表現論シンポジウム講演集, 1999
pp.87-98

On the differential equations satisfied by weighted orbital integrals

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Introduction

Orbital integrals have played an important role in the work of Harish-Chandra about harmonic analysis on reductive groups, and they appear in the geometric terms of the Selberg trace formula. They are distributions on a reductive group G over a local field F. We suppose that G is connected in the Zariski topology and denote the set of its F-rational points by the same letter. Given $g \in G$ and a test function f on G, the orbital integral is defined as

$$J_G(g,f) = |D(g)|^{1/2} \int_{G/G_g^0} f(xgx^{-1})\, d\dot{x},$$

where G_g^0 denotes the connected component of the unit element in the centralizer of g in G and $d\dot{x}$ stands for a fixed invariant measure on G/G_g^0. Moreover, we have set

$$D(g) = \det_{\mathfrak{g}/\mathfrak{g}_s}(\mathrm{Id} - \mathrm{Ad}(s)),$$

where s is the semisimple component in the Jordan decomposition of g and $\mathfrak{g}_s$ the centralizer of s in the Lie algebra $\mathfrak{g}$ of G.

In the Selberg trace formula for noncompact quotients, which was generalized by Arthur to groups of arbitrary rank, not all geometric contributions can be expressed in terms of orbital integrals. Here, one needs a generalization, the so-called weighted orbital integrals. They depend on a Levi subgroup M of G and an element m of M, and reduce to orbital integrals when $M = G$. If $G_m^0 \subset M$, the weighted orbital integral is defined as

$$J_M(m,f) = |D(g)|^{1/2} \int_{G/G_m^0} f(xmx^{-1}) v_M(x)\, d\dot{x}. \qquad (1)$$

The weight factor v_M, which depends implicitly on the choice of a special maximal compact subgroup of G, is defined as follows. Let H_M be the natural map from M to the vector space $\mathfrak{a}_M := \mathrm{Hom}(X(M)_F, \mathbb{R})$. Then $v_M(x)$ is the volume of the convex hull of the points $H_P(x)$ in $\mathfrak{a}_M/\mathfrak{a}_G$, where P runs through the set of parabolic F-subgroups of G with Levi component M and where $H_P(x) = H_M(m)$ if we write $x = nmk$ with $k \in K$, $m \in M$ and n in the unipotent radical of P. If necessary, we indicate the ambient group as a superscript, thus $J_M(m,f) = J_M^G(m,f)$.

In this report we shall be concerned with the differential equations satisfied by weighted orbital integrals in the case when G is defined over the field of real numbers. In order to state these differential equations, we have to fix a maximal $\mathbb{R}$-torus T of M, which is, of course, also a maximal torus of G. Denote by $Z(\mathfrak{g})$ be the centre of $U(\mathfrak{g})$, the universal enveloping algebra of $\mathfrak{g}$, and by G_{reg} the set of regular semisimple elements of G. If $f \in C_c^\infty(G)$ and the invariant measure on $G/G_t^0 = G/T$ is chosen independently of $t \in T \cap G_{\mathrm{reg}}$, then $J_M(t,f)$ depends smoothly on t. Proposition 11.1 of [3] asserts the following. For each pair (L,M), where $L \supset M$ are Levi $\mathbb{R}$-subgroups of G containing T, there is a linear map ∂_M^L from $Z(\mathfrak{l})$ to the algebra of differential operators on $T \cap L_{\mathrm{reg}}$ such that

$$J_M^{L'}(t,zf) = \sum_{M \subset L \subset L'} \partial_M^L(t,z_L) J_L^{L'}(t,f) \tag{2}$$

for all pairs (L',M) as above, $t \in T \cap L'_{\mathrm{reg}}$, $z \in Z(\mathfrak{l}')$ and $f \in C_c^\infty(L')$. Here the summation runs over all Levi subgroups L sandwiched between M and L', and $z \mapsto z_L$ denotes the Harish-Chandra homomorphism $Z(\mathfrak{l}') \to Z(\mathfrak{l})$. Moreover, the family of maps ∂_M^L with the stated properties is unique. We have inserted t as an argument in $\partial_M^L(t,z)$ because we usually regard the latter as an element of $U(\mathfrak{t})$ depending on t.

The motivation for our research is connected with the notion of invariant Fourier transform. If f belongs to Harish-Chandra's Schwarz space $\mathcal{C}(G)$ and π is an irreducible tempered representation of G, we denote by $\hat{f}(\pi)$ the trace of the operator $\pi(f) = \int_G f(x)\pi(x)\,dx$. This defines a function $\hat{f}$ on the tempered dual $\Pi_{\mathrm{temp}}(G)$, and the space $\mathcal{I}(G)$ of all such functions can be endowed with a topology such that $f \mapsto \hat{f}$ is a continuous open surjective map $\mathcal{C}(G) \to \mathcal{I}(G)$. If I is a tempered invariant distribution on G, i.e., a continuous linear functional on $\mathcal{C}(G)$, then one can define its Fourier transform $\hat{I}$ as a continuous linear functional on $\mathcal{I}(G)$ by requiring $\hat{I}(\hat{f}) = I(f)$ for $f \in \mathcal{C}(G)$.

The distributions $J_M(m,f)$ are tempered (at least for $m \in M \cap G_{\mathrm{reg}}$), but not invariant, so they would require the more complicated notion of operator-valued Fourier transform. However, in the paper [2] Arthur has constructed invariant distributions $I_M(m,f)$ out of the weighted orbital integrals and their counterparts, the weighted characters. Their importance lies in the fact that the trace formula can also be given in terms of these invariant distributions. Arthur has also shown in [5] that their Fourier transforms $\hat{I}_M(m)$ are regular distributions on $\Pi_{\mathrm{temp}}(G)$ in the sense that there exist functions $\hat{I}_M(m,\pi)$ such that

$$\hat{I}_M(m,\psi) = \int_{\Pi_{\mathrm{temp}}(G)} \hat{I}_M(m,\pi)\psi(\pi)\,d\pi$$

for $\psi \in \mathcal{I}(G)$, where $d\pi$ is a certain canonical measure.

It is desirable to calculate these Fourier transforms. At present, two methods exist for this purpose. One is to use the local trace formula with test functions approximating a delta distribution. This yields the restriction of $\hat{I}_M(m,\pi)$ to the discrete part of the tempered dual (cf. [4]). The other method is based on the fact that (2) remains valid if one replaces J_M by I_M (cf. [5], § 7), which implies that

$$\chi(z)\hat{I}_M^{L'}(t,\pi) = \sum_{M\subset L\subset L'} \partial_M^L(t,z_L)\hat{I}_L^{L'}(t,\pi) \tag{3}$$

if χ is the infinitesimal character of π. The idea is to find the general solution of this system of equations and then to determine $\hat{I}_M(t,\pi)$ from its behaviour when t tends to infinity or to the singular set, respectively. However, one knows a general formula for the operator $\partial_M^L(t,z)$ only under the assumption that $M = L$ or that z is the Casimir element. Accordingly, the second method has been realized so far only for the cases where the only nonzero term on the right-hand side of the differential equation is that with $M = L$ (see [8], [9]) or the real rank of G is one (see [10]).

In the following we describe the results of a manuscript in preparation where we take the first step towards the solution of these equations in the general case. We show that this system of differential equations has a simple singularity at infinity. This implies that the solution has an expansion into an absolutely convergent series on each component of the regular set. We find that the coefficients in this series satisfy a certain recursion formula. Finally, we give the complete solution for a number of groups of small rank.

1 The radial decomposition

As a prerequisite to our main result, we want to establish the vanishing at infinity of certain coefficients in the radial decomposition of invariant differential operators on reductive groups. The radial decomposition we consider comes from the local product structure of G_{reg} given by G-orbits and maximal tori, and it has already been studied in [7].

Let G be a linear algebraic $\mathbb{R}$-group. For each $y \in G$, there is a unique representation of $U(\mathfrak{g})$ on itself, written as $\Gamma_y : U(\mathfrak{g}) \otimes U(\mathfrak{g}) \to U(\mathfrak{g})$, such that

$$uF(x,y) = 1 \otimes \Gamma_y(u)F(x,y)$$

for all $u \in U(\mathfrak{g}) \otimes U(\mathfrak{g}) \cong U(\mathfrak{g}\times\mathfrak{g})$ whenever $F \in C^\infty(G\times G)$ is given by

$$F(x,y) = f(xyx^{-1}) \tag{4}$$

for some $f \in C^\infty(G)$.

¿From now on, we suppose that G is connected and reductive, and we fix a maximal $\mathbb{R}$-torus T of G. Let $\mathfrak{q}$ be the T-stable complement of $\mathfrak{t}$ in $\mathfrak{g}$, and let $\mathfrak{Q}$ denote the image of the symmetric algebra $S(\mathfrak{q})$ under the canonical G-equivariant bijection $S(\mathfrak{g}) \to U(\mathfrak{g})$. By the Poincare-Birkhoff-Witt theorem, the product map $\mathfrak{Q} \otimes U(\mathfrak{t}) \to U(\mathfrak{g})$ is bijective. Harish-Chandra has shown in [7] that for $t \in T \cap G_{\mathrm{reg}}$ the map Γ_t restricts to a T-equivariant bijection

$$\mathfrak{Q} \otimes U(\mathfrak{t}) \to U(\mathfrak{g})$$

compatible with filtrations. Moreover, if one denotes the preimage of an element v by $\gamma(t,v)$, then $D_T(t)^r\gamma(t,v)$ extends analytically to $t \in T$, where $r = \deg v$ and

$$D_T(t) = \det_{\mathfrak{g}/\mathfrak{t}}(\mathrm{Id} - \mathrm{Ad}(t)).$$

If we set $\mathfrak{Q}' = \mathfrak{Q} \cap U(\mathfrak{g})\mathfrak{g}$, we can write

$$\gamma(t,v) = 1 \otimes \beta(t,v) + \delta(t,v),$$

where $\beta(t,v) \in U(\mathfrak{t})$ (the so-called radial component of v) and $\delta(t,v) \in \mathfrak{Q}' \otimes U(\mathfrak{t})$ are uniquely determined. We are mainly interested in the case when $v = z \in Z(\mathfrak{g})$. Then $\gamma(t,z)$ is centralized by T. Theorem 2 of [7] says that

$$|D_T|^{1/2}\beta(z) \circ |D_T|^{-1/2} = z_T,$$

where the factors on the left-hand side are considered as differential operators on $T \cap G_{\mathrm{reg}}$ and z_T is the image of z under the Harish-Chandra isomorphism.

We want to study $\delta(t,z)$ when t tends to infinity in a sense which we are now going to make precise. Let A denote the greatest $\mathbb{R}$-split subtorus of T and $\mathcal{P}$ the set of all parabolic $\mathbb{R}$-subgroups of G which have the centralizer of A as Levi component. For each such P, let Σ_P denote the set of roots of $\mathfrak{t}_{\mathbb{C}}$ in the unipotent radical of $\mathfrak{p}_{\mathbb{C}}$. If f is a map from a subset of T to a topological space X, we write $\lim\limits_{t \to_P \infty} f(t) = x$ if for each neighbourhood U of x there exists $s \in \mathbb{R}$ such that for all t in the domain of f satisfying $|t^\alpha| > s$ for all $\alpha \in \Sigma_P$ we have $f(t) \in U$.

Proposition 1 *If $z \in Z(\mathfrak{g})$ and $P \in \mathcal{P}$, then $\lim\limits_{t \to_P \infty} \delta_T(t,z) = 0$ in the finite-dimensional subspace of $\mathfrak{Q}' \otimes U(\mathfrak{t})$ of elements of degree not exceeding that of z.*

By an algebraic argument, we reduce the proof of this result to the case when T is split (and hence P is Borel). In that case, we exploit the fact that, roughly speaking, the fibering of G_{reg} in a neighbourhood of t by conjugacy classes, when we transfer it to a neighbourhood of 1 by left multiplication with t^{-1}, will approach the fibering of the big Bruhat cell by double $(\bar{N}, N)$-cosets as $t \to_P \infty$.

2 Arthur's differential operators

We want to prove that the system of partial differential equations (3) has a simple singularity as $t \to_P \infty$. This comes down to four properties of the differential operators $\partial_M^G(t,z)$ defined by equation (2), which we are now going to formulate in four lemmas. In the situation of the previous section, let M be a Levi $\mathbb{R}$-subgroup of G containing T. The case $M = G$ presents no problems, since

$$\partial_G^G(t,z) = z_T \tag{5}$$

for all $z \in Z(\mathfrak{g})$ and $t \in T \cap G_{\text{reg}}$ by Lemma 12.4 of [3]. For the other Levi subgroups, the following assertion is an easy consequence of Proposition 1, because the operator $\partial_M^G(t,z)$ can be calculated from the radial decomposition of z as described in Lemma 12.1 of [3].

Lemma 1 *If $M \neq G$, then $\lim\limits_{t \to_P \infty} \partial_M^G(t,z) = 0$ for all $z \in Z(\mathfrak{g})$ and $P \in \mathcal{P}$.*

Let Σ be the root system of $(\mathfrak{g}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}})$ and Λ the free abelian group generated by Σ. For each $\alpha \in \Sigma$, we have a character $t \mapsto t^\alpha$ of T defined by $\mathrm{Ad}(t)X = t^\alpha X$ for $X \in \mathfrak{g}_{\mathbb{C},\alpha}$, and this gives rise to characters $t \mapsto t^\mu$ for $\mu \in \Lambda$ such that $t^{\mu_1+\mu_2} = t^{\mu_1}t^{\mu_2}$. Let us denote by Σ_M the set of weights of T in $\mathfrak{g}/\mathfrak{m}$ and by Λ_M the subgroup of Λ generated by Σ_M. Moreover, let

$$D_M(t) = \det_{\mathfrak{g}/\mathfrak{m}}(\mathrm{Id} - \mathrm{Ad}(t)) = \prod_{\alpha \in \Sigma_M} (1 - t^\alpha).$$

Lemma 2 *If $z \in Z(\mathfrak{g})$ with $\deg z \leq r$, then there exist $u_\mu \in U(\mathfrak{t}_{\mathbb{C}})$ for all $\mu \in \Lambda_M$, only finitely many of them nonzero, such that*

$$\partial_M^G(t,z) = D_M(t)^{-r} \sum_{\mu \in \Lambda_M} t^\mu u_\mu.$$

A similar assertion about $\gamma(t,z)$ has been proved in Lemma 23 of [7], and just as in the proof of Lemma 1 this implies our result in the case $M = T$, provided the latter is a Levi subgroup. For the proof in the general case, we have to define a relative variant $\gamma_M(t,z)$ and use its transitivity. The next result, however, is almost obvious.

Lemma 3 *If $M \neq G$, then $\deg \partial_M^G(t,z) < \deg z$.*

Given a Levi $\mathbb{R}$-subgroup L of G containing M, let $\mathcal{L}_M^L$ denote the set of all Levi $\mathbb{R}$-subgroups of L containing M. This set is contained in $\mathcal{L}_M := \mathcal{L}_M^G$. The last result in this section follows easily from the uniqueness of the operators ∂_M^L.

Lemma 4 *For all z_1, $z_2 \in Z(\mathfrak{g})$ we have*

$$\partial_M^G(t, z_1 z_2) = \sum_{L \in \mathcal{L}_M} \partial_M^L(t, z_{1,L}) \partial_L^G(t, z_2).$$

3 A system with simple singularity

It is customary in the theory of differential equations with regular singular points to transfer the singularity to the origin by a substitution of variables and to allow complex arguments. In our case this leads to a toric variety, which we are now going to introduce. Given a parabolic $\mathbb{R}$-subgroup P of G containing T, we consider the variety

$$V_P = \mathrm{Hom}_{\text{rings}}(\mathbb{Z}[\Lambda_P], \mathbb{C}),$$

where Λ_P denotes the submonoid of Λ generated by Σ_P, the set of roots of $\mathfrak{t}_\mathbb{C}$ in the unipotent radical $\mathfrak{n}_\mathbb{C}$ of $\mathfrak{p}_\mathbb{C}$. The value of an element $v \in V_P$ at $\mu \in \Lambda_P$ will be denoted by v_μ. There is a natural map $m_P : T_\mathbb{C} \to V_P$ given by $m_P(t) = t^{-\mu}$. The operation $(v'v'')_\mu := v'_\mu v''_\mu$ gives V_P the structure of a monoid, and thereby m_P defines an action of $T_\mathbb{C}$ on V_P. This action factors through $\mathrm{Ad}(T_\mathbb{C})$ and turns V_P into an affine toric variety. The corresponding fan consists of the single cone spanned by Σ_P.

In V_P, we consider the subset

$$B_P = \{v \in V_P \mid |v_\alpha| < 1 \quad \forall \alpha \in \Sigma_P\}.$$

Let $\mathcal{O}_P$ be the algebra of regular functions on this analytic space, i.e., the subalgebra of $\mathbb{C}[[\Lambda_P]]$ consisting of those formal series which become convergent if we choose any $v \in B_P$ and replace each symbol μ in the series by the complex number v_μ. Although the action of $T_\mathbb{C}$ via m_P does not leave B_P invariant, the associated action of $\mathfrak{t}_\mathbb{C}$ does leave $\mathcal{O}_P$ invariant. Explicitly, if η_μ denotes the coordinate function $\eta_\mu(v) = v_\mu$, then $X\eta_\mu = -\mu(X)\eta_\mu$ for $X \in \mathfrak{t}_\mathbb{C}$. Thereby $U(\mathfrak{t}_\mathbb{C})$ may be identified with an algebra of differential operators on B_P. In $\mathcal{O}_P$, we have the $t_\mathbb{C}$-invariant ideal

$$\mathcal{I}_P = \sum_{\alpha \in \Sigma_P} \eta_\alpha \mathcal{O}_P$$

of functions vanishing on the complement of $m_P(T_\mathbb{C})$.

Our main object of study is the system of differential equations

$$\chi(z)\phi_M(t) = \sum_{L \in \mathcal{L}_M} \partial_M^L(t, z_L)\phi_L(t) \qquad \forall M \in \mathcal{L}_{M_0},\ z \in Z(\mathfrak{g}_\mathbb{C}). \tag{6}$$

Here, a character χ of $Z(\mathfrak{g}_\mathbb{C})$ and a Levi $\mathbb{R}$-subgroup M_0 containing T are given, and we are looking for smooth functions ϕ_M on the subset of T where the coefficients of all occurring differential operators are regular. This set is the union of the open sets

$$T_P := \{t \in T \mid |t^\alpha| > 1 \quad \forall \alpha \in \Sigma_P\}$$

parametrized by the groups $P \in \mathcal{P}_{M_0}$. Let $\mathcal{D}(T_P)$ denote the algebra of differential operators on T_P. Given $z \in Z(\mathfrak{g}_\mathbb{C})$, we define $\partial_P(z) \in \mathcal{D}(T_P) \otimes \mathrm{End}\,\mathbb{C}^{\mathcal{L}_{M_0}}$ by

$$(\partial_P(z)\phi)_M = \sum_{L \in \mathcal{L}_M} \partial_M^L(z_L)\phi_L$$

for all $M \in \mathcal{L}_{M_0}$ and all tuples $\phi \in C^\infty(T_P)^{\mathcal{L}_{M_0}}$ of functions indexed by $\mathcal{L}_{M_0}$. We want to extend both differential operators and solutions to the complex domain

$$T_{\mathbb{C},P} := \{t \in T_\mathbb{C} \mid |t^\alpha| > 1 \quad \forall \alpha \in \Sigma_P\} = \{t \in T_\mathbb{C} \mid m_P(t) \in B_P\}.$$

Let $\mathcal{O}(T_{\mathbb{C},P})$ denote the algebra of holomorphic functions and $\mathcal{D}(T_{\mathbb{C},P})$ the algebra of holomorphic differential operators on $T_{\mathbb{C},P}$. On the space $\mathcal{O}(T_{\mathbb{C},P})^{\mathcal{L}_{M_0}}$, we have a natural action of $U(\mathfrak{t}_{\mathbb{C}})$, and we have an action of $\mathcal{D}_P^0 := \mathcal{O}_P \otimes \operatorname{End} \mathbb{C}^{\mathcal{L}_{M_0}}$ by multiplication with the function pulled back via m_P. Let $\mathcal{D}_P$ be the $\mathcal{D}_P^0$-submodule generated by $U(\mathfrak{t}_{\mathbb{C}})$ in the left $\mathcal{D}_P^0$-module $\mathcal{D}(T_{\mathbb{C},P}) \otimes \operatorname{End} \mathbb{C}^{\mathcal{L}_{M_0}}$. The latter module is, of course, an algebra, and $\mathcal{D}_P$ is a subalgebra.

Proposition 2 *Let P be a parabolic $\mathbb{R}$-subgroup of G containing T.*

(i) For each $z \in Z(\mathfrak{g}_{\mathbb{C}})$, the operator $\partial_P(z)$ extends to a holomorphic differential operator belonging to $\mathcal{D}_P$, and $\partial_P(z) - z_T \in \mathcal{I}_P \mathcal{D}_P$.

(ii) The map $\partial_P : Z(\mathfrak{g}_{\mathbb{C}}) \to \mathcal{D}_P$ is a homomorphism of $\mathbb{C}$-algebras.

(iii) If I is an ideal of finite codimension in $Z(\mathfrak{g}_{\mathbb{C}})$, then $\mathcal{D}_P / \mathcal{D}_P \partial_P(I)$ is a finitely generated $\mathcal{D}_P^0$-module.

The series expansions of the coefficients of $\partial_P(z)$ can be obtained from Lemma 2, and Lemma 1 implies the other assertions of (i). Item (ii) follows from Lemma 4, and (iii) can be deduced from Lemma 3.

The system of differential equations (6) is a special case of the system

$$\partial_P(z)\phi = 0 \qquad \forall\, z \in I, \tag{7}$$

where I is an ideal of finite codimension in $Z(\mathfrak{g}_{\mathbb{C}})$. This system also makes sense if applied to $\phi \in \mathcal{O}(T_{\mathbb{C},P})^{\mathcal{L}_{M_0}}$. If P is a Borel subgroup and T is split, then m_P embeds $T_{\mathbb{C}}$ into the vector space V_P. In this case, Proposition 2 says that the system (7), extended to the polycylinder B_P, has a simple singularity. Now Theorem B.16 of [11] is applicable and provides immediately the assertions of Proposition 3 below. In general, however, V_P is only a toric variety. Although singular analytic spaces are allowed in the more general theory of [6], the latter cannot be applied here because we are not able to check that the connection obtained from (7) is integrable. Therefore, we adapt the simple theory exposed in Appendix B of [11] to our situation. This allows us to prove the main result, which is as follows.

Proposition 3 *Let $\phi \in C^\infty(T_P)^{\mathcal{L}_{M_0}}$ be a solution of (7). Then there exist a finite set $E \subset \mathfrak{t}_{\mathbb{C}}^*$ and functions a_M on $(E - \Lambda_P) \times (T \cap T_0) \times \mathfrak{t}_P$ such that*

$$\phi_M(t \exp X) = \sum_{\mu \in E - \Lambda_P} a_M(\mu, t, X) e^{\mu(X)}$$

(absolutely convergent series). Moreover, there is a natural number r such that $a_M(\mu, t, X)$ is a polynomial in X of degree at most r for each μ and t.

Here T_0 denotes the maximal compact subgroup of $T_{\mathbb{C}}$. This is a connected group intersecting every connected component of T. In the proof, we reduce the system (7) to a system of first-order differential equations using Proposition 2(ii) and (iii). We lift the solution to the universal cover of T_P and extend it holomorphically. This projects down to a multivalued function on $T_{\mathbb{C},P}$, which has the same monodromy as a suitable complex power of characters of $T_{\mathbb{C}}$. Then we prove and apply a lemma about removable singularities of functions on B_P as a substitute for the usual results about functions on smooth manifolds. The Proposition then follows by mimicking the classical argument.

The second assertion of Proposition 2(i) implies that replacing $\partial_P(z)$ by z_T yields the corresponding Euler system in the sense of [11], p. 706. This has the usual consequence, which we now formulate. In the notation of the last proposition, let $a(\mu)$ denote the tuple of the functions $a_M(\mu)$ on $(T \cap T_0) \times \mathfrak{t}_P$ parametrized by $\mathcal{L}_{M_0}$. One calls $\mu_0 \in \mathfrak{t}_{\mathbb{C}}$ a leading exponent of ϕ if $a(\mu_0) \neq 0$ and $a(\mu_0 + \mu) = 0$ for all nonzero $\mu \in \Lambda_P$. The set of leading exponents is, of course, the smallest set E making the statement of Proposition 3 true. Let I_T denote the image of the ideal I under the Harish-Chandra isomorphism.

Proposition 4 *Let E be the set of leading exponents of a solution ϕ of the system (7). Then E is contained in the zero set of the ideal $U(\mathfrak{t}_{\mathbb{C}})I_T$ of $U(\mathfrak{t}_{\mathbb{C}})$ considered as the algebra of polynomial functions on $\mathfrak{t}_{\mathbb{C}}^*$.*

Given $\lambda \in \mathfrak{t}_{\mathbb{C}}^*$, we have the character χ_λ of $Z(\mathfrak{g}_{\mathbb{C}})$ given by $\chi_\lambda(z) = z_T(\lambda)$. This character depends only on the orbit of λ under the Weyl group W of $(\mathfrak{g}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}})$. If I is the kernel of χ_λ, then I_T is the set of all W-invariant elements of $U(\mathfrak{t}_{\mathbb{C}})$ vanishing at λ. Thus we get:

Corollary 1 *Let E be the set of leading exponents of a solution ϕ of the system (6) for $\chi = \chi_\lambda$. Then E is contained in the W-orbit of λ.*

4 The recursion formula

Fix, as above, $G \supset M_0 \supset T$ and P with Levi component M_0, and let $\lambda \in \mathfrak{t}_{\mathbb{C}}^*$ be T-integral. We look for solutions of the system

$$\chi_\lambda(z)\phi_M(t) = \sum_{L \in \mathcal{L}_M} \partial_M^L(t, z_L)\phi_L(t) \qquad \forall\, z \in Z(\mathfrak{g}_{\mathbb{C}}),\ M \in \mathcal{L}_{M_0}$$

in the form

$$\phi_M(t) = \sum_{\mu \in \Lambda_{MP}} a_M(\mu) t^{\lambda - \mu}$$

such that $a_M(0) = 0$ for $M \neq G$. Note that $\Lambda_G = \{0\}$, and we can w.l.o.g. assume that $a_G(0) = 1$. We shall see that the equation for z equal to the Casimir element ω alone then determines all other coefficients. Therefore we shall write $\phi_M(t, \lambda)$ and $a_M(\mu, \lambda)$.

Before we recall the formula for $\partial_M^G(t, \omega)$, let us fix the normalization of the differential operators. If one uses the same invariant measure on G/T in the definition of the weighted orbital integrals $J_M(t, f)$ for $t \in T \cap G_{\mathrm{reg}}$ and all Levi

$\mathbb{R}$-subgroups M of G containing T, then $\partial_M^G(t,z)$ does not depend on the choice of this measure. However, $v_M(x)$ still depends on the choice of an invariant measure on A_M/A_G (or rather its Lie algebra). Let η_M^G be a top degree real alternating form on $\mathfrak{a}_M/\mathfrak{a}_G =: \mathfrak{a}_M^G$ whose absolute value defines this volume element. We denote the pull-back of η_M^G to a_M and its $\mathbb{C}$-linear extension to $a_{M,\mathbb{C}}^*$ by the same letter. Replacing G by $L \in \mathcal{L}_M$, we can choose η_M^L on $\mathfrak{a}_M$ so that

$$\eta_M^G = \eta_M^L \wedge \eta_L^G.$$

Now let $\langle\,.\,,\,.\,\rangle$ be a nondegenerate symmetric Ad-invariant bilinear form on $\mathfrak{g}$. By restriction and $\mathbb{C}$-linear extension, we get a nondegenerate symmetric W-invariant form on $\mathfrak{t}_\mathbb{C}$ and thus, on $\mathfrak{t}_\mathbb{C}^*$. The latter bilinear form can be regarded as an element of $S(\mathfrak{t}_\mathbb{C}) = U(\mathfrak{t}_\mathbb{C})$, which is of the form ω_T for a unique element $\omega \in Z(\mathfrak{g}_\mathbb{C})$ called the Casimir element. We suppose that $\langle\alpha,\alpha\rangle > 0$ for $\alpha \in \Sigma$. We recall from (5) that

$$\partial_G^G(\omega) = \omega_T.$$

By calculations similar to [1], p. 571, it can easily be deduced from [3], Lemma 12.1 that we have

$$\partial_M^G(\omega) = 2 \sum_{\alpha\in\Sigma_M/\pm 1} \frac{|\langle\eta_M^G,\alpha\rangle|}{(t^\alpha-1)(t^{-\alpha}-1)}$$

if M is maximal and $\partial_M^G(\omega) = 0$ otherwise. For $t \in T_P$, our differential equation takes the form

$$(\langle\lambda,\lambda\rangle - \omega_T)\phi_M(t,\lambda) = 2 \sum_{\substack{L\supset M\\ \dim \mathfrak{a}_M^L=1}} \phi_L(t,\lambda) \sum_{\alpha\in\Sigma_{MP}^{LP}} |\langle\eta_M^G,\alpha\rangle| \sum_{n=1}^{\infty} n t^{-n\alpha}.$$

Inserting the series expansion for ϕ_M and ϕ_L, we obtain the recursion formula

$$\langle\mu-2\lambda,\mu\rangle a_M(\mu,\lambda) = 2 \sum_{\substack{L\supset M\\ \dim \mathfrak{a}_M^L=1}} \sum_{\alpha\in\Sigma_{MP}^{LP}} |\langle\eta_M^G,\alpha\rangle| \sum_{n=1}^{\infty} n a_L(\mu-n\alpha,\lambda). \quad (8)$$

It is clear that thereby $a_M(\mu,\lambda)$ is determined for generic λ. An attempt to solve these equations explicitly runs soon into combinatorial difficulties.

Although we cannot state a general formula, there is some evidence that the following constructions exhibit some features of the general picture. Here, for a subset A of Σ, we set $\check{A} := \{\check{\alpha} \mid \alpha \in A\}$.

Definition 1 *A root cone in Σ is a triple (A,B,c), where A, B are subsets of Σ spanning the same real vector space V (hence $\check{A}$, $\check{B}$ span the same vector space $\check{V}$) and $c : V \times \check{V} \to \mathbb{R}$ is a bilinear pairing such that* *(i) the rays $\mathbb{R}_+\alpha$ for $\alpha \in A$ (resp. $\mathbb{R}_+\check{\beta}$ for $\beta \in B$) are the edges of a polyhedral cone C_A (resp. $C_{\check{B}}$),*

(ii)

$$C_A = \{\lambda \in V \mid c(\lambda, \check{\beta}) > 0 \quad \forall \beta \in B\},$$
$$C_{\check{B}} = \{v \in \check{V} \mid c(\alpha, v) > 0 \quad \forall \alpha \in A\},$$

(iii) there exists a natural number s such that, for all $\alpha \in A$ and $\beta \in B$,

$$c(\alpha, \check{\beta}) \in \{0, s\},$$

(iv) for all β, $\beta' \in B$ we have

$$2\langle \beta, \beta' \rangle = c(\beta, \check{\beta}')\langle \beta', \beta' \rangle + c(\beta', \check{\beta})\langle \beta, \beta \rangle.$$

The special case $\beta = \beta'$ of the last condition just means that $c(\beta, \check{\beta}) = 1$ for all $\beta \in B$. It is easy to see that if B, c satisfy condition (iv) alone, then c is nondegenerate.

Notice that by condition (ii) the pairing c puts the cones C_A and $C_{\check{B}}$ in duality. In view of conditions (i) and (iii), each element of A corresponds to a maximal face of $C_{\check{B}}$, and $c(\alpha, \check{\beta})$ is basically the incidence matrix between maximal faces and edges of that cone. Since each edge contains at most one (co)root, one can recover B (or, dually, A) from the other data.

Let M^1 be the kernel of H_M. We denote by $\pi_M : \mathfrak{t} \to \mathfrak{a}_M$ the projection along $\mathfrak{t} \cap \mathfrak{m}^1$ as well as its $\mathbb{C}$-linear extension. Let (A, B, c) be a root cone in Σ. We call it a root cone for the parabolic subgroup MP if A and B are subsets of Σ_{MP} and π_M maps $\check{V}$ bijectively on $\mathfrak{a}_M^G$. If, moreover, there is no other root cone (A', B', c) for MP such that B is a proper subset of B', then we call it a maximal root cone for MP.

If (A, B, c) is a root cone for MP, then the pullback of η_M^G under π_M is a volume form $\eta_{\check{V}}$ on $\check{V}$. Given $\mu \in \Lambda \cap \bar{C}_A$, we define a function on $C_{\check{B}}^* := \{\lambda \in \mathfrak{t}_{\mathbb{C}}^* \mid \operatorname{Re}\lambda(X) < 0 \quad \forall X \in C_{\check{B}}\}$ by the absolutely convergent integral

$$a_{B,c}(\mu, \lambda) = \int_{C_{\check{B}}} e^{\lambda(X) - c(\mu, X)} |d\eta_{\check{V}}(X)|.$$

If $C_{\check{B}}$ (and therefore C_A) is a simplicial cone, then we can choose a numbering $B = \{\beta_1, \ldots, \beta_l\}$, and an easy calculation shows that

$$a_{B,c}(\mu, \lambda) = \frac{|\eta_{\check{V}}(\check{\beta}_1, \ldots, \check{\beta}_l)|}{\prod_{\beta \in B}(c(\mu, \check{\beta}) - \lambda(\check{\beta}))}.$$

A general cone $C_{\check{B}}$ can be triangulated by simplicial cones, and $a_{B,c}$ is then a linear combination of functions of the aforementioned type, hence a rational function in λ with at most simple poles along the hyperplanes $\lambda(\check{\beta}) \in \mathbb{N}_0$ for $\beta \in B$.

Given $\Psi \subset \Lambda \cap \bar{C}_A$, we define a function on $T_{\mathbb{C},MP} \times C^*_{\check{B}}$ by

$$\phi_{B,c,\Psi}(t,\lambda) = \sum_{\mu \in \Psi} a_{B,c}(\mu,\lambda) t^{-\mu}.$$

If Π is a linearly independent subset of $\Lambda \cap \bar{C}_A$ and Λ_Π the submonoid of Λ generated by Π, then

$$\phi_{B,c,\Lambda_\Pi}(t,\lambda) = \int_{C_{\check{B}}} \frac{e^{\lambda(X)}}{\prod_{\gamma \in \Pi}(1 - t^{-\gamma} e^{-c(\gamma,X)})} |d\eta_{\check{V}}(X)|,$$

and both integral and series are absolutely convergent. The latter is true in general, because Ψ is contained in a finite union of sets of the form Λ_Π. Thus $\phi_{B,c,\Psi}(t,\lambda)$ is holomorphic for $\lambda \in C^*_{\check{B}}$. If λ varies in a bounded subset of $\mathfrak{t}_{\mathbb{C}}$, then for sufficiently large n_0 the functions

$$a_{B,c}(\mu,\lambda) \prod_{\beta \in B} \prod_{n=0}^{n_0} (n - \lambda(\check{\beta}))$$

are holomorphic and uniformly bounded by a polynomial in $||\mu||$, hence $\phi_{B,c,\Psi}(t,\lambda)$ extends meromorphically to $\mathfrak{t}_{\mathbb{C}}$ with at most simple poles along the hyperplanes $\lambda(\check{\beta}) \in \mathbb{N}_0$ for $\beta \in B$.

Proposition 5 *Suppose that G is one of the following:*

(i) a group of real rank one,

(ii) a split group of real rank two, or

(iii) the group GL(4).

Then we have

$$\phi_M(t,\lambda) = t^\lambda \sum_{(A,B,c)} \phi_{B,c,\Psi}(t,\lambda),$$

as meromorphic functions in λ, where the sum is over all maximal root cones (A,B,c) for MP. For each such root cone, Ψ is a subset of $\Lambda_A \cap C_A \cap \bar{C}_B$. It can be a proper subset only in case (iii).

Except for a few root cones in case (iii), all other ones occurring here are simplicial. We can prove an identity for simplicial root cones in any root system, which reduces the proof of the Proposition to a combinatorial problem. In case (iii), one has to triangulate the nonsimplicial root cones and observe a number of cancellations, which might be the precursors of more complicated structures in the general case.

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