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# Uncertainty principle for the Dunkl transform

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**Abstract**

Analogues of Heisenberg's inequality and Hardy's theorem for the classical Fourier transform are obtained for the Dunkl transform associated with a root system.

## Contents



## 1 Uncertainty principle for the Fourier transform

The Fourier transform of a function $f \in L^1(\mathbb{R})$ is the function defined by

$$\hat{f}(\lambda) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-\sqrt{-1}\lambda x} dx.$$

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### Heisenberg's inequality

Heisenberg's inequality states that for $f \in L^2(\mathbb{R})$,

$$\int_{-\infty}^{\infty} x^2|f(x)|^2dx \int_{-\infty}^{\infty} \lambda^2|\hat{f}(\lambda)|^2d\lambda \geq \frac{1}{4}\left[\int_{-\infty}^{\infty} |f(x)|^2dx\right]^2$$

with equality only if $f(x)$ is almost everywhere equal to a constant multiple of $e^{-px^2}$ for some $p > 0$.

A proof is given in Weyl [10, Appendix 1]. de Bruijn [1] proved Heisenberg's inequality by using expansions with respect to Hermite polynomials.

### Hardy's theorem

Hardy [4] proved a theorem concerning the decay of $f$ and $\hat{f}$ at infinity; let $p$ and $q$ be positive constants and assume that $f$ is a function on the real line satisfying $|f(x)| \leq Ce^{-p|x|^2}$ and $|\hat{f}(\lambda)| \leq Ce^{-q|x|^2}$ for some positive constant $C$. Then (i) $f = 0$ if $pq > 1/4$; (ii) $f = Ae^{-px^2}$ for some constant $A$ if $pq = 1/4$; (iii) there are infinitely many $f$ if $pq < 1/4$.

There is a proof based on the Phragmén-Lindelöf theorem.

## 2 Review on the Dunkl transform

In this section, we review on results of Dunkl [3] and de Jeu [2].

Let $\mathfrak{a} = \mathbb{R}^N$ be a $N$-dimensional real vector space with inner product $(\cdot,\cdot)$. The norm is denoted by $|x| = (x,x)^{1/2}$. For $\alpha \in \mathfrak{a} \setminus \{0\}$ let $r_\alpha$ denote the orthogonal reflection with respect to the hyperplane orthogonal to $\alpha$. Let $G \subset O(\mathfrak{a})$ be a finite reflection group. Let $R$ be the corresponding root system. We will assume that $R$ is a normalized root system, i.e. $(\alpha,\alpha) = 2$ for all $\alpha \in R$. Choose and fix a positive system $R_+ \subset R$.

A complex valued function $k : \alpha \mapsto k_\alpha$ on $R$ which is $G$-invariant is called a multiplicity function. In this article we always assume that $\operatorname{Re} k_\alpha \geq 0$ for all $\alpha \in R$.

Let $\mathfrak{h} = \mathfrak{a} \otimes_{\mathbb{R}} \mathbb{C}$ be the complexification. For $\xi \in \mathfrak{h}$ let $\partial_\xi$ denote the corresponding directional derivative. Define the *Dunkl operator* $T_\xi$ by

$$(T_\xi f)(x) = (\partial_\xi f)(x) + \sum_{\alpha \in R_+} k_\alpha(\alpha,\xi)\frac{f(x) - f(r_\alpha x)}{(\alpha,x)}.$$

We have $[T_\xi, T_\eta] = 0$ for any $\xi$, $\eta \in \mathfrak{h}$. Given $\lambda \in \mathfrak{h}^*$ consider the following system of differential-difference equations on $\mathfrak{a}$:

$$T_\xi(k)f = \lambda(\xi)f, \quad \xi \in \mathfrak{a}. \tag{2.1}$$

**Theorem 2.1 (de Jeu, [2] Theorem 2.6)** *Assume* $\operatorname{Re} k_\alpha \geq 0$ *for all* $\alpha \in R$. *Then there is a unique solution* $\mathrm{Exp}_G(\lambda, k, \cdot)$ *of* (2.1) *such that*
(i) $\mathrm{Exp}_G(\lambda, k, 0) = 1$,
(ii) $\mathrm{Exp}_G(\lambda, k, x)$ *is holomorphic in* $\lambda \in \mathfrak{h}$ *and analytic in* $x \in \mathfrak{a}$.

Let

$$h(x) = \prod_{\alpha \in R_+} |(\alpha, x)|^{k_\alpha}$$

and

$$\gamma = \sum_{\alpha \in R_+} k_\alpha.$$

Then $h$ is a homogeneous $G$-invariant function of degree $\gamma$. Define the normalization constant

$$c_h = \left( (2\pi)^{-N/2} \int_{\mathbb{R}^N} h(x)^2 e^{-|x|^2/2} dx \right)^{-1}.$$

de Jeu [2, Corollary 4.17] proved that the constant $c_h$ is strictly positive (see also [2, Remark 4.12]).

For $f \in L^1(\mathbb{R}^N, h^2 dx)$, let

$$\hat{f}(\lambda) = (2\pi)^{-N/2} c_h \int_{\mathbb{R}^N} f(x) \mathrm{Exp}_G(-\sqrt{-1}\lambda, k, x) h(x)^2 dx,$$

the *Dunkl transform* of $f$. If $k_\alpha = 0$ for all $\alpha \in R$, then it is nothing but the Fourier transform on $\mathbb{R}^N$.

We recall main results of de Jeu [2], the inversion formula and the Plancherel theorem for the Dunkl transform.

**Theorem 2.2 (de Jeu [2], Theorem 4.20, 4.26)** (1) *Assume* $\operatorname{Re} k_\alpha \geq 0$ *for all* $\alpha \in R$. *Let* $f \in L^1(\mathbb{R}^N, h^2 dx)$ *and suppose that* $\hat{f} \in L^1(\mathbb{R}^N, h^2 dx)$. *Then*

$$f(x) = (2\pi)^{-N/2} c_h \int_{\mathbb{R}^N} \hat{f}(\lambda) \mathrm{Exp}_G(\sqrt{-1}\lambda, k, x) h(\lambda)^2 d\lambda.$$

(2) *Assume* $k_\alpha \geq 0$ *for all* $\alpha \in R$. *If* $f \in L^1(\mathbb{R}^N, h^2 dx) \cap L^2(\mathbb{R}^N, h^2 dx)$, *then* $\hat{f} \in L^2(\mathbb{R}^N, h^2 dx)$ *and*

$$\int_{\mathbb{R}^N} |f(x)|^2 h(x)^2 dx = \int_{\mathbb{R}^N} |\hat{f}(\lambda)|^2 h(\lambda)^2 d\lambda.$$

# 3 Uncertainty principle for the Dunkl transform

### Heisenberg's inequality

**Theorem 3.1** *Assume that $k_\alpha \geq 0$ for all $\alpha \in R$. For $f \in L^2(\mathbb{R}^N, h^2dx)$,*

$$\int_{\mathbb{R}^N} |x|^2 |f(x)|^2 h(x)^2 dx \int_{\mathbb{R}^N} |\lambda|^2 |\hat{f}(\lambda)|^2 h(\lambda)^2 d\lambda \geq \left(\gamma + \frac{N}{2}\right)^2 \left[\int_{\mathbb{R}^N} |f(x)|^2 h(x)^2 dx\right]^2$$

*with equality only if $f(x)$ is almost everywhere equal to a constant multiple of $e^{-p|x|^2}$ for some $p > 0$.*

After we proved Theorem 3.1, we noticed that Theorem 3.1 was already proved by Rösler [6].

Our proof of Theorem 3.1 is similar to but simpler than that of Rösler [6]. Rösler used expansions in terms of generalized Hermite polynomials and recurrence relations among them, which were given in [5]. We used expansions in terms of the basis given by Dunkl [3] and recurrence relations for the classical Laguerre polynomial.

### Hardy's theorem

**Theorem 3.2** *Assume $\operatorname{Re} k_\alpha \geq 0$ for all $\alpha \in R$. Let $p$ and $q$ be positive constants. Suppose $f$ is a measurable function on $\mathbb{R}^N$ satisfying*

$$|f(x)| \leq C \exp(-p|x|^2) \quad x \in \mathbb{R}^N \tag{3.1}$$

*and*

$$|\hat{f}(\lambda)| \leq C \exp(-q|\lambda|^2) \quad \lambda \in \mathbb{R}^N, \tag{3.2}$$

*where $C$ is a positive constant. Then we have following results:*

(1) *If $pq > 1/4$, then $f = 0$ almost everywhere.*

(2) *If $pq = 1/4$, then $\hat{f}(\lambda) = A\exp(-q|\lambda|^2)$, where $A$ is an arbitrary constant.*

(3) *If $pq < 1/4$, then there are infinitely many such functions $f$.*

The proof follows closely that of Sitaram and Sundari [9], where analogues of Hardy's theorem were proved for certain function spaces on semisimple Lie groups.

# 4 Concluding remarks

(1) Heisenberg's inequality and Hardy's theorem for the following special cases follow from our results.

  (i) If $N = 1$ and $G = \mathbb{Z}_2$, then the restriction of the Dunkl transform to symmetric functions coincides with the classical Hankel transform.

  (ii) The Dunkl transform sometimes appears "in nature" as the spherical Fourier transform on Riemannian symmetric spaces $X$ of the Euclidean type.

(2) An important property of the Dunkl transform is that it maps $e^{-p|x|^2}$ ($p > 0$) to itself. The property no longer holds for the Heckman-Opdam transform, which is the trigonometric counterpart of the Dunkl transform.

(3) In [8] the author gives an analogue of Theorem 3.2 (1) for the Heckman-Opdam transform.

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