表現論シンポジウム講演集, 2000
pp.45-61

TWINING CHARACTER FORMULA FOR DEMAZURE MODULES

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0. Introduction.

Let $\mathfrak{g}$ be a finite-dimensional complex semi-simple Lie algebra with Cartan subalgebra $\mathfrak{h}$ and Borel subalgebra $\mathfrak{b} \supset \mathfrak{h}$. Let $\Delta \subset \mathfrak{h}^*$ be the set of roots of $\mathfrak{g}$ relative to $\mathfrak{h}$. We choose the set of positive roots Δ_+ such that the roots of $\mathfrak{b}$ are $-\Delta_+$. Let $\{\alpha_i \mid i \in I\}$ be the set of simple roots in Δ_+, $\{h_i \mid i \in I\}$ the set of simple coroots in $\mathfrak{h}$, $A = (a_{ij})_{i,j \in I}$ the Cartan matrix with $a_{ij} = \alpha_j(h_i)$, and $W = \langle r_i \mid i \in I\rangle \subset GL(\mathfrak{h}^*)$ the Weyl group. We take and fix a Chevalley basis $\{e_\alpha, f_\alpha \mid \alpha \in \Delta_+\} \cup \{h_i \mid i \in I\}$ of $\mathfrak{g}$, and let $\mathfrak{h}_{\mathbb{Z}} = \sum_{i \in I} \mathbb{Z} h_i$.

A bijection ω (of order N) of the index set I such that $a_{\omega(i),\omega(j)} = a_{ij}$ for all $i, j \in I$ induces a unique automorphism ω, called a (Dynkin) diagram automorphism, of the Lie algebra $\mathfrak{g}$ such that $\omega(e_{\alpha_i}) = e_{\alpha_{\omega(i)}}$, $\omega(f_{\alpha_i}) = f_{\alpha_{\omega(i)}}$, and $\omega(h_i) = h_{\omega(i)}$ for $i \in I$. We denote by $\langle\omega\rangle$ the cyclic subgroup (of order N) of $\mathrm{Aut}(\mathfrak{g})$ generated by the diagram automorphism ω. The restriction of ω to $\mathfrak{h}$ induces a transposed map $\omega^* \colon \mathfrak{h}^* \to \mathfrak{h}^*$, which stabilizes the integral weight lattice $\mathfrak{h}^*_{\mathbb{Z}} = \{\lambda \in \mathfrak{h}^* \mid \lambda(h_i) \in \mathbb{Z} \quad \text{for all } i \in I\} \simeq \mathrm{Hom}(\mathfrak{h}_{\mathbb{Z}}, \mathbb{Z})$. We set $\mathfrak{g}^0 = \{x \in \mathfrak{g} \mid \omega(x) = x\}$, $\mathfrak{h}^0 = \{h \in \mathfrak{h} \mid \omega(h) = h\}$, $W^\omega = \{w \in W \mid \omega^* w = w\omega^*\}$, $(\mathfrak{h}^*)^0 = \{\lambda \in \mathfrak{h}^* \mid \omega^*(\lambda) = \lambda\} \simeq (\mathfrak{h}^0)^*$, and $(\mathfrak{h}^*_{\mathbb{Z}})^0 = \{\lambda \in \mathfrak{h}^*_{\mathbb{Z}} \mid \omega^*(\lambda) = \lambda\}$.

Let $\widehat{\mathfrak{g}}$ be the orbit Lie algebra, which is the dual complex semi-simple Lie algebra of the fixed point (semi-simple) subalgebra $\mathfrak{g}^0$ of $\mathfrak{g}$, i.e., a complex semi-simple Lie algebra with the opposite Dynkin diagram to that of $\mathfrak{g}^0$. Let $\widehat{\mathfrak{h}}$ be the Cartan subalgebra of $\widehat{\mathfrak{g}}$, $\widehat{\mathfrak{b}} \supset \widehat{\mathfrak{h}}$ the Borel subalgebra, and $\widehat{\Delta}_+ \subset \widehat{\mathfrak{h}}^*$ the set of positive roots chosen so that the roots of $\widehat{\mathfrak{b}}$ are $-\widehat{\Delta}_+$. Let $\{\widehat{\alpha}_i \mid i \in \widehat{I}\}$ be the set of simple roots in $\widehat{\Delta}_+$ and $\widehat{W} = \langle \widehat{r}_i \mid i \in \widehat{I}\rangle \subset GL(\widehat{\mathfrak{h}}^*)$ the Weyl group, where the index set $\widehat{I}$ is a set of representatives of the ω-orbits in I. It is known that there exist an isomorphism of groups $\Theta : \widehat{W} \to W^\omega$ and a $\mathbb{C}$-linear isomorphism $P_\omega \colon \mathfrak{h}^0 \to \widehat{\mathfrak{h}}$ such that if $P_\omega^* \colon \widehat{\mathfrak{h}}^* \tilde{\to} (\mathfrak{h}^0)^* \simeq (\mathfrak{h}^*)^0$ is the transposed map of P_ω, then $\Theta(\widehat{w})|_{(\mathfrak{h}^*)^0} = P_\omega^* \circ \widehat{w} \circ (P_\omega^*)^{-1}$ for all $\widehat{w} \in \widehat{W}$. We set $w_i = \Theta(\widehat{r}_i) \in W^\omega$ for $i \in \widehat{I}$. In particular, $(W^\omega, \{w_i \mid i \in \widehat{I}\})$ forms a Coxeter system.

Our preprint is available from Naito's home page:
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For dominant $\lambda \in (\mathfrak{h}_{\mathbb{Z}}^*)^0$, let $L(\lambda)$ be the simple $\mathfrak{g}$-module of highest weight λ. It admits a unique $\mathbb{C}$-linear $\langle\omega\rangle$-action such that $\omega \cdot (xv) = \omega(x)(\omega \cdot v)$ for each $x \in \mathfrak{g}$, $v \in L(\lambda)$ and such that $\omega \cdot v_\lambda = v_\lambda$, where v_λ is a (nonzero) highest weight vector of $L(\lambda)$. So therefore does its dual module $L(\lambda)^* \simeq L(-w_0(\lambda))$ with w_0 the longest element in W. Let $\mathfrak{U}(\mathfrak{b})$ be the universal enveloping algebra of $\mathfrak{b}$, and for each $w \in W^\omega$, let $J_w(\lambda) = \mathfrak{U}(\mathfrak{b})v_{w(\lambda)}^* \subset L(\lambda)^*$ be Joseph's module of highest weight $-w(\lambda)$ in $L(\lambda)^*$, with $v_{w(\lambda)}^*$ a (nonzero) weight vector in $L(\lambda)^*$ of weight $-w(\lambda)$. Since $w \in W^\omega$, the weight vector $v_{w(\lambda)}^* \in L(\lambda)^*$ turns out to be fixed by the action of $\langle\omega\rangle$, and hence Joseph's module $J_w(\lambda) \subset L(\lambda)^*$ is $\langle\omega\rangle$-invariant. In the talk we will prove a formula of Demazure type for the twining character $\mathrm{ch}^\omega(J_w(\lambda))$ of $J_w(\lambda)$ defined by

$$\mathrm{ch}^\omega(J_w(\lambda)) = \sum_{\mu \in (\mathfrak{h}_{\mathbb{Z}}^*)^0} \mathrm{Tr}(\omega|_{J_w(\lambda)_\mu})\, e(\mu)$$

in the group algebra $\mathbb{C}[(\mathfrak{h}_{\mathbb{Z}}^*)^0]$ over $\mathbb{C}$ of $(\mathfrak{h}_{\mathbb{Z}}^*)^0$ with basis $e(\mu)$, $\mu \in (\mathfrak{h}_{\mathbb{Z}}^*)^0$. As a corollary, we will find a striking relation:

$$\mathrm{ch}^\omega(J_w(\lambda)) = P_\omega^*\Big(\mathrm{ch}\, \widehat{J}_{\widehat{w}}(\widehat{\lambda})\Big),$$

where $\widehat{w} = \Theta^{-1}(w) \in \widehat{W}$, $\widehat{\lambda} = (P_\omega^*)^{-1}(\lambda) \in \widehat{\mathfrak{h}}^*$, and $\mathrm{ch}\, \widehat{J}_{\widehat{w}}(\widehat{\lambda}) \in \mathbb{C}[\widehat{\mathfrak{h}}^*]$ is the ordinary character of Joseph's module $\widehat{J}_{\widehat{w}}(\widehat{\lambda})$ of highest weight $-\widehat{w}(\widehat{\lambda})$ over the orbit Lie algebra $\widehat{\mathfrak{g}}$.

Although our problem can be stated purely algebraically as above, it seems very difficult (at least for us) to solve it only by algebraic methods. Hence we resort to (algebro-) geometric methods. For that purpose, we introduce more notation. Let G be a connected, simply connected semi-simple linear algebraic group over $\mathbb{C}$ with maximal torus T and Borel subgroup $B \supset T$ such that $\mathrm{Lie}(G) = \mathfrak{g}$, $\mathrm{Lie}(T) = \mathfrak{h}$, and $\mathrm{Lie}(B) = \mathfrak{b}$. Then the character group $\Lambda = \mathrm{Hom}(T, GL_1)$ of T may be identified with $\mathfrak{h}_{\mathbb{Z}}^*$ by taking the differential at the identity element, i.e., by the map $\lambda \mapsto \mathrm{d}\lambda$. For each $i \in I$ and $\lambda \in \Lambda$, we will write $\langle\lambda, \alpha_i^\vee\rangle = (\mathrm{d}\lambda)(h_i)$, where $\alpha_i^\vee \in \mathrm{Hom}(GL_1, T)$ is the coroot of $\alpha_i \in \Lambda$. There exists an automorphism of G whose differential at the identity element coincides with the diagram automorphism ω of $\mathfrak{g}$ above. By abuse of notation, we will denote by ω this automorphism of G and by $\langle\omega\rangle$ the cyclic subgroup (of order N) of $\mathrm{Aut}(G)$ generated by ω. We will also denote the induced action of $\omega \in \langle\omega\rangle$ on Λ by the same letter ω, and set $\Lambda^\omega = \{\lambda \in \Lambda \mid \omega \cdot \lambda = \lambda\}$, $\Lambda_+^\omega = \{\lambda \in \Lambda^\omega \mid \langle\lambda, \alpha_i^\vee\rangle \geq 0 \quad \text{for all } i \in I\}$.

By a $G \rtimes \langle\omega\rangle$-module M, we will mean a finite-dimensional rational G-module that admits a $\mathbb{C}$-linear $\langle\omega\rangle$-action such that $\omega \cdot (gm) = \omega(g)(\omega \cdot m)$ for each $g \in G$ and $m \in M$. Regarding the semi-direct product $G \rtimes \langle\omega\rangle$ of G and $\langle\omega\rangle$ as a linear algebraic group, this is the same as a rational $G \rtimes \langle\omega\rangle$-module. Likewise for $B \rtimes \langle\omega\rangle$- and $T \rtimes \langle\omega\rangle$-modules. Let $\mathbb{C}[\Lambda^\omega]$ be the group algebra over $\mathbb{C}$ of Λ^ω with basis $e(\mu)$, $\mu \in \Lambda^\omega$. For a $T \rtimes \langle\omega\rangle$-module

V, we define the twining character $\mathrm{ch}^{\omega}(V) \in \mathbb{C}[\Lambda^{\omega}]$ of V to be

$$\mathrm{ch}^{\omega}(V) = \sum_{\mu \in \Lambda^{\omega}} \mathrm{Tr}(\omega|_{V_{\mu}})\, e(\mu),$$

where $V_{\mu} = \{v \in V \mid t\,v = \mu(t)v \quad \text{for all} \quad t \in T\}$ is the μ-weight space of V.

Recall that $W \simeq N_G(T)/T$, where $N_G(T)$ is the normalizer of T in G. Fix $w \in W^{\omega}$, and let $X(w)$ be the associated Schubert variety over $\mathbb{C}$, which is the Zariski closure in the flag variety G/B of the Bruhat cell $B\dot{w}B/B$, where $\dot{w}$ denotes a right coset representative of w in $N_G(T)$ fixed by $\omega \in \mathrm{Aut}(G)$. If M is a $B \rtimes \langle\omega\rangle$-module, then the B-equivariant $\mathcal{O}_{X(w)}$-module $\mathcal{L}_{X(w)}(M)$ associated to M carries a structure of "$(B, \langle\omega\rangle)$-equivariant" ($\doteqdot B \rtimes \langle\omega\rangle$-equivariant) sheaf, so that its cohomology groups $H^{\bullet}(X(w), \mathcal{L}_{X(w)}(M))$ are $B \rtimes \langle\omega\rangle$-modules. (The precise definition of a $(B, \langle\omega\rangle)$-equivariant sheaf will be given in the talk.)

For each $\lambda \in \Lambda^{\omega}$, we let $\mathbb{C}_{\lambda}$ denote the one-dimensional $B \rtimes \langle\omega\rangle$-module on which B acts via λ through the quotient $B \to T$ and $\langle\omega\rangle$ trivially. We call $H^0(X(w), \mathcal{L}_{X(w)}(\mathbb{C}_{\lambda}))$ for $\lambda \in \Lambda^{\omega}_{+}$ a Demazure module. Joseph's module $J_w(\lambda)$ admits a structure of $B \rtimes \langle\omega\rangle$-module, and we have an isomorphism of $B \rtimes \langle\omega\rangle$-modules

$$J_w(\lambda)^* \simeq H^0(X(w), \mathcal{L}_{X(w)}(\mathbb{C}_{\lambda})),$$

where $J_w(\lambda)^*$ is the dual $B \rtimes \langle\omega\rangle$-module of $J_w(\lambda)$.

For $i \in \widehat{I}$, we define the ω-Demazure operator $\widehat{D}_i$ to be a $\mathbb{C}$-linear endomorphism of $\mathbb{C}[\Lambda^{\omega}]$ such that

$$\widehat{D}_i(e(\mu)) = \frac{e(\mu) - e(-s_i\beta_i)e(w_i(\mu))}{1 - e(-s_i\beta_i)} \quad \text{for } \mu \in \Lambda^{\omega},$$

where $\beta_i = \sum_{k=0}^{N_i-1} \alpha_{\omega^k(i)}$ and $s_i = 2/\sum_{k=0}^{N_i-1} a_{i,\omega^k(i)}$ with N_i the number of elements of the ω-orbit of $i \in I$.

The following is our main result.

THEOREM 0.1. *Let M be a finite-dimensional rational $B \rtimes \langle\omega\rangle$-module, $w \in W^{\omega}$, and let $w = w_{i_1} w_{i_2} \cdots w_{i_n}$ be a reduced expression in the Coxeter system $(W^{\omega}, \{w_i \mid i \in \widehat{I}\})$. Then we have in $\mathbb{C}[\Lambda^{\omega}]$,*

$$\begin{aligned}\chi^{\omega}(M) &= \sum_{j \geq 0} (-1)^j\, \mathrm{ch}^{\omega}(H^j(X(w), \mathcal{L}_{X(w)}(M))) \\ &= \widehat{D}_{i_1} \widehat{D}_{i_2} \cdots \widehat{D}_{i_n}\Big(\mathrm{ch}^{\omega}(M)\Big).\end{aligned}$$

In particular, for $\lambda \in \Lambda_+^\omega$, we have

$$\mathrm{ch}^\omega(H^0(X(w), \mathcal{L}_{X(w)}(\mathbb{C}_\lambda))) = \widehat{D}_{i_1}\widehat{D}_{i_2}\cdots\widehat{D}_{i_n}(e(\lambda)).$$

There is thus revealed a striking relation between twining characters for $\mathfrak{g}$ and ordinary characters for the orbit Lie algebra $\widehat{\mathfrak{g}}$. Let $\widehat{\mathfrak{h}}_\mathbb{Z} = \sum_{i\in\widehat{I}} \mathbb{Z}\widehat{h}_i$ and $\widehat{\mathfrak{h}}_\mathbb{Z}^* = \mathrm{Hom}(\widehat{\mathfrak{h}}_\mathbb{Z}, \mathbb{Z}) \subset \widehat{\mathfrak{h}}^*$. For dominant $\widehat{\lambda} \in \widehat{\mathfrak{h}}_\mathbb{Z}^*$, let $\widehat{L}(\widehat{\lambda})$ be the simple $\widehat{\mathfrak{g}}$-module of highest weight $\widehat{\lambda}$, and for each $\widehat{w} \in \widehat{W}$, let $\widehat{J}_{\widehat{w}}(\widehat{\lambda}) = \mathfrak{U}(\widehat{\mathfrak{b}})\,\widehat{v}^*_{\widehat{w}(\widehat{\lambda})} \subset \widehat{L}(\widehat{\lambda})^*$ be Joseph's module of highest weight $-\widehat{w}(\widehat{\lambda})$, with $\widehat{v}^*_{\widehat{w}(\widehat{\lambda})} \in \widehat{L}(\widehat{\lambda})^*$ a (nonzero) weight vector of weight $-\widehat{w}(\widehat{\lambda})$.

COROLLARY 0.2. *Let $\lambda \in (\mathfrak{h}_\mathbb{Z}^*)^0$ be dominant and $w \in W^\omega$. We set $\widehat{w} = \Theta^{-1}(w) \in \widehat{W}$ and $\widehat{\lambda} = (P_\omega^*)^{-1}(\lambda) \in \widehat{\mathfrak{h}}_\mathbb{Z}^*$. Then we have in $\mathbb{C}[(\mathfrak{h}_\mathbb{Z}^*)^0]$,*

$$\mathrm{ch}^\omega(J_w(\lambda)) = P_\omega^*\left(\mathrm{ch}\,\widehat{J}_{\widehat{w}}(\widehat{\lambda})\right),$$

where P_ω^* on the right-hand side is a $\mathbb{C}$-algebra isomorphism $\mathbb{C}[\widehat{\mathfrak{h}}_\mathbb{Z}^*] \tilde{\rightarrow} \mathbb{C}[(\mathfrak{h}_\mathbb{Z}^*)^0]$ defined by $P_\omega^*(e(\widehat{\mu})) = e(P_\omega^*(\widehat{\mu}))$ for each basis element $e(\widehat{\mu})$, $\widehat{\mu} \in \widehat{\mathfrak{h}}_\mathbb{Z}^*$, of the group algebra $\mathbb{C}[\widehat{\mathfrak{h}}_\mathbb{Z}^*]$ over $\mathbb{C}$ of $\widehat{\mathfrak{h}}_\mathbb{Z}^*$.

1. PRELIMINARIES.

1.1. **Diagram automorphisms.** Let $\mathfrak{g}$ be a finite-dimensional complex semi-simple Lie algebra with Cartan subalgebra $\mathfrak{h}$ and Borel subalgebra $\mathfrak{b} \supset \mathfrak{h}$. Let $\Delta \subset \mathfrak{h}^*$ be the set of roots of $\mathfrak{g}$ relative to $\mathfrak{h}$. We choose the set of positive roots Δ_+ such that the roots of $\mathfrak{b}$ are $-\Delta_+$. Let $\{\alpha_i \mid i \in I\}$ be the set of simple roots in Δ_+, $\{h_i \mid i \in I\}$ the set of simple coroots in $\mathfrak{h}$, $A = (a_{ij})_{i,j\in I}$ the Cartan matrix with $a_{ij} = \alpha_j(h_i)$, and $W = \langle r_i \mid i \in I\rangle \subset GL(\mathfrak{h}^*)$ the Weyl group. We take and fix a Chevalley basis $\{e_\alpha, f_\alpha \mid \alpha \in \Delta_+\} \cup \{h_i \mid i \in I\}$ of $\mathfrak{g}$, and let $\mathfrak{h}_\mathbb{Z} = \sum_{i\in I} \mathbb{Z}h_i$.

We fix a bijection $\omega\colon I \to I$ of the index set I such that

$$a_{\omega(i),\omega(j)} = a_{ij} \quad \text{for all } i, j \in I.$$

Let N be the order of ω, and N_i the number of elements of the ω-orbit of $i \in I$. This ω can be extended in a unique way to an automorphism (also denoted by ω) of order N of the Lie algebra $\mathfrak{g}$ in such a way that

$$\begin{cases} \omega(e_{\alpha_i}) = e_{\alpha_{\omega(i)}}, & i \in I, \\ \omega(f_{\alpha_i}) = f_{\alpha_{\omega(i)}}, & i \in I, \\ \omega(h_i) = h_{\omega(i)}, & i \in I. \end{cases}$$

Note that the restriction of ω to the Cartan subalgebra $\mathfrak{h}$ induces a transposed map $\omega^*\colon \mathfrak{h}^* \to \mathfrak{h}^*$ such that $\omega^*(\lambda)(h) = \lambda(\omega(h))$ for $\lambda \in \mathfrak{h}^*$, $h \in \mathfrak{h}$. We set

$$(\mathfrak{h}^*)^0 = \{\lambda \in \mathfrak{h}^* \mid \omega^*(\lambda) = \lambda\} \quad \text{and} \quad (\mathfrak{h}^*_{\mathbb{Z}})^0 = \{\lambda \in \mathfrak{h}^*_{\mathbb{Z}} \mid \omega^*(\lambda) = \lambda\},$$

where $\mathfrak{h}^*_{\mathbb{Z}} = \{\lambda \in \mathfrak{h}^* \mid \lambda(h_i) \in \mathbb{Z} \ \text{ for all } i \in I\} \simeq \mathrm{Hom}(\mathfrak{h}_{\mathbb{Z}}, \mathbb{Z})$. Note that the Weyl vector $\rho = (1/2) \cdot \sum_{\alpha \in \Delta_+} \alpha$ is in $(\mathfrak{h}^*_{\mathbb{Z}})^0$.

1.2. **Orbit Lie algebras.** We choose and fix a set $\widehat{I}$ of representatives of the ω-orbits in I, and set $\widehat{A} = (\widehat{a}_{ij})_{i,j \in \widehat{I}}$, where $\widehat{a}_{ij}$ is given by

$$\widehat{a}_{ij} = s_j \times \sum_{k=0}^{N_j - 1} a_{i,\omega^k(j)} \quad \text{for } i, j \in \widehat{I} \quad \text{with} \quad s_j = \frac{2}{\sum_{k=0}^{N_j-1} a_{j,\omega^k(j)}} \quad \text{for } j \in \widehat{I}.$$

Set for each $i \in \widehat{I}$, $I_i = \{\omega^k(i) \mid 0 \le k \le N_i - 1\} \subset I$. We know that for each $i \in \widehat{I}$,

$$\sum_{k \in I_i} a_{ik} = 1 \ \text{ or } \ 2.$$

Moreover, there are only two possibilities:

(a) if $\sum_{k \in I_i} a_{ik} = 1$, then N_i is even and the subgraph of the Dynkin diagram corresponding to the subset $I_i \subset I$ is of type $A_2 \times \cdots \times A_2$ (where A_2 appears $N_i/2$ times);
(b) if $\sum_{k \in I_i} a_{ik} = 2$, then the subgraph of the Dynkin diagram corresponding to the subset $I_i \subset I$ is totally disconnected and of type $A_1 \times \cdots \times A_1$ (where A_1 appears N_i times).

The orbit Lie algebra associated to the diagram automorphism $\omega \in \mathrm{Aut}(\mathfrak{g})$ is defined to be the complex semi-simple Lie algebra $\widehat{\mathfrak{g}}$ associated to the Cartan matrix $\widehat{A} = (\widehat{a}_{ij})_{i,j \in \widehat{I}}$ with the Cartan subalgebra $\widehat{\mathfrak{h}}$, the Borel subalgebra $\widehat{\mathfrak{b}} \supset \widehat{\mathfrak{h}}$, the set of positive roots $\widehat{\Delta}_+ \subset \widehat{\mathfrak{h}}^*$ chosen so that the roots of $\widehat{\mathfrak{b}}$ are $-\widehat{\Delta}_+$, the set of simple roots $\{\widehat{\alpha}_i \mid i \in \widehat{I}\} \subset \widehat{\mathfrak{h}}^*$, the set of simple coroots $\{\widehat{h}_i \mid i \in \widehat{I}\} \subset \widehat{\mathfrak{h}}$, and the Weyl group $\widehat{W} = \langle \widehat{r}_i \mid i \in \widehat{I} \rangle \subset GL(\widehat{\mathfrak{h}}^*)$.

Remark 1.2.1. We can easily deduce that the orbit Lie algebra $\widehat{\mathfrak{g}}$ is the dual complex semi-simple Lie algebra of the fixed point (semi-simple) subalgebra $\mathfrak{g}^0 = \{x \in \mathfrak{g} \mid \omega(x) = x\}$ of $\mathfrak{g}$, i.e., a complex semi-simple Lie algebra which has the opposite Dynkin diagram to that of $\mathfrak{g}^0$.

We set $\mathfrak{h}^0 = \{h \in \mathfrak{h} \mid \omega(h) = h\}$. Then there exists a linear isomorphism $P_\omega : \mathfrak{h}^0 \to \widehat{\mathfrak{h}}$ given by

$$P_\omega\left(\sum_{k \in I_i} h_k\right) = N_i \widehat{h}_i \quad \text{for each } i \in \widehat{I}.$$

This map $P_\omega : \mathfrak{h}^0 \tilde{\to} \widehat{\mathfrak{h}}$ induces a transposed map $P_\omega^* : \widehat{\mathfrak{h}}^* \tilde{\to} (\mathfrak{h}^0)^* \simeq (\mathfrak{h}^*)^0$ such that

$$P_\omega^*(\widehat{\lambda})(h) = \widehat{\lambda}(P_\omega(h)) \quad \text{for } \widehat{\lambda} \in \widehat{\mathfrak{h}}^*,\ h \in \mathfrak{h}^0.$$

Note that, if $\widehat{\mathfrak{h}}_{\mathbb{Z}} = \sum_{i \in \widehat{I}} \mathbb{Z}\widehat{h}_i$ and $\widehat{\mathfrak{h}}_{\mathbb{Z}}^* = \mathrm{Hom}(\widehat{\mathfrak{h}}_{\mathbb{Z}}, \mathbb{Z}) \subset \widehat{\mathfrak{h}}^*$, then $P_\omega^*(\widehat{\mathfrak{h}}_{\mathbb{Z}}^*) = (\mathfrak{h}_{\mathbb{Z}}^*)^0$.

We now define the subgroup W^ω of W by

$$W^\omega = \{w \in W \mid \omega^* w = w \omega^*\}.$$

It is known that there exists an isomorphism of groups $\Theta : \widehat{W} \to W^\omega$ from the Weyl group $\widehat{W}$ of the orbit Lie algebra $\widehat{\mathfrak{g}}$ onto the group W^ω such that the following diagram commutes for each $\widehat{w} \in \widehat{W}$:

$$\begin{array}{ccc} \widehat{\mathfrak{h}}^* & \xrightarrow{P_\omega^*} & (\mathfrak{h}^*)^0 \\ {\scriptstyle \widehat{w}}\downarrow & & \downarrow{\scriptstyle \Theta(\widehat{w})|_{(\mathfrak{h}^*)^0}} \\ \widehat{\mathfrak{h}}^* & \xrightarrow[P_\omega^*]{} & (\mathfrak{h}^*)^0. \end{array}$$

For each $i \in \widehat{I}$, set $w_i = \Theta(\widehat{r}_i) \in W^\omega$. Explicitly,

$$w_i = \begin{cases} \displaystyle\prod_{k=0}^{N_i/2-1} \left(r_{\omega^k(i)} r_{\omega^{k+N_i/2}(i)} r_{\omega^k(i)}\right) & \text{if } \displaystyle\sum_{k=0}^{N_i-1} a_{i,\omega^k(i)} = 1, \\ \displaystyle\prod_{k=0}^{N_i-1} r_{\omega^k(i)} & \text{if } \displaystyle\sum_{k=0}^{N_i-1} a_{i,\omega^k(i)} = 2. \end{cases}$$

Hence each w_i is the longest element of the subgroup W_{I_i} of the Weyl group W generated by the r_k's for $k \in I_i$. Furthermore, $(W^\omega, \{w_i \mid i \in \widehat{I}\})$ forms a Coxeter system as $(\widehat{W}, \{\widehat{r}_i \mid i \in \widehat{I}\})$ is. We will denote the length function of the Coxeter system $(W, \{r_i \mid i \in I\})$ (resp. $(W^\omega, \{w_i \mid i \in \widehat{I}\})$) by $\ell : W \to \mathbb{Z}_{\geq 0}$ (resp. $\widehat{\ell} : W^\omega \to \mathbb{Z}_{\geq 0}$).

Remark 1.2.2. Note that the longest element $w_0 \in W$ belongs to W^ω. In fact, we can easily show that the isomorphism $\Theta : \widehat{W} \tilde{\to} W^\omega$ maps the longest element $\widehat{w}_0 \in \widehat{W}$ to the longest element $w_0 \in W$.

1.3. **The ω-Demazure operators.** Recall the ordinary Demazure operator D_i for $i \in I$ on the group ring $\mathbb{Z}[\mathfrak{h}_{\mathbb{Z}}^*] = \coprod_{\lambda \in \mathfrak{h}_{\mathbb{Z}}^*} \mathbb{Z}\, e(\lambda)$:

$$D_i : e(\lambda) \mapsto \frac{e(\lambda) - e(-\alpha_i) e(r_i(\lambda))}{1 - e(-\alpha_i)}.$$

Let $\mathbb{C}[\widehat{\mathfrak{h}}_{\mathbb{Z}}^{*}]$ be the group algebra over $\mathbb{C}$ of $\widehat{\mathfrak{h}}_{\mathbb{Z}}^{*}$ with basis $e(\widehat{\lambda})$, $\widehat{\lambda} \in \widehat{\mathfrak{h}}_{\mathbb{Z}}^{*}$. Define likewise the Demazure operator $D_{\widehat{r}_i}$, $i \in \widehat{I}$, on $\mathbb{C}[\widehat{\mathfrak{h}}_{\mathbb{Z}}^{*}]$ to be the $\mathbb{C}$-linear endomorphism of $\mathbb{C}[\widehat{\mathfrak{h}}_{\mathbb{Z}}^{*}]$ given by

$$D_{\widehat{r}_i}(e(\widehat{\lambda})) = \frac{e(\widehat{\lambda}) - e(-\widehat{\alpha}_i)e(\widehat{r}_i(\widehat{\lambda}))}{1 - e(-\widehat{\alpha}_i)}.$$

Then transfer $D_{\widehat{r}_i}$ via P_{ω}^{*} onto the group algebra $\mathbb{C}[(\mathfrak{h}_{\mathbb{Z}}^{*})^{0}]$ to define the ω-Demazure operator

$$\widehat{D}_i = P_{\omega}^{*} \circ D_{\widehat{r}_i} \circ (P_{\omega}^{*})^{-1} \quad \text{for } i \in \widehat{I}. \tag{1.3.1}$$

Thus we can easily check the following.

LEMMA 1.3.1. *Let $i \in \widehat{I}$. For each $\lambda \in (\mathfrak{h}_{\mathbb{Z}}^{*})^{0}$, we have*

$$\widehat{D}_i(e(\lambda)) = \frac{e(\lambda) - e(-s_i\beta_i)e(w_i(\lambda))}{1 - e(-s_i\beta_i)},$$

and moreover

$$\widehat{D}_i(e(\lambda)) = \begin{cases} e(\lambda) + e(\lambda - s_i\beta_i) + \cdots + e(w_i(\lambda)) & \text{if } \lambda(h_i) \in \mathbb{Z}_{\geq 0}, \\ 0 & \text{if } \lambda(h_i) = -1, \\ -\Big(e(\lambda + s_i\beta_i) + e(\lambda + 2s_i\beta_i) + \cdots + e(w_i(\lambda + s_i\beta_i))\Big) & \text{if } \lambda(h_i) \in \mathbb{Z}_{\leq -2}. \end{cases}$$

Remark 1.3.2. Let $w = w_{i_1}w_{i_2}\cdots w_{i_n}$ be a reduced expression of $w \in W^{\omega}$ in the Coxeter system $(W^{\omega}, \{w_i \mid i \in \widehat{I}\})$, i.e., $\ell(w) = n$. We set $\widehat{D}_w = \widehat{D}_{i_1}\widehat{D}_{i_2}\cdots\widehat{D}_{i_n} \in \mathrm{End}_{\mathbb{C}}(\mathbb{C}[(\mathfrak{h}_{\mathbb{Z}}^{*})^{0}])$, which does not depend on the choice of the reduced expression of $w \in W^{\omega}$.

1.4. Twining characters.

Let G be a connected, simply connected semi-simple linear algebraic group over $\mathbb{C}$ with maximal torus T and Borel subgroup $B \supset T$ such that $\mathrm{Lie}(G) = \mathfrak{g}$, $\mathrm{Lie}(T) = \mathfrak{h}$, and $\mathrm{Lie}(B) = \mathfrak{b}$. Then the character group $\Lambda = \mathrm{Hom}(T, GL_1)$ of T may be identified with $\mathfrak{h}_{\mathbb{Z}}^{*}$ by taking the differential at the identity element, i.e., by the map $\lambda \mapsto \mathrm{d}\lambda$. For each $i \in I$ and $\lambda \in \Lambda$, we will write $\langle \lambda, \alpha_i^{\vee} \rangle = (\mathrm{d}\lambda)(h_i)$, where $\alpha_i^{\vee} \in \mathrm{Hom}(GL_1, T)$ is the coroot of $\alpha_i \in \Lambda$. Let $\Lambda_{+} = \{\lambda \in \Lambda \mid \langle \lambda, \alpha_i^{\vee} \rangle \geq 0 \quad \text{for all } i \in I\}$ be the set of dominant weights of Λ.

There exists an automorphism of G whose differential at the identity element coincides with the diagram automorphism ω of $\mathfrak{g}$. By abuse of notation, we will denote still by ω this automorphism of G and by $\langle \omega \rangle$ the cyclic subgroup (of order N) of $\mathrm{Aut}(G)$ generated by the ω. Whenever there can be ambiguity, we will write $\mathrm{d}\omega$ for the automorphism of $\mathfrak{g}$. Recall also that the Weyl group $W \subset GL(\mathfrak{h}^{*})$ may be identified with $N_G(T)/T$, $N_G(T)$ the normalizer of T in G. Each $w \in W^{\omega}$ lifts to an element of $N_G(T)$ fixed by

$\omega \in \mathrm{Aut}(G)$, which will be denoted by $\dot{w}$. We will also denote the induced action of ω on Λ by the same letter ω, and set $\Lambda^{\omega} = \{\lambda \in \Lambda \mid \omega \cdot \lambda = \lambda\}$, $\Lambda^{\omega}_{+} = \Lambda^{\omega} \cap \Lambda_{+}$. Note that, under the identification $\Lambda \simeq \mathfrak{h}^*_{\mathbb{Z}} \subset \mathfrak{h}^*$, this action of ω on Λ coincides with the restriction of $((\mathrm{d}\omega)^{-1})^* = ((\mathrm{d}\omega)^*)^{-1}$ to $\mathfrak{h}^*_{\mathbb{Z}}$.

By a $G \rtimes \langle\omega\rangle$-module M, we will always mean a finite-dimensional rational G-module that admits a $\mathbb{C}$-linear $\langle\omega\rangle$-action such that

$$\omega \cdot (g m) = \omega(g)(\omega \cdot m) \quad \text{for all } g \in G,\ m \in M.$$

Regarding the semi-direct product $G \rtimes \langle\omega\rangle$ of G and $\langle\omega\rangle$ as a linear algebraic group, this is the same as a finite-dimensional rational $G \rtimes \langle\omega\rangle$-module. Likewise for $B \rtimes \langle\omega\rangle$- and $T \rtimes \langle\omega\rangle$-modules. Let $\mathbb{C}[\Lambda^{\omega}]$ be the group algebra over $\mathbb{C}$ of Λ^{ω} with basis $e(\lambda)$, $\lambda \in \Lambda^{\omega}$. Let M be a $T \rtimes \langle\omega\rangle$-module, and let

$$M = \coprod_{\lambda \in \Lambda} M_{\lambda} \quad \text{with} \quad M_{\lambda} = \{m \in M \mid t m = \lambda(t) m \quad \text{for all } t \in T\}$$

be the weight space decomposition with respect to T. Now we define the twining character $\mathrm{ch}^{\omega}(M)$ of M to be

$$\mathrm{ch}^{\omega}(M) = \sum_{\lambda \in \Lambda^{\omega}} \mathrm{Tr}(\omega|_{M_{\lambda}})\, e(\lambda) \in \mathbb{C}[\Lambda^{\omega}].$$

Remark 1.4.1. It easily follows that for each $t \in T$,

$$\mathrm{Tr}((t, \omega)\ ;\ M) = \sum_{\lambda \in \Lambda^{\omega}} \mathrm{Tr}(\omega|_{M_{\lambda}})\, \lambda(t) \in \mathbb{C}$$

since $\omega \cdot M_{\lambda} = M_{\omega(\lambda)}$ for $\lambda \in \Lambda$.

1.5. **An important example.** Let $\lambda \in \Lambda^{\omega}_{+}$ and $L(\lambda)$ the simple rational G-module of highest weight λ. We can make $L(\lambda)$ into a $G \rtimes \langle\omega\rangle$-module as follows. Let v_{λ} be a (nonzero) highest weight vector of $L(\lambda)$. If $\mathfrak{U}(\mathfrak{g})$ is the universal enveloping algebra of $\mathfrak{g}$, there is an isomorphism of $\mathfrak{U}(\mathfrak{g})$-modules

$$\mathfrak{U}(\mathfrak{g})/\mathfrak{J}(\lambda) \simeq L(\lambda) \quad \text{via } x \mapsto x\, v_{\lambda},$$

where $\mathfrak{J}(\lambda)$ is the left ideal of $\mathfrak{U}(\mathfrak{g})$ given by

$$\mathfrak{J}(\lambda) = \sum_{i \in I} \Big(\mathfrak{U}(\mathfrak{g}) e_i + \mathfrak{U}(\mathfrak{g})(h_i - \langle \lambda, \alpha_i^{\vee} \rangle) + \mathfrak{U}(\mathfrak{g}) f_i^{\langle \lambda, \alpha_i^{\vee} \rangle + 1} \Big).$$

Since $\lambda \in \Lambda^{\omega}$, the left ideal $\mathfrak{J}(\lambda)$ of $\mathfrak{U}(\mathfrak{g})$ is ω-invariant, i.e., $\mathrm{d}\omega$-invariant, and hence $L(\lambda)$ admits a structure of $\langle\omega\rangle$-module such that

$$\omega \cdot (x\, v_{\lambda}) = \Big((\mathrm{d}\omega)(x) \Big) v_{\lambda} \quad \text{for all } x \in \mathfrak{U}(\mathfrak{g}).$$

Therefore, the $L(\lambda)$ admits a structure of $G \rtimes \langle\omega\rangle$-module such that $\omega \cdot v_\lambda = v_\lambda$. Note that a $G \rtimes \langle\omega\rangle$-module structure on $L(\lambda)$ such that $\omega \cdot v_\lambda = v_\lambda$ is unique since $L(\lambda)$ is a cyclic G-module generated by v_λ.

On the other hand, for each $i \in \widehat{I}$, we have $(P_\omega^*)^{-1}(\lambda)(\widehat{h}_i) = \lambda(h_i)$. Hence $\widehat{\lambda} = (P_\omega^*)^{-1}(\lambda) \in \widehat{\mathfrak{h}}^*$ is dominant integral. If $\widehat{L}(\widehat{\lambda})$ is the simple $\widehat{\mathfrak{g}}$-module of highest weight $\widehat{\lambda}$, we know that

$$\mathrm{ch}^\omega(L(\lambda)) = P_\omega^*\Big(\mathrm{ch}\,\widehat{L}(\widehat{\lambda})\Big), \tag{1.5.1}$$

where P_ω^* on the right-hand side is a $\mathbb{C}$-algebra isomorphism $\mathbb{C}[\widehat{\mathfrak{h}}_{\mathbb{Z}}^*] \tilde{\to} \mathbb{C}[(\mathfrak{h}_{\mathbb{Z}}^*)^0]$ defined by

$$P_\omega^*(e(\widehat{\mu})) = e(P_\omega^*(\widehat{\mu})) \quad \text{for } \widehat{\mu} \in \widehat{\mathfrak{h}}_{\mathbb{Z}}^*.$$

Assume now that $J = I_i = \{\omega^k(i) \mid 0 \le k \le N_i - 1\} \subset I$, $i \in \widehat{I}$, and let P_J be the standard parabolic subgroup of G associated to J. Let $\nu \in \Lambda^\omega$ with $\langle \nu, \alpha_i^\vee \rangle \ge 0$ (hence $\langle \nu, \alpha_j^\vee \rangle \ge 0$ for all $j \in J$). If $L_J(\nu)$ is the simple rational P_J-module of highest weight ν, then it remains simple as a rational module over the Levi factor L_J of P_J with the unipotent radical U_J of P_J acting trivially. We can make $L_J(\nu)$ into a $P_J \rtimes \langle\omega\rangle$-module in the same way as $L(\lambda)$ above.

The following lemma is a first (but important) step towards our main result (Theorem 0.1).

LEMMA 1.5.1. *With the notation and assumption as above, we have in* $\mathbb{C}[\Lambda^\omega]$,

$$\mathrm{ch}^\omega(L_J(\nu)) = \widehat{D}_i(e(\nu)).$$

2. Proof of the main result.

Since the proof of our main result is so simple and clear modulo some algebro-geometric arguments, we give a "detailed outline" of it in this section. Fix $w \in W^\omega$ and let $X(w)$ be the associated Schubert variety over $\mathbb{C}$, i.e., the Zariski closure of the Bruhat cell $B\dot{w}B/B$ in the flag variety G/B. For a $B \rtimes \langle\omega\rangle$-module M, the ω-Euler characteristic $\chi_w^\omega(M)$ is defined to be

$$\chi_w^\omega(M) = \sum_{j \ge 0} (-1)^j\, \mathrm{ch}^\omega(H^j(X(w), \mathcal{L}_{X(w)}(M))) \in \mathbb{C}[\Lambda^\omega].$$

Here recall that, since M is a $B \rtimes \langle\omega\rangle$-module, the $\mathcal{O}_{X(w)}$-module $\mathcal{L}_{X(w)}(M)$ associated to M is a $(B, \langle\omega\rangle)$-equivariant ($\doteqdot B \rtimes \langle\omega\rangle$-equivariant) sheaf, and hence the cohomology groups $H^j(X(w), \mathcal{L}_{X(w)}(M))$, $j \ge 0$, are $B \rtimes \langle\omega\rangle$-modules.

Let $w = w_{i_1} \cdots w_{i_n}$ be a reduced expression of $w \in W^{\omega}$ in the Coxeter system $(W^{\omega}, \{w_i \mid i \in \widehat{I}\})$, i.e., $\widehat{\ell}(w) = n$. Note that we have $\ell(w) = \ell(w_{i_1}) + \cdots + \ell(w_{i_n})$. We want to show that

$$\chi_w^{\omega}(M) = \widehat{D}_{i_1} \cdots \widehat{D}_{i_n}(\mathrm{ch}^{\omega}(M)),$$

where $\widehat{D}_j$ for $j = i_1, \ldots, i_n$ is the ω-Demazure operator defined in §1.3. In particular, we will obtain a twining character formula of the Demazure module $H^0(X(w), \mathcal{L}_{X(w)}(\mathbb{C}_\lambda))$ for $\lambda \in \Lambda_+^{\omega}$, where $\mathbb{C}_\lambda$ is the one-dimensional $B \rtimes \langle\omega\rangle$-module on which B acts by the weight λ through the quotient $B \to T$ and $\langle\omega\rangle$ trivially.

2.1. **Formula for the ω-Euler characteristics.** Set $\widehat{D}_w = \widehat{D}_{i_1} \cdots \widehat{D}_{i_n}$. Then we are to show

$$\chi_w^{\omega}(M) = \widehat{D}_w(\mathrm{ch}^{\omega}(M)). \tag{2.1.1}$$

Let us first make some reductions. Since both sides of (2.1.1) are additive in M, we may assume that M is one-dimensional of weight $\mu \in \Lambda^{\omega}$ on which ω is acting by a scalar ζ^k for a primitive N-th root of unity ζ in $\mathbb{C}$ and $k \in \mathbb{Z}$. We will denote such M by $\mathbb{C}_{\mu,k}$. Thus we are reduced to showing that

$$\chi_w^{\omega}(\mathbb{C}_{\mu,k}) = \widehat{D}_w(\mathrm{ch}^{\omega}(\mathbb{C}_{\mu,k})),$$

where $\mathrm{ch}^{\omega}(\mathbb{C}_{\mu,k}) = \zeta^k e(\mu)$.

Put for simplicity $z_j = w_{i_j}$, $1 \le j \le n$. We have an isomorphism of $B \rtimes \langle\omega\rangle$-modules

$$H^{\bullet}(X(w), \mathcal{L}_{X(w)}(\mathbb{C}_{\mu,k})) \simeq H^{\bullet}(X(z_1, \ldots, z_n), \mathcal{L}_{X(z_1,\ldots,z_n)}(\mathbb{C}_{\mu,k})), \tag{2.1.2}$$

and for each s with $1 \le s \le n-1$, a $B \rtimes \langle\omega\rangle$-equivariant spectral sequence

$$H^i(X(z_s), \mathcal{L}(H^j(X(z_{s+1}, \ldots, z_n), \mathcal{L}(\mathbb{C}_{\mu,k})))) \Rightarrow H^{i+j}(X(z_s, \ldots, z_n), \mathcal{L}(\mathbb{C}_{\mu,k})). \tag{2.1.3}$$

Here $X(z_s, \ldots, z_t)$ for $1 \le s \le t \le n$ is the so-called Bott-Samelson variety, and $\mathcal{L}_{X(z_s,\ldots,z_t)}(\mathbb{C}_{\mu,k})$ is the sheaf of $\mathcal{O}_{X(z_s,\ldots,z_t)}$-modules associated to the $B \rtimes \langle\omega\rangle$-module $\mathbb{C}_{\mu,k}$. Note that, since $z_s, \ldots, z_t \in W^{\omega}$ and their right coset representatives $\dot{z}_s, \ldots, \dot{z}_t \in N_G(T)$ are fixed by $\omega \in \mathrm{Aut}(G)$, the Bott-Samelson variety $X(z_s, \ldots, z_t)$ is an $\langle\omega\rangle$-invariant subvariety of $(G/B)^{t-s+1}$. (The proofs of (2.1.2) and (2.1.3) are not so difficult, but rather long. For details, see our preprint on Naito's home page.)

Remark 2.1.1. There are several equivalent (or inequivalent) definitions of a Bott-Samelson variety, but in the talk, we stick to that of [Ja]:

$$X(y_1, \ldots, y_n) = \{(g_1B, \ldots, g_nB) \in (G/B)^n \mid g_{i-1}^{-1}g_i \in \overline{B\dot{y}_iB} \quad \text{for all } i\}$$

for $y_1, \ldots, y_n \in W$. If J_i for $1 \le i \le n$ is a subset of I and z_{J_i} is the longest element of the subgroup $W_{J_i} = \langle r_k \mid k \in J_i\rangle$ of the Weyl group W, then the Bott-Samelson variety

$X(z_{J_1},\dots,z_{J_n})$ is smooth. Moreover, if we assume that $\ell(z_{J_1}\cdots z_{J_n}) = \ell(z_{J_1})+\dots+\ell(z_{J_n})$, then the restriction ϕ of the n-th projection $\pi_n : (G/B)^n \to G/B$ to the Bott-Samelson variety $X(z_{J_1},\dots,z_{J_n}) \subset (G/B)^n$ gives a Demazure-Hansen desingularization

$$\phi\colon X(z_{J_1},\dots,z_{J_n}) \to X(z_{J_1}\cdots z_{J_n})$$

of the Schubert variety $X(z_{J_1}\cdots z_{J_n})$. Note that the ϕ induces an isomorphism of suitable open and dense subvarieties.

Now it follows that

$$\begin{aligned}
\chi_w^\omega(\mathbb{C}_{\mu,k}) &= \sum_{j\geq 0}(-1)^j \operatorname{ch}^\omega(H^j(X(z_1,\dots,z_n),\mathcal{L}_{X(z_1,\dots,z_n)}(\mathbb{C}_{\mu,k}))) \quad \text{by (2.1.2)}\\
&= \sum_{j\geq 0}(-1)^j \left(\sum_{i\geq 0}(-1)^i \operatorname{ch}^\omega(H^i(X(z_1),\mathcal{L}(H^j(X(z_2,\dots,z_n),\mathcal{L}(\mathbb{C}_{\mu,k})))))\right) \quad \text{by (2.1.3)}\\
&= \sum_{j\geq 0}(-1)^j \chi_{z_1}^\omega(H^j(X(z_2,\dots,z_n),\mathcal{L}_{X(z_2,\dots,z_n)}(\mathbb{C}_{\mu,k}))).
\end{aligned}$$

By induction on n, we may assume that $w = w_i$ for some $i \in \widehat{I}$ in proving (2.1.1). So put $J = I_i$ and let $P = P_J$ be the standard parabolic subgroup of G associated to J. We are to show

$$\chi_{w_i}^\omega(\mathbb{C}_{\mu,k}) = \widehat{D}_i(\zeta^k e(\mu)). \tag{2.1.4}$$

Assume first that $\langle\mu,\alpha_i^\vee\rangle \geq 0$ (and hence that $\langle\mu,\alpha_k^\vee\rangle \geq 0$ for all $k\in J$). Let $L_J(\mu)$ be the simple rational P_J-module of highest weight μ admitting an $\langle\omega\rangle$-action as in §1.5, and let ζ^k be the one-dimensional trivial P_J-module with ω acting by the scalar ζ^k.

LEMMA 2.1.2. *Let the notation and assumption be as above. Then we have the following isomorphism of $P_J \rtimes \langle\omega\rangle$-modules.*

$$H^0(P_J/B,\mathcal{L}_{P_J/B}(\mathbb{C}_{\mu,k})) \simeq L_J(\mu)\otimes_{\mathbb{C}}\zeta^k.$$

(This lemma is, in a sense, crucial to the proof of our main result. Although no one doubts the truth of this lemma, its complete proof would be rather long.)

Now we deduce that

$$\begin{aligned}\chi^{\omega}_{w_i}(\mathbb{C}_{\mu,k}) &= \mathrm{ch}^{\omega}(H^0(P/B, \mathcal{L}_{P/B}(\mathbb{C}_{\mu,k}))) \quad \text{by Kempf's vanishing theorem}\\ &= \mathrm{ch}^{\omega}(L_J(\mu) \otimes_{\mathbb{C}} \zeta^k) \quad \text{by Lemma 2.1.2}\\ &= \zeta^k \mathrm{ch}^{\omega}(L_J(\mu))\\ &= \zeta^k\, \widehat{D}_i(e(\mu)) \quad \text{by Lemma 1.5.1}\\ &= \widehat{D}_i(\zeta^k e(\mu)).\end{aligned}$$

If $\langle \mu, \alpha_i^{\vee} \rangle = -1$ (and hence $\langle \mu, \alpha_k^{\vee} \rangle = -1$ for all $k \in J$), then both sides of (2.1.4) vanish.

Assume finally that $\langle \mu, \alpha_i^{\vee} \rangle \le -2$ (and hence that $\langle \mu, \alpha_k^{\vee} \rangle \le -2$ for all $k \in J$). Set $\rho_J = \frac{1}{2} \sum_{\alpha \in \Delta_J^+} \alpha$ with $\Delta_J^+ = \Delta_+ \cap \sum_{k \in J} \mathbb{Z}\alpha_k$ the positive root system of P_J. By direct checking, using the $T \rtimes \langle \omega \rangle$-module isomorphism $(\mathrm{Lie}(P)/\mathrm{Lie}(B))^* \simeq \bigoplus_{\alpha \in \Delta_J^+} \mathbb{C} f_{\alpha}$, we see that as $B \rtimes \langle \omega \rangle$-modules,

$$\textstyle\bigwedge_{\mathbb{C}}^{\ell(w_i)} (\mathrm{Lie}(P)/\mathrm{Lie}(B))^* \simeq \mathbb{C}_{-2\rho_J, 0} \otimes_{\mathbb{C}} (-1)^{\ell(w_i)-1},$$

where $\ell(w_i) = \dim_{\mathbb{C}}(P/B)$ and $(-1)^{\ell(w_i)-1}$ is the one-dimensional $B \rtimes \langle \omega \rangle$-module with B acting trivially and ω by the scalar $(-1)^{\ell(w_i)-1}$. Then the $B \rtimes \langle \omega \rangle$-equivariant Serre duality reads

(2.1.5)

$$\begin{aligned}H^j(P/B, \mathcal{L}_{P/B}(\mathbb{C}_{\mu,k}))^* &\simeq H^{\ell(w_i)-j}(P/B, \mathcal{L}_{P/B}(\mathbb{C}_{-\mu-2\rho_J,-k} \otimes_{\mathbb{C}} (-1)^{\ell(w_i)-1}))\\ &\simeq \begin{cases} H^0(P/B, \mathcal{L}_{P/B}(\mathbb{C}_{-\mu-2\rho_J,-k})) \otimes_{\mathbb{C}} (-1)^{\ell(w_i)-1} & \text{if } j = \ell(w_i),\\ 0 & \text{otherwise (by Kempf).}\end{cases}\end{aligned}$$

(The use above of the $B \rtimes \langle \omega \rangle$-equivariant Serre duality is the most essential part of the proof of our main result.)

Remark 2.1.3. Put $X = P/B$ and $m = \dim_{\mathbb{C}} X$. Let $\mathcal{M}$ be a $(B, \langle \omega \rangle)$-equivariant $\mathcal{O}_X$-module that is locally free of finite rank over $\mathcal{O}_X$. The $B \rtimes \langle \omega \rangle$-equivariant Serre duality (see our preprint on Naito's home page) asserts that, as $B \rtimes \langle \omega \rangle$-modules,

$$H^i(X, \mathcal{M}^{\vee} \otimes_X \Omega_X^m) \simeq H^{m-i}(X, \mathcal{M})^* \quad \text{for all } 0 \le i \le m,$$

where $\mathcal{M}^{\vee} = \mathcal{H}om_X(\mathcal{M}, \mathcal{O}_X)$ is the dual sheaf of $\mathcal{M}$, $\Omega_X^m = \bigwedge_X^m \Omega_X^1$ is the canonical sheaf on X, and $H^{m-i}(X, \mathcal{M})^*$ is the dual $B \rtimes \langle \omega \rangle$-module of $H^{m-i}(X, \mathcal{M})$. This Serre duality will be a consequence of the triviality of the $B \rtimes \langle \omega \rangle$-action on the one-dimensional vector space $H^m(X, \Omega_X^m)$. Since the triviality of the B-action on it is known, it remains to show the triviality of the $\langle \omega \rangle$-action. There are many ways to show it, but the way we take here is (we think) purely algebro-geometric and elementary: first take a $P \rtimes \langle\omega\rangle$-equivariant closed immersion $\imath: X \to \mathbb{P} = \mathbb{P}(L(\lambda))$ for sufficiently dominant $\lambda \in \Lambda^{\omega}_{+}$; then use the fact that the full automorphism group $PGL(L(\lambda))$ of $\mathbb{P}(L(\lambda))$ acts trivially on the one-dimensional vector space $H^l(\mathbb{P}, \Omega^l_{\mathbb{P}})$ with $l = \dim_{\mathbb{C}} \mathbb{P}$, where (though not so trivial)

$$H^l(\mathbb{P}, \Omega^l_{\mathbb{P}}) \simeq H^m(\mathbb{P}, \imath_* \Omega^m_X) \simeq H^m(X, \Omega^m_X)$$

as $P \rtimes \langle\omega\rangle$-modules.

The proof of the following lemma is easy.

LEMMA 2.1.4. *Let J be an ω-invariant subset of I, w_J the longest element of the Weyl group W_J of P_J, and let $\nu \in \Lambda^{\omega}$ be such that $\langle \nu, \alpha_k^{\vee} \rangle \geq 0$ for all $k \in J$. Then we have the following isomorphism of $P_J \rtimes \langle\omega\rangle$-modules.*

$$L_J(\nu)^* \simeq L_J(-w_J(\nu)).$$

The isomorphism (2.1.5) together with Lemmas 2.1.2 and 2.1.4 implies that, as $B \rtimes \langle\omega\rangle$-modules,

$$H^{\ell(w_i)}(P/B, \mathcal{L}_{P/B}(\mathbb{C}_{\mu,k})) \simeq \Big(L_J(-\mu - 2\rho_J)^* \otimes_{\mathbb{C}} \zeta^k\Big) \otimes_{\mathbb{C}} (-1)^{\ell(w_i)-1} \tag{2.1.6}$$
$$\simeq L_J(w_i(\mu + 2\rho_J)) \otimes_{\mathbb{C}} \zeta^k \otimes_{\mathbb{C}} (-1)^{\ell(w_i)-1}.$$

Then, setting $\widehat{\mu} = (P^*_{\omega})^{-1}(\mu)$,

$$\begin{aligned}
\chi^{\omega}_{w_i}(\mathbb{C}_{\mu,k}) &= (-1)^{\ell(w_i)} \operatorname{ch}^{\omega}(L_J(w_i(\mu + 2\rho_J)) \otimes_{\mathbb{C}} \zeta^k \otimes_{\mathbb{C}} (-1)^{\ell(w_i)-1}) \quad \text{by (2.1.6)} \\
&= -\zeta^k \operatorname{ch}^{\omega}(L_J(w_i(\mu + 2\rho_J))) \\
&= -\zeta^k \widehat{D}_i(e(w_i(\mu + 2\rho_J))) \quad \text{by Lemma 1.5.1} \\
&= -\zeta^k \Big(P^*_{\omega} \circ D_{\widehat{r}_i} \circ (P^*_{\omega})^{-1}\Big)(e(w_i(\mu + 2\rho_J))) \\
&= -\zeta^k \Big(P^*_{\omega} \circ D_{\widehat{r}_i}\Big)(e(\widehat{r}_i(\widehat{\mu} + \widehat{\alpha}_i))) \quad \text{since } (P^*_{\omega})^{-1}(2\rho_J) = \widehat{\alpha}_i \\
&= -\zeta^k P^*_{\omega}(-D_{\widehat{r}_i}(e(\widehat{\mu}))) \\
&= \zeta^k \Big(\widehat{D}_i \circ P^*_{\omega}\Big)(e(\widehat{\mu})) \\
&= \zeta^k \widehat{D}_i(e(\mu)) \\
&= \widehat{D}_i(\zeta^k e(\mu)).
\end{aligned}$$

Thus in all cases (2.1.4) holds, and we are done.

If $\lambda \in \Lambda^{\omega}_{+}$, then for any Schubert variety $X(w)$,

$$H^j(X(w), \mathcal{L}_{X(w)}(\lambda)) = 0 \quad \text{for all } j \geq 1$$

by the Demazure vanishing theorem of Andersen et al. Hence we have proved

THEOREM 2.1.5. *Let M be a finite-dimensional rational $B \rtimes \langle\omega\rangle$-module and $w \in W^\omega$. Then we have in $\mathbb{C}[\Lambda_\omega]$,*

$$\chi_w^\omega(M) = \sum_{j\geq 0}(-1)^j \operatorname{ch}^\omega(H^j(X(w), \mathcal{L}_{X(w)}(M))) = \widehat{D}_w(\operatorname{ch}^\omega(M)),$$

where $\widehat{D}_w = \widehat{D}_{i_1}\widehat{D}_{i_2}\cdots\widehat{D}_{i_n}$ for any reduced expression $w = w_{i_1}w_{i_2}\cdots w_{i_n}$ of $w \in W^\omega$ in the Coxeter system $(W^\omega, \{w_i \mid i \in \widehat{I}\})$. In particular, for $\lambda \in \Lambda_+^\omega$, we have

$$\operatorname{ch}^\omega(H^0(X(w), \mathcal{L}_{X(w)}(\mathbb{C}_\lambda))) = \widehat{D}_w(e(\lambda)),$$

where $\mathbb{C}_\lambda$ is the one-dimensional $B \rtimes \langle\omega\rangle$-module on which B acts by the weight λ through the quotient $B \to T$ and ω trivially.

Theorem 2.1.5 above reveals that there exists a striking relation between the ω-Euler characteristic $\chi_w^\omega(\mathbb{C}_\lambda) \in \mathbb{C}[(\mathfrak{h}_\mathbb{Z}^*)^0]$ for $\mathfrak{g}$ and the ordinary Euler characteristic for the orbit Lie algebra $\widehat{\mathfrak{g}}$. To state the relation, we need some notation. Recall that the orbit Lie algebra $\widehat{\mathfrak{g}}$ is the dual complex semi-simple Lie algebra of the fixed point subalgebra $\mathfrak{g}^0 = \{x \in \mathfrak{g} \mid \omega(x) = x\}$ of $\mathfrak{g}$. Let $\widehat{G}$ be a connected, simply connected semi-simple linear algebraic group over $\mathbb{C}$ with maximal torus $\widehat{T}$ and Borel subgroup $\widehat{B} \supset \widehat{T}$ such that $\operatorname{Lie}(\widehat{G}) = \widehat{\mathfrak{g}}$, $\operatorname{Lie}(\widehat{T}) = \widehat{\mathfrak{h}}$, and $\operatorname{Lie}(\widehat{B}) = \widehat{\mathfrak{b}}$. For $\widehat{w} \in \widehat{W} \simeq N_{\widehat{G}}(\widehat{T})/\widehat{T}$, we take a right coset representative $\dot{\widehat{w}} \in N_{\widehat{G}}(\widehat{T})$ of $\widehat{w}$, and define the Schubert variety $\widehat{X}(\widehat{w})$ over $\mathbb{C}$ by

$$\widehat{X}(\widehat{w}) = \overline{\widehat{B}\dot{\widehat{w}}\widehat{B}/\widehat{B}} = \overline{\widehat{B}\widehat{w}\widehat{B}}/\widehat{B} \subset \widehat{G}/\widehat{B}.$$

For each $\widehat{\lambda} \in \widehat{\mathfrak{h}}_\mathbb{Z}^*$, we denote by $\mathcal{L}_{\widehat{X}(\widehat{w})}(\mathbb{C}_{\widehat{\lambda}})$ the (locally free) $\widehat{B}$-equivariant sheaf of $\mathcal{O}_{\widehat{X}(\widehat{w})}$-modules associated to the one-dimensional $\widehat{B}$-module $\mathbb{C}_{\widehat{\lambda}}$ on which $\widehat{B}$ acts by the weight $\widehat{\lambda}$ through the quotient $\widehat{B} \to \widehat{T}$.

Now we are ready to state the following

COROLLARY 2.1.6. *Let $\lambda \in (\mathfrak{h}_\mathbb{Z}^*)^0$ and $w \in W^\omega$. We set $\widehat{w} = \Theta^{-1}(w) \in \widehat{W}$ and $\widehat{\lambda} = (P_\omega^*)^{-1}(\lambda) \in \widehat{\mathfrak{h}}_\mathbb{Z}^*$. Then we have in the algebra $\mathbb{C}[(\mathfrak{h}_\mathbb{Z}^*)^0]$.*

$$\begin{aligned}\chi_w^\omega(\mathbb{C}_\lambda) &= \sum_{j\geq 0}(-1)^j \operatorname{ch}^\omega(H^j(X(w), \mathcal{L}_{X(w)}(\mathbb{C}_\lambda)))\\ &= P_\omega^*\left(\sum_{j\geq 0}(-1)^j \operatorname{ch} H^j(\widehat{X}(\widehat{w}), \mathcal{L}_{\widehat{X}(\widehat{w})}(\mathbb{C}_{\widehat{\lambda}}))\right),\end{aligned}$$

where $\operatorname{ch} H^j(\widehat{X}(\widehat{w}), \mathcal{L}_{\widehat{X}(\widehat{w})}(\mathbb{C}_{\widehat{\lambda}})) \in \mathbb{C}[\widehat{\mathfrak{h}}_\mathbb{Z}^*]$ for $j \in \mathbb{Z}_{\geq 0}$ is the ordinary character of the j-th cohomology group $H^j(\widehat{X}(\widehat{w}), \mathcal{L}_{\widehat{X}(\widehat{w})}(\mathbb{C}_{\widehat{\lambda}}))$ of $\widehat{X}(\widehat{w})$.

(This immediately follows from Theorem 2.1.5 and the ordinary Demazure character formula for the orbit Lie algebra $\widehat{\mathfrak{g}}$.)

2.2. **Joseph's modules.** Let us finally return to Joseph's module $J_w(\lambda)$, with $w \in W^\omega$ and $\lambda \in \Lambda_+^\omega$. Thus let v_λ^* be a (nonzero) lowest weight vector of the dual module $L(\lambda)^*$ (which is the dual element of a (nonzero) highest weight vector v_λ of $L(\lambda)$), and let $\dot{w} \in N_G(T)^\omega$ representing $w \in W^\omega$. Since v_λ^* is fixed by ω, so is $\dot{w}\, v_\lambda^*$. Joseph's module $J_w(\lambda)$ of highest weight $-w(\lambda)$ in $L(\lambda)^*$ is defined to be

$$J_w(\lambda) = \mathfrak{U}(\mathfrak{b})\,(\dot{w}\, v_\lambda^*) \subset L(\lambda)^*,$$

where $\mathfrak{U}(\mathfrak{b})$ is the universal enveloping algebra of $\mathfrak{b} = \mathrm{Lie}(B)$. Note that, since $\omega \cdot (\dot{w}\, v_\lambda^*) = \dot{w}\, v_\lambda^*$, Joseph's module $J_w(\lambda)$ is a $B \rtimes \langle\omega\rangle$-submodule of $L(\lambda)^*$. Moreover, since $\dot{w}_0\, v_\lambda^*$ is a (nonzero) highest weight vector of $L(\lambda)^*$ fixed by ω, there is an isomorphism of $G \rtimes \langle\omega\rangle$-modules

$$L(\lambda)^* \simeq L(-w_0(\lambda)), \tag{2.2.1}$$

which enables us to regard $J_w(\lambda)$ as a $B \rtimes \langle\omega\rangle$-submodule of $L(-w_0(\lambda))$. Then we obtain a short exact sequence of $B \rtimes \langle\omega\rangle$-modules

$$0 \leftarrow J_w(\lambda)^* \leftarrow L(-w_0(\lambda))^* \leftarrow J_w(\lambda)^\perp \leftarrow 0,$$

with $J_w(\lambda)^\perp = \{\phi \in L(-w_0(\lambda))^* \mid \phi(J_w(\lambda)) = 0\}$. On the other hand, Lemma 2.1.2 for the case $J = I$ combined with (2.2.1) yields an isomorphism of $G \rtimes \langle\omega\rangle$-modules

$$H^0(G/B, \mathcal{L}_{G/B}(\mathbb{C}_\lambda)) \simeq L(-w_0(\lambda))^*.$$

Since the restriction map

$$H^0(G/B, \mathcal{L}_{G/B}(\mathbb{C}_\lambda)) \to H^0(X(w), \mathcal{L}_{X(w)}(\mathbb{C}_\lambda))$$

is known to be a ($B \rtimes \langle\omega\rangle$-equivariant) surjection, we obtain an isomorphism of $B \rtimes \langle\omega\rangle$-modules

$$J_w(\lambda)^* \simeq H^0(X(w), \mathcal{L}_{X(w)}(\mathbb{C}_\lambda)),$$

or equivalently

$$J_w(\lambda) \simeq H^0(X(w), \mathcal{L}_{X(w)}(\mathbb{C}_\lambda))^*. \tag{2.2.2}$$

We now define a $\mathbb{C}$-linear conjugation $\bar{\ } : \mathbb{C}[\Lambda^\omega] \to \mathbb{C}[\Lambda^\omega]$ by

$$\overline{\sum_{\mu \in \Lambda^\omega} a_\mu\, e(\mu)} = \sum_{\mu \in \Lambda^\omega} a_\mu\, e(-\mu) \quad \text{with } a_\mu \in \mathbb{C} \text{ for } \mu \in \Lambda^\omega.$$

Then we obtain the following theorem from the $B \rtimes \langle\omega\rangle$-module isomorphism (2.2.2).

THEOREM 2.2.1. *Let $\lambda \in \Lambda_+^\omega$ and $w \in W^\omega$. Then we have in $\mathbb{C}[\Lambda^\omega]$,*

$$\mathrm{ch}^\omega(J_w(\lambda)) = \overline{\mathrm{ch}^\omega(H^0(X(w), \mathcal{L}_{X(w)}(\mathbb{C}_\lambda)))}.$$

By combining Theorems 2.1.5 and 2.2.1, we obtain the following

COROLLARY 2.2.2. *Let $\lambda \in \Lambda_+^\omega$ and $w \in W^\omega$. Then we have in $\mathbb{C}[\Lambda^\omega]$,*

$$\mathrm{ch}^\omega(J_w(\lambda)) = \overline{\widetilde{D}_w(e(\lambda))}.$$

Finally, by combining Corollary 2.1.6 and Theorem 2.2.1, we obtain a remarkable relation between the twining character $\mathrm{ch}^\omega(J_w(\lambda))$ of Joseph's module $J_w(\lambda)$ for $\mathfrak{g}$ and the ordinary character of Joseph's module for the orbit Lie algebra $\widehat{\mathfrak{g}}$, which is the dual complex semi-simple Lie algebra of $\mathfrak{g}^0$. For each $\widehat{w} \in \widehat{W}$, let

$$\widehat{J}_{\widehat{w}}(\widehat{\lambda}) = \mathfrak{U}(\widehat{\mathfrak{b}})\,(\dot{\widehat{w}}\,\widehat{v}^*_{\widehat{\lambda}}) \subset \widehat{L}(\widehat{\lambda})^*$$

be Joseph's module of highest weight $-\widehat{w}(\widehat{\lambda})$, with $\widehat{v}^*_{\widehat{\lambda}} \in \widehat{L}(\widehat{\lambda})^*$ a lowest weight vector of $\widehat{L}(\widehat{\lambda})^*$.

COROLLARY 2.2.3. *Let $\lambda \in (\mathfrak{h}^*_{\mathbb{Z}})^0$ be dominant and $w \in W^\omega$. We set $\widehat{w} = \Theta^{-1}(w) \in \widehat{W}$ and $\widehat{\lambda} = (P^*_\omega)^{-1}(\lambda) \in \widehat{\mathfrak{h}}^*_{\mathbb{Z}}$. Then we have in $\mathbb{C}[(\mathfrak{h}^*_{\mathbb{Z}})^0]$,*

$$\mathrm{ch}^\omega(J_w(\lambda)) = P^*_\omega\left(\mathrm{ch}\,\widehat{J}_{\widehat{w}(\widehat{\lambda})}(\widehat{\lambda})\right).$$

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