表現論シンポジウム講演集, 1999
pp.135-138

Matrix coefficients of the large discrete series of $SU(2,2)$

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For Tateyama Worsshop, November 1999

1 Formulation of problem

Our concern is to have an explicit formula for the A-radial part of the matrix coefficients with minimal K-types of the large discrete series representations of the Lie group $G = SU(2,2)$.

1.0.1 Notation and definitions

Our Lie group G is the real special unitary group of signature $(2+, 2-)$, i.e. its defining Hermitian matrix is $I_{2,2} = \mathrm{diag}(1,1,-1,-1)$. A Cartan involution is given by $\theta : g \mapsto (g^*)^{-1} = {}^t\bar{g}^{-1}$, and the fixed group $K = G^\theta = G \cap U(4)$ is a maximal compact subgroup.

We denote by X_{ij} the matrix unit in $M_4(\mathbf{C})$ with unique non-zero entry 1 at the (i,j)-th component $(1 \leq i, j \leq 4)$. The Lie algebra of a compact Cartan subgroup is generated by

$$\sqrt{-1}(X_{11} - X_{22}), \sqrt{-1}(X_{33} - X_{44}), \text{ and } \sqrt{-1}I_{2,2}.$$

The absolute roots or the weights β of the Cartan algebra is represented their values on this basis, i.e.

$$\beta = [r, s; u] \quad \text{if } \beta(X_{11} - X_{22}) = r, \beta(X_{33} - X_{44}) = s, \beta(I_{2,2}) = u.$$

We fix a postive system of roots

$$\Delta^+ = \{[2,0;0],[0,2;0],[\pm,\pm;2]\}.$$

with comapct positive roots $\Delta_c^+ = \{[2,0;0],[0,2;0]\}$.

1.1 Discrete series of $SU(2,2)$

Since the half sum of postive roots is integral weight of the compact Cartan subgroup for G, the discrete series representation of G are parametrized by the orbit space of the regular integral weights Ξ by the Weyl group W_K of K. Or we may take the fundametal set Ξ_c cocnsisting of regular integral weights positive with respect to Δ^+c. Let Ξ_I be the subset of Ξ_c consisiting of weights postive with resprect to Δ^+. The the sets

$$\Xi_{II} = \sigma_2(\Xi_I), \Xi_V = \sigma_2\sigma_3\sigma_1(\Xi_I)$$

are parameter spaces of the large discrete series representations of G. Here σ_i $(i = 1, 2, 3)$ are the elements of Weyl group W of G given by

$$\sigma_1([r, s, ; u]) = [-r, s; u], \ \sigma_3([r, s; u]) = [r, -s; u]$$

and

$$\sigma_2([r, s; u]) = [\frac{1}{2}(r - s + u), \frac{1}{2}(-r + s + u); r + s].$$

We consider the case when the Harish-Chandra paramter Λ of the discrete series representation π_Λ belongs to Ξ_{II}. The highest weight of the contragradient representation τ of the minimal K-type τ^* of π_Λ is denoted by $[r, s; u_1]$. Then we have $r > s$.

1.2 Matrix coefficients

The irreducible representation of the maximal compact subgroup $K = S(U(2) \times U(2))$ with highest weight $[r, s; u]$ has dimension $(r + 1)(s + 1)$. The irreducible representaions of the group $SU(2)$ or rather its complexified Lie algebra has the canonical basis determined upto $\pm$ multiple. Then the representation $\tau_{[r,s,;u]}$ of K has the cannonical basis obtained from the canonical basis of the representation of $SU(2)$ by exterior tensor product. We denote this by $\{f_{k,l}^{[r,s;u]} \ (1 \leq k \leq r, 1 \leq l \leq s)\}$. The dual basis of this is denoted by $\{(f_{k,l}^{[r,s;u]})^*\}$.

Let *tau* be an element of $\hat{K}$ such that τ^* is the minimal K-type of π_λ with $\Lambda \in \Xi_{II}$. The representation space V^* of *tau** is considered as a subspace of the representation space H_Λ of π_Λ. Simlilarly the representation space V of τ as a subspace of the representation space H_Λ^* of the contragradient representation π_Λ^* of π_Λ.

We are interested in the functions

$$c_{[k_1,l_1;k_2,l_2]}(g) =< \pi_\Lambda(g)((f_{[k_1,l_1]}^{[r,s,;u]})^*), (f_{[k_2,l_2]}^{[r,s;-u]})^*).$$

Or rather the vector valued function

$$\Phi(g) = \sum_{k_1,l_1,k_2,l_2} c_{[k_1,l_1;k_2,l_2]} f_{[k_1,l_1]}^{[r,s;u]} \otimes f_{[k_2,l_2]}^{[r,s;-u]}.$$

Note here that the contragradient representation *tau** has the highest weight $[r, s, ; -u]$.

To determine Φ it suffices to determine its A-radial part by the Cartan decomposition $G = KAK$.

1.3 The method to obtain the solution

It is the same as other papers of ours. We have explicit formulae of the right and left Schmid operators. This give a holonomic system of rank 4 over A. We can reduced this holonomic system to the modified F_2 system of Appell.

2 The solution

Firstly we show the power series solution. After that it is rewritten as integral expression.

2.1 Macro symbols

Definition We denote by $\mathcal{M}(r,s)$ the set

$\{M=(k_1,l_1;k_2,l_2)|0\leq k_i\leq r, 0\leq l_i\leq s(i=1,2), k_1+l_1+k_2+l_2=r+s\}$.

Moreover for any $M\in\mathcal{M}(r,s)$, we set

$$s(M) = \frac{1}{2}\{(k_1+k_2-r)-(l_1+l_2-s)\}, \quad (1)$$

$$c(M) = \frac{1}{2}\{(k_1-k_2)-(l_1-l_2)\}, \quad (2)$$

$$w(M) = |s(M)|. \quad (3)$$

We call $w(M)$ the *weight* of M.

Definition (Multiplicators) For each $M\in\mathcal{M}(r,s)$, we set

$$m(M)=\binom{r}{k_1}\binom{r}{k_2}\binom{s}{l_1}\binom{s}{l_2}$$

and

$$\mu_M^{\pm}(a)=\{sh(a_1)sh(a_2)\}^{\pm s(M)}\{ch(a_1)ch(a_2)\}^{\pm u_1/2}\{ch(a_1)/ch(a_2)\}^{c(M)}.$$

2.2 Change of functions and change of variables

Definition We write

$$c_M(a)=m(M)\mu_M^{\pm}(a)h_M^{\pm}(a).$$

Definition We introduce new variables x_1, x_2 by

$$x_i=-sh^2(a_i) \quad (i=1,2).$$

2.3 The power series solution

Theorem 1 *Assume* $r > s$ *and* $s(M) \geq 0$. *Then*

$$h_M^+(x) = c^+(r,s)(-1)^{s(M)+c(M)} \sum_{m_1,m_2 \geq 0} \frac{(m_1+k_1)!(m_2+k_2)!}{m_1!m_2!}$$

$$\times \prod_{i_1=1}^{l_2}(m_1+s(M)+i_1)\prod_{i_2=1}^{l_1}(m_2+s(M)+i_2)\xi^+(m_1+m_2-l_1-l_2)x_1^{m_1}x_2^{m_2}.$$

Here $c^+(r,s)$ *is a constant independent of* M, *and*

$$\xi^+(k) = \frac{(a_0)_k(b_0)_k}{(c_0)_k(r+s+2)_k} \quad (\textit{ for } k \in \mathbf{Z},\ -s < k)$$

with

$$a_0 = s+1, \quad b_0 = \beta+s+2 = \frac{1}{2}(u_1+r+s), \quad c_0 = 2(s+1).$$

Remark $(\alpha)_k$ means the Gaussian symbol

$$(\alpha)_k = \prod_{i=0}^{k-1}(\alpha+i).$$

2.4 Ingtegral expression if solution

Theorem 2 *Set*

$$\tilde{h}_M^+(x) = c^+(r,s)(-1)^{s(M)+c(M)}(k_1+l_1)!(k_2+l_2)!$$

$$\times \sum_{m_1,m_2 \geq 0} \frac{(k_1+l_1+1)_{m_1-l_1}(k_2+l_2+1)_{m_2-l_2}}{m_1!m_2!}\xi^+(m_1+m_2-l_1-l_2)x_1^{m_1}x_2^{m_2}.$$

Then

$$h_M^+(x) = (x_1x_2)^{-s(M)}\left(\frac{\partial}{\partial x_1}\right)^{l_1}\left(\frac{\partial}{\partial x_2}\right)^{l_2}\{(x_1x_2)^{s(M)}x_1^{l_2}x_2^{l_1}\tilde{h}_M^+(x)\},$$

and

$$\tilde{h}_M^+(x) = \textit{const.} \times \int_0^1 {}_2F_1(s(M)+1, \beta+s(M)+2; s+s(M)+2; tx_1+(1-t)x_2)t^{k_1}(1-t)^{k_2}dt.$$

Remark We can determine the constant before the integral symbol. But it is omitted here.

Hayata, T., Koseki,H., Oda, T., *Matrix coeeficients of the middle discrete series of* $SU(2,2)$, Oct. 1999