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On the Cowling-Price theorem for $SU(1,1)$

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Abstract

M. G. Cowling and J. F. Price showed a kind of uncertainty principle on Fourier analysis: If v and w grow very rapidly then the finiteness of $\|vf\|_p$ and $\|w\hat{f}\|_q$ implies that $f=0$, where $\hat{f}$ denotes the Fourier transform of f. We give an analogue of this theorem for $SU(1,1)$.

1 Introduction

The Hardy theorem asserts that if a measurable function f on $\mathbf{R}$ satisfies $|f(x)| \leq Ce^{-ax^2}$ and $|\hat{f}(y)| \leq Ce^{-by^2}$ and $ab > \frac{1}{4}$ then $f=0$ (a.e.). Here we use the Fourier transform defined by $\hat{f}(y)=(1/\sqrt{2\pi})\int_{-\infty}^{\infty} f(x)e^{\sqrt{-1}xy}\,dx$. M. G. Cowling and J. F. Price [3] generalized the Hardy theorem as follows: Suppose that $1 \leq p,q \leq \infty$ and one of them is finite. If a measurable function f on $\mathbf{R}$ satisfies $\|\exp\{ax^2\}f(x)\|_{L^p(\mathbf{R})} < \infty$ and $\|\exp\{by^2\}\hat{f}(y)\|_{L^q(\mathbf{R})} < \infty$ and $ab \geq 1/4$ then $f=0$ (a.e.). The case where $p=q=\infty$ and $ab > 1/4$ is covered by the Hardy theorem. S. C. Bagchi and S. K. Ray [1] showed that if $ab > 1/4$, then the Hardy theorem is equivalent to the Cowling-Price theorem.

A.Sitaram and M.Sundari [10] obtained the Hardy theorem in the case of noncompact semisimple Lie groups with one conjugacy class of Cartan subgroups, $SL(2,\mathbf{R})$ and Riemannian symmetric spaces of the noncompact type. Recently J. Sengupta [8] and M. Ebata et al. [5] obtained the Hardy theorem for all Lie groups of Harish-Chandra class and all connected semisimple Lie groups with finite center respectively. Also, M. Cowling, A. Sitaram and M. Sundari [4] gave another simple proof of the Hardy theorem for connected real semisimple Lie groups with finite center. On the other hand, S. C. Bagchi and S. K. Ray [1] obtained the Cowling-Price theorem for some Lie groups and M. Eguchi, S. Koizumi and K. Kumahara [6] also obtained the Cowling-Price theorem for motion groups. Further, J. Sengupta [9] obtained the Cowling-Price theorem on Riemannian symmetric spaces of the noncompact type. In this note, we prove the Cowling-Price theorem for $SU(1,1)$ under the assumption that $1 \leq p,q \leq \infty$ and $ab > 1/4$.

2 Notation and preliminaries

If $\mathcal{H}$ is a complex separable Hilbert space, $\mathbf{B}(\mathcal{H})$ denotes the Banach space comprised of all bounded operators on $\mathcal{H}$ with operator norm $\|\cdot\|_\infty$. For $T \in \mathbf{B}(\mathcal{H})$ and

$1 \le p < \infty$, we indicate its Schatten norm by $\|T\|_p$, that is, $\|T\|_p = (\mathrm{tr}(T^*T)^{p/2})^{1/p}$, T^* being the adjoint operator of T. For a complex separable Hilbert space $\mathcal{H}$ and a σ-finite measure space (X, μ), we denote by $L^p(X, \mathbf{B}(\mathcal{H}))$ the Banach space comprised of all $\mathbf{B}(\mathcal{H})$-valued L^p functions on X. Here the L^p-norm $\|F\|_{L^p(X,\mathbf{B}(\mathcal{H}))}$ of $F \in L^p(X, \mathbf{B}(\mathcal{H}))$ is given by the following:

$$\begin{aligned}
\|F\|_{L^p(X,\mathbf{B}(\mathcal{H}))} &= \left(\int_X \|F(x)\|_p^p d\mu(x) \right)^{1/p}, \ (1 \le p < \infty), \\
\|F\|_{L^\infty(X,\mathbf{B}(\mathcal{H}))} &= \operatorname*{ess.\,sup}_{x \in X} \|F(x)\|_\infty .
\end{aligned}$$

Here G denotes the matrix group $SU(1,1)$, that is,

$$G = \left\{ g = \begin{pmatrix} \alpha & \beta \\ \bar{\beta} & \bar{\alpha} \end{pmatrix} \ ; \ |\alpha|^2 - |\beta|^2 = 1 \ , \ \alpha, \beta \in \mathbf{C} \right\}.$$

For $\varepsilon = 0, 1$ and $\nu \in \mathbf{R}$, let

$$\mathcal{H}_{\varepsilon,\nu} = \left\{ \varphi \in L^2(K) \ ; \ \varphi(k(\pm I)) = (\pm 1)^\varepsilon \varphi(k), \ k \in K \right\}.$$

We define the action $\pi_{\varepsilon,\nu}$ on $\mathcal{H}_{\varepsilon,\nu}$ by

$$(\pi_{\varepsilon,\nu}(g)\varphi)(k) = e^{(\sqrt{-1}\nu - 1/2)t(g^{-1}k)} \varphi(k_{\theta(g^{-1}k)}).$$

Then $\pi_{\varepsilon,\nu}$ is a unitary representation on $\mathcal{H}_{\varepsilon,\nu}$ and is called a principal series representation. Let $I_{\varepsilon,\nu}$ be the standard intertwining operator defined by Knapp and Stein. For each $\varepsilon = 0, 1$, it is satisfied that

$$I_{\varepsilon,\nu} \pi_{\varepsilon,\nu}(g) = \pi_{\varepsilon,-\nu}(g) I_{\varepsilon,\nu}$$

for all $\nu \in \mathbf{R}$ and $g \in G$. We also need another representation. For $\lambda \in \mathbf{Z} \backslash \{0\}$, we put the discrete series representation $(\pi_\lambda, \mathcal{H}_\lambda)$.

For $f \in L^1(G)$, its Fourier transform on G is defined by

$$\mathcal{F}^c f(\varepsilon, \nu) = \int_G f(g) \pi_{\varepsilon,\nu}(g)\, dg, \tag{2.1}$$

$$\mathcal{F}^d f(\lambda) = \int_G f(g) \pi_\lambda(g)\, dg. \tag{2.2}$$

We write $\mathcal{F} = (\mathcal{F}^c, \mathcal{F}^d)$. If $f \in C_0^\infty(G)$, then the following inversion formula holds

$$\begin{aligned}
f(g) &= \sum_{\varepsilon=0}^{1} \int_0^\infty \mathrm{tr}(\mathcal{F}^c f(\varepsilon,\nu) \pi_{\varepsilon,\nu}(g^{-1})) \mu(\varepsilon,\nu) d\nu \\
&+ \sum_{\lambda \in \mathbf{Z} \backslash \{0\}} d(\lambda) \left\{ \mathrm{tr}(\mathcal{F}^d f(\lambda) \pi_\lambda(g^{-1})) \right\},
\end{aligned} \tag{2.3}$$

where $\mu(0,\nu) = \pi\nu\tanh\pi\nu$, $\mu(1,\nu) = \pi\nu\coth\pi\nu$ and $d(\lambda) = |\lambda|/(4\pi)$. For convenience we write $\mathcal{L}^p_\varepsilon(\mathfrak{a}^*) = L^p_\varepsilon(\mathfrak{a}^*, \mathbf{B}(\mathcal{H}_{\varepsilon,\nu}), \mu(\varepsilon,\nu)d\nu)$ and $L^p_\varepsilon(\mathfrak{a}^*) = L^p_\varepsilon(\mathfrak{a}^*, \mu(\varepsilon,\nu)d\nu)$.

If a function f satisfies $\|e^{a\sigma(g)^2} f(g)\|_{L^p(G)} \leq C$ for $a > 0$ and $1 \leq p \leq \infty$, we call that f is very rapidly decreasing. Such functions belong to $L^1(G)$. The Schwartz space on G is defined by

$$\mathcal{C}(G) = \{\phi \in C^\infty(G) \; ; \; \|\phi\|_{r,D,E} < \infty \quad \text{for all } r \in \mathbf{Z}_{\geq 0}, \; D, E \in U(\mathfrak{g}_c)\}$$

$$\text{where} \quad \|\phi\|_{r,D,E} = \sup_{g\in G} |(1+\sigma(g))^r \Xi(g)^{-1}\phi(D;g;E)|.$$

As is well known, the system of seminorms $\|\cdot\|_{r,D,E}$ makes $\mathcal{C}(G)$ into a Fréchet space.

Let $\mathcal{C}_c(\hat{G})$ be the set of operator valued functions $F : \{0,1\} \times \mathbf{R} \to \oplus_{\varepsilon=0}^1 \mathbf{B}(\mathcal{H}_{\varepsilon,\nu})$ such that

(i) $F(\varepsilon,\nu) \in \mathbf{B}(\mathcal{H}_{\varepsilon,\nu})$ for each $\varepsilon = 0, 1$, $\nu \in \mathbf{R}$

(ii) $\nu \mapsto F(\varepsilon,\nu)$ is smooth on $\mathbf{R}$

(iii) $I_{\varepsilon,\nu}F(\varepsilon,\nu) = F(\varepsilon,-\nu)I_{\varepsilon,\nu}$ for each $\varepsilon = 0, 1$, $\nu \in \mathbf{R}$

(iv) $$\sup_{\substack{\varepsilon=0,1,\nu\in\mathbf{R}\\ \ell_1,\ell_2\in\mathbf{Z}(\varepsilon)}} \left| \left(\frac{d}{d\nu}\right)^r \langle F(\varepsilon,\nu)e_{\ell_2}, e_{\ell_1}\rangle \right| (1+|\nu|)^{r_1}(1+|\ell_1|)^{r_2}(1+|\ell_2|)^{r_3} < \infty$$

for all $r_1, r_2, r_3, r \in \mathbf{Z}_{\geq 0}$.

The system of seminorms given by (iv) makes $\mathcal{C}_c(\hat{G})$ into a Fréchet space.

Let $\mathcal{C}_d(\hat{G})$ be the set of all $F : \mathbf{Z}\backslash\{0\} \to \oplus_{\lambda\in\mathbf{Z}\backslash\{0\}}\mathbf{B}(\mathcal{H}_\lambda)$ such that

(i) $F(\lambda) \in \mathbf{B}(\mathcal{H}_\lambda)$ for each $\lambda \in \mathbf{Z}\backslash\{0\}$

(ii) $$\sup_{\substack{\lambda\in\mathbf{Z}\backslash\{0\}\\ \ell_1,\ell_2\in\mathbf{Z}_\lambda}} |(F(\lambda)\psi_{\ell_2}, \psi_{\ell_1})_\lambda| (1+|\lambda|)^{r_1}(1+|\ell_1|)^{r_2}(1+|\ell_2|)^{r_3} < \infty$$

for all $r_1, r_2, r_3 \in \mathbf{Z}_{\geq 0}$.

The system of seminorms given by (ii) makes $\mathcal{C}_d(\hat{G})$ into a Fréchet space. Put $\mathcal{C}(\hat{G}) = \mathcal{C}_c(\hat{G}) \oplus \mathcal{C}_d(\hat{G})$. Then $\mathcal{C}(\hat{G})$ is a Fréchet space in an obvious manner.
We put $\mathcal{S}^c = (\mathcal{F}^c)^{-1}$ and $\mathcal{S}^d = (\mathcal{F}^d)^{-1}$. Then they are given by

$$\begin{aligned} \mathcal{S}^c F(g) &= \int_0^\infty \mathrm{tr}(F(\varepsilon,\nu)\pi_{\varepsilon,\nu}(g^{-1}))\mu(\varepsilon,\nu)d\nu, \text{ for } F \in \mathcal{C}_c(\hat{G}), \\ \mathcal{S}^d F(g) &= \sum_{\lambda\in\mathbf{Z}\backslash\{0\}} d(\lambda)\mathrm{tr}(F(\lambda)\pi_\lambda(g^{-1})), \text{ for } F \in \mathcal{C}_d(\hat{G}). \end{aligned}$$

PROPOSITION 2.1 (cf. [7]) *The Fourier transform $\mathcal{F}$ is a topological isomorphism from $\mathcal{C}(G)$ onto $\mathcal{C}(\hat{G})$. And its inverse transform is given by (2.3).*

Let

$$\begin{aligned} \mathcal{C}_c(G) &= \{\phi \in \mathcal{C}(G) \; ; \; \mathcal{F}^d\phi(\lambda) = 0, \; \lambda \in \mathbf{Z}\backslash\{0\}\}, \\ \mathcal{C}_d(G) &= \{\phi \in \mathcal{C}(G) \; ; \; \mathcal{F}^c\phi(\varepsilon,\nu) = 0, \; \varepsilon = 0, 1, \; \nu \in \mathbf{R}\}, \end{aligned}$$

and $\mathcal{C}_{c,mn}(G)$ (resp. $\mathcal{C}_{d,mn}(G)$) denote the subset of $\mathcal{C}_c(G)$ (resp. $\mathcal{C}_d(G)$) consisting of the (m,n)-spherical functions.

Let $m,n\in\mathbf{Z}$. If $m-n\in 2\mathbf{Z}+1$, we set $\mathcal{C}_{c,mn}(\hat{G})=\emptyset$. If $m-n\in 2\mathbf{Z}$, we choose ε so that $m,n\in\mathbf{Z}(\varepsilon)$ and let $\mathcal{C}_{c,mn}(\hat{G})$ be the set of C^∞ functions $F:\mathbf{R}\to\mathbf{C}$ such that

$$\begin{aligned}
&\text{(i)} \quad F(-\nu)=c_n(\nu)^{-1}c_m(\nu)F(\nu) \quad \text{for each } \nu\in\mathbf{R},\\
&\text{(ii)} \quad \sup_{\nu\in\mathbf{R}}\left|(1+|\nu|)^r\left(\frac{d}{d\nu}\right)^s F(\nu)\right|<\infty \quad \text{for all } r,s\in\mathbf{Z}_{\geq 0}.
\end{aligned}$$

The system of seminorms given by (ii) makes $\mathcal{C}_{c,mn}(\hat{G})$ into a Fréchet space.

Let $\mathcal{C}_{d,mn}(\hat{G})$ be the set of all functions $F:\mathbf{Z}\backslash\{0\}\to\mathbf{C}$ such that

$$F(\lambda)=0 \quad \text{for all } \lambda\notin L(m,n).$$

We equip $\mathcal{C}_{d,mn}(\hat{G})$ with the topology induced by the system of seminorms $\|F\|_\ell=\sup_{\lambda\in L(m,n)}|F(\lambda)|(1+|\lambda|)^\ell$ for $\ell\in\mathbf{Z}_{\geq 0}$. Then $\mathcal{C}_{d,mn}(\hat{G})$ becomes a Fréchet space. It is also known that $\mathcal{C}(G)\subseteq L^2(G)$ and $\mathcal{C}_{c,mn}(\hat{G})\subseteq L^p_\varepsilon(\mathfrak{a}^*)$ for all $p\in[1,\infty]$.

For $f\in L^1(G)$, we define its (m,n)-spherical transforms $\mathcal{F}^c_{mn}f$ and $\mathcal{F}^d_{mn}f$ by

$$\begin{aligned}
(\mathcal{F}^c_{mn}f)(\varepsilon,\nu)&=\int_G f(g)\Phi^{\varepsilon,\nu}_{mn}(g)dg,\\
(\mathcal{F}^d_{mn}f)(\lambda)&=\int_G f(g)\Psi^{\lambda}_{mn}(g)dg.
\end{aligned}$$

For $\phi\in L^1_\varepsilon(\mathfrak{a}^*)$ and $m,n\in\mathbf{Z}(\varepsilon)$, we set

$$(\mathcal{S}^c_{mn}\phi)(g) \;=\; \int_0^\infty \phi(\nu)\Phi^{\varepsilon,\nu}_{nm}(g^{-1})\mu(\varepsilon,\nu)d\nu\ .$$

For an arbitrary function $\phi:\mathbf{Z}\backslash\{0\}\to\mathbf{C}$, we put

$$(\mathcal{S}^d_{mn}\phi)(g) \;=\; \sum_{\lambda\in L(m,n)} d(\lambda)\phi(\lambda)\Psi^\lambda_{nm}(g^{-1})\ .$$

PROPOSITION 2.2 (cf. [7]) *The (m,n)-spherical transform $\mathcal{F}^c_{mn}$ (resp. $\mathcal{F}^d_{mn}$) is a topological isomorphism of $\mathcal{C}_{c,mn}(G)$ (resp. $\mathcal{C}_{d,mn}(G)$) onto $\mathcal{C}_{c,mn}(\hat{G})$ (resp. $\mathcal{C}_{d,mn}(\hat{G})$). And inverse transform of $\mathcal{F}^c_{mn}$ (resp. $\mathcal{F}^d_{mn}$) is given by $\mathcal{S}^c_{mn}$ (resp. $\mathcal{S}^d_{mn}$).*

For $\phi\in\mathcal{C}(G)$, we define the wave packets $\phi_{c,mn}\in\mathcal{C}_{c,mn}(G)$ and $\phi_{d,mn}\in\mathcal{C}_{d,mn}(G)$ by

$$\begin{aligned}
\phi_{c,mn}(g) &= \mathcal{S}^c_{mn}(\mathcal{F}^c_{mn}\phi)(g)=\int_0^\infty(\mathcal{F}^c_{mn}\phi)(\varepsilon,\nu)\Phi^{\varepsilon,\nu}_{nm}(g^{-1})\mu(\varepsilon,\nu)d\nu,\\
\phi_{d,mn}(g) &= \mathcal{S}^d_{mn}(\mathcal{F}^d_{mn}\phi)(g)=\sum_{\lambda\in L(m,n)} d(\lambda)(\mathcal{F}^d_{mn}\phi)(\lambda)\Psi^\lambda_{nm}(g^{-1}).
\end{aligned}$$

PROPOSITION 2.3 (cf. [2]) *For each $\phi \in \mathcal{C}(G)$, there is a unique expansion*

$$\phi = \sum_{m,n\in\mathbf{Z}} \phi_{c,mn} + \sum_{m,n\in\mathbf{Z}} \phi_{d,mn}.$$

The series converges absolutely to ϕ in $\mathcal{C}(G)$, and the mappings $\phi \to \phi_{c,mn}$ and $\phi \to \phi_{d,mn}$ are continuous.

For a tempered distribution $T \in \mathcal{C}'(G)$, we define $T_{c,mn}, T_{d,mn} \in \mathcal{C}'(G)$ by

$$T_{c,mn}[\phi] = T[\phi_{c,mn}] \ , \ T_{d,mn}[\phi] = T[\phi_{d,mn}] \quad (\phi \in \mathcal{C}(G)).$$

Similarly, we also define $T_{mn} \in \mathcal{C}'(G)$ by

$$T_{mn}[\phi] = T[\phi_{mn}],$$

where ϕ_{mn} is (m,n)-spherical function in $\mathcal{C}(G)$.

PROPOSITION 2.4 (cf. [2]) *Retain the above notation.*

$$T = \sum_{m,n\in\mathbf{Z}} T_{c,mn} + \sum_{m,n\in\mathbf{Z}} T_{d,mn},$$

where the series converges absolutely to T in the weak topology of $\mathcal{C}'(G)$.

Here we give some lemmas.

LEMMA 2.5 (cf. [2]) *Let $T \in \mathcal{C}'(G)$. Then*

$$\mathcal{F}^c T_{c,mn} = \mathcal{F}^c_{mn} T, \ \mathcal{F}^d T_{c,mn} = 0, \ \mathcal{F}^c T_{d,mn} = 0, \ \mathcal{F}^d T_{d,mn} = \mathcal{F}^d_{mn} T.$$

LEMMA 2.6 *Let f be very rapidly decreasing and (m,n)-spherical. Then*

$$\begin{aligned} (T_f)_{c,rs} &= \delta_{r,-m}\delta_{s,-n}(T_f)_{c,(-m)(-n)}, \\ (T_f)_{d,rs} &= \delta_{r,-m}\delta_{s,-n}(T_f)_{d,(-m)(-n)}, \end{aligned}$$

for $r, s \in \mathbf{Z}$.

Let $F \in L^p_\varepsilon(\mathfrak{a}^*)$ and fix $m, n \in \mathbf{Z}(\varepsilon)$. If we set

$$T_F[\Phi] = \int_0^\infty F(\nu)\Phi(\nu)\mu(\varepsilon,\nu)d\nu, \text{ for } \Phi \in \mathcal{C}_{c,mn}(\hat{G}),$$

then $T_F \in \mathcal{C}'_{c,mn}(\hat{G})$.

For an arbitrary function $F : \mathbf{Z}\backslash\{0\} \to \mathbf{C}$, we put

$$T_F[\Phi] = \sum_{\lambda\in L(m,n)} d(\lambda)F(\lambda)\Phi(\lambda), \text{ for } \Phi \in \mathcal{C}_{d,mn}(\hat{G}).$$

Then $T_F \in \mathcal{C}'_{d,mn}(\hat{G})$.

LEMMA 2.7 *Let f be very rapidly decreasing and (m,n)-spherical, and $\mathcal{F}^c_{mn}f \in L^1_\varepsilon(\mathfrak{a}^*)$. Then*

$$\begin{aligned} \mathcal{F}^{-1}\mathcal{F}^c_{(-m)(-n)}T_f &= T_{(\mathcal{S}^c_{(-n)(-m)}\mathcal{F}^c_{(-n)(-m)}\check{f})^\vee}, \\ \mathcal{F}^{-1}\mathcal{F}^d_{(-m)(-n)}T_f &= T_{(\mathcal{F}^d_{(-n)(-m)}\mathcal{F}^d_{(-n)(-m)}\check{f})^\vee}, \end{aligned}$$

where $\check{f}(g) = f(g^{-1})$.

PROPOSITION 2.8 *Let f be very rapidly decreasing and (m,n)-spherical, and $\mathcal{F}^c_{mn}f \in L^1_\varepsilon(\mathfrak{a}^*)$. Then*

$$f(g) = (\mathcal{S}^c_{mn}\mathcal{F}^c_{mn}f)(g) + (\mathcal{S}^d_{mn}\mathcal{F}^d_{mn}f)(g) \quad \text{(a.e.)}.$$

3 The main theorem

We need the following lemma of Cowling-Price [3].

LEMMA 3.1 *Let $1 \le p \le \infty$ and $A > 0$. Let g be an entire function such that*

$$\begin{aligned} |g(x+\sqrt{-1}y)| &\le Ae^{\pi x^2}, \\ \left(\int_{\mathbf{R}} |g(x)|^p dx\right)^{1/p} &\le A. \end{aligned}$$

Then g is a constant function on $\mathbf{C}$. Moreover, if $p < \infty$ then $g = 0$.

By using Proposition 2.8, Lemma 3.1 and a similar argument of [6], we obtain the following proposition.

PROPOSITION 3.2 *Let $1 \le p, q \le \infty$. Let f be a (m,n)-spherical measurable function on G such that*

$$\begin{aligned} \left\| e^{a\sigma(g)^2} f(g) \right\|_{L^p(G)} &\le C, \\ \left\| e^{b\nu^2} (\mathcal{F}^c_{mn}f)(\varepsilon,\nu) \right\|_{L^q_\varepsilon(\mathfrak{a}^*)} &\le C, \end{aligned}$$

for $C > 0$, $a > 0$ and $b > 0$. If $ab > 1/4$ then $f = 0$ (a.e.).

The following main theorem is an easy consequence of Proposition 3.2.

THEOREM 3.3 (the Cowling-Price theorem for $SU(1,1)$) *Let $1 \leq p, q \leq \infty$. Let f be a measurable function on G such that*

$$\begin{aligned} \left\| e^{a\sigma(g)^2} f(g) \right\|_{L^p(G)} &\leq C, \\ \left\| e^{b\nu^2} \mathcal{F}^c f(\varepsilon, \nu) \right\|_{\mathcal{L}^q_\varepsilon(\mathfrak{a}^*)} &\leq C_\varepsilon, \end{aligned}$$

for $C > 0$, $a > 0$ and $b > 0$. If $ab > 1/4$ then $f = 0$ (a.e.).

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