group SL(2,**R**) where **R** is a ring of functions. *Uspekhi Mat. Nauk, 28 (5)*,1973, pp. 83–128. English transl.: Russian Math. Surv. 28 (5), 1973, 1973.

[40] A.M. Vershik, I.M. Gel'fand and M.I. Graev, Representations of the group SL(2,**R**) where **R** is a ring of functions. In: *Representation Theory*, London Math. Soc. Lecture Notes 69, Cambridge Univ. Press, 1982.

[41] G. Zhang, Berezin transform on line bundles over bounded symmetric domains. Preprint 1998.

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