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A construction of an automoprhic sheaf for GL_r on the moduli space of parabolic structures.

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1 Introduction

Let $k \subset \mathbf{C}$ be an algebraically closed field of characteristic 0. Let X be a smooth proper algebraic curve over k. Let $S \subset X$ be a finite subset of X consists of closed points. Let $C = X \backslash S$. Let G be a split connected reductive algebraic group over k. Let be an irreducible G-local system on C.
Let $G^\vee$ be a dual group of G, i.e. a connected algebraic group over k which corresponds to the dual root datum to that of G. There is a well-known identification due to Weil,

$$Bun_G \cong G(k(X)) \backslash G(A_X)/G(O).$$

Here Bun_G is the moduli of principal G-bundle, A_X is the adele ring of $k(X)$, and O is its ring of integer. An elementary deformation theory says that $T^*_P Bun_G \cong H^1(X, ad(P))^* \cong H^0(ad(P) \otimes \omega)$. Here $ad(P) := \mathfrak{g} \times_{\text{adjoint action}} P$. Let $Nilp$ be the set of its nilpotent vectors. Another well-known fact is the Satake isomorphism

$$k[G(\mathcal{O}_x) \backslash G(k(X)_x)/G(\mathcal{O}_x)] \cong k[X^*(T^\vee)]^W.$$

Here LHS is a convolution algebra of double coset space, RHS is a algebra which is semi-groupring of the set of dominant weight of $G^\vee$.
A sheaf-theoretic version of the above result is due to Ginzburg[5] and can be realized as follows. First, consider the category $P_{G(\mathcal{O}_x)}(Gr)$ consists of $G(\mathcal{O}_x)$-equivariant irreducible perverse sheaves with finite dimensional support and its direct summands. Here $\mathcal{O}_x := \mathcal{O}_{X,x}$, Gr is infinite grassmanian. Second consider a functor $H^* : P_{G(\mathcal{O}_x)}(Gr) \to Vect$. This functor is known to be compatible with forgetful functor $RepG^\vee \to Vect$ if we use the identification of $G(\mathcal{O}_x) \backslash G(k(X))/G(\mathcal{O}_x)$ with the set of dominant weights of $G^\vee$.
We can define a convolution $* : P_{G(\mathcal{O}_x)}(Gr) \times \mathcal{D}^b_c(Bun_G) \to \mathcal{D}^b_c(Bun_G)$. Then, the geometric Langlands correspondence asserts that there exists an irreducible local system $\mathcal{A}$

on $Bun_{G^\vee}X$ which satisfies the so-called Hecke eigenvalue property.
Hecke Eigenvalue Property (Local version)

$$IC_\lambda * \mathcal{A} \cong V_\lambda \otimes_{\mathbf{C}} \mathcal{A}$$

Hecke Eigenvalue Property (Global version)
V_λ constructed above makes a local system as x moves along X. Moreover, it is restriction of a Whittaker sheaf associated to to the diagonal embedding of X of type λ. (GL_r case.)

We knows that if $\mathcal{A}$ satisfies the local Hecke eigenvalue property and $Ch\mathcal{A} \subset Nilp$, then $\mathcal{A}$ satisfies the global Hecke eigenvalue property.[5] So, we want to study $Nilp$ and the local Hecke eigenvalue property.
There is a conjectural construction of $\mathcal{A}$ due to Beilinson and Drinfeld. It is constructed using a D-module which comes from a quantization of the Hitchin morphism

$$h : T^*Bun_G \to Hitch = \oplus_{m_k:\text{exponents of } G} H^0\left(X, \omega^{m_k}\right).$$

Here h is constructed by a homogeneous generator of the ring of invariants $\{i_1, i_2, \cdots, i_{\mathrm{rk}\,\mathfrak{g}}\} \subset k[\mathfrak{g}^*]^G$. Precisely, let $m_k = \deg i_k$. Then, substituting $H^0\left(ad\left(P\right) \otimes \omega\right)$ gives an homomorphism

$$h_i : T^*Bun_G \to H^0\left(X, \omega^{m_k}\right).$$

Moreover, summing them up gives h. It is obvious that $h^{-1}\left(0\right) = Nilp$. It is known[3] that this is always a Lagrangian substack of T^*Bun_G.
For GL_r, there is another formulation of the same result due to Drinfeld[1], Laumon[6][8]. Their approach depends on the fact that we have an explicit formula for a Whittaker function for GL_r and that the Fourier transform of a Whittaker function is the only candidate for $\mathcal{A}$. Their construction supports the conjecture $Ch\mathcal{A} \subset Nilp$. In Frenkel[4], he asserts that the sheaf corresponding to $\mathcal{A}$ in $X = \mathbf{P}^1$, $S \neq \emptyset$ is the Gaudin model and execute an inverse Radon transform to get symmetric power(Whittaker sheaf) for degree 0 case.
In this article, we give a characterizations of the open stratum of the moduli stack of rank r bundle with parabolic structures on $\mathbf{P}^1$. Since the sheaf $\mathcal{A}$ must be irreducible, it should be a minimal extension from this locus. Next, we give a construction of a candidate of an automorphic sheaf, which is, roughly speaking, the Fourier dual of a Whittaker function. Finally, we give an estimate for the characteristic variety of our sheaf and see it contains $Nilp$ for sufficiently high degree case. Moreover if an appropriate local triviality of monodromy is proved, we can see that it is exactly $Nilp$. Since we knows that our moduli for $\mathbf{P}^1$ case can be easily written, we can expect the whole picture can be described explicitly. Of course we can describe the moduli of rank r bundle via spectral cover and compute it in the Jacobian of a spectral curve, we do not know how to describe its Hecke correspondence and stratification. I do not know what the moduli of rank r bundle is except for the case $r = 2$, X hyperelliptic.

2 Rough Stratification

Let us fix a finite subset $S \subset X$.

Definition 1 ([9]) *A rank r bundle on X with a quasi-parabolic structure at S is a pair $(E, \{F_*E \otimes k_x\}_{x \in S})$ consists of a rank r locally free sheaf E on X, and a decreasing filtration $\{F_*E \otimes k_x\}_{x \in S}$ of vector spaces at each $x \in S$ as follows*

$$E \otimes k_x = F_{x,0}E \otimes k_x \supset F_{x,1}E \otimes k_x \supset \cdots \supset F_{x,s_x}E \otimes k_x = \{0\}.$$

A rank r bundle on X with parabolic structure is a quasi-parabolic structure with an extra data

$$0 \leq \alpha_{x,1} < \alpha_{x,2} < \cdots < \alpha_{x,s_x} < 1$$

associated to $\{F_*E \otimes k_x\}_{x \in S}$.

Remark 1 *Unless stated otherwise, we use only parabolic structures with $s_x = r$, $dim F_{x,i}E \otimes k_x / F_{x,i-1}E \otimes k_x = 1$ for all i. In addition, we do not specify $\alpha_{x,i}$, but assume that $\alpha_{x,i} = \alpha_{y,i}$ for all $x, y \in S$. Finally, we assume that $\alpha_{x,i}$ is common to all of vector bundle with parabolic structures.*

Definition 2 *An extended filtration of parabolic bundle is defined as follows.*

$$F_{x,i}E_x = \begin{cases} \{\phi \in E_x \mid \phi \otimes k_x \in F_{x,i}E \otimes k_x\} & \text{for } 1 \leq i \leq r \\ F_{x,i+js_x}E_x(-jx) & \text{otherwise} \end{cases}$$

Here we set $\alpha_{x,i+s_x} = \alpha_{x,i} + 1$.

Definition 3 ([9]) *A morphism of vector bundle with parabolic structures at S is a morphism $f : E \to F$ of base sheaves satisfying the following conditions*

$$f(F_{x,i}E \otimes k_x) \subset F_{x,j}F \otimes k_x \tag{1}$$

$$\alpha_{E,x,i} > \alpha_{F,x,j} \tag{2}$$

We denote the set of morphism between vector bundles with parabolic structures by $Hom_{par}(E, F)$.

Lemma 1 *Let L and M be parabolic line bundles with parameters $\{\alpha_x\}$, and $\{\beta_x\}$ respectively. Then*

$$Hom_{par}(L, M) \cong Hom(L, M(-H)).$$

Here H is a divisor which is the sum of points $x \in S$ such that $\alpha_x \leq \beta_x$ with single multiplicities.

Definition 4 *A rank r bundle with a level structure is a pair $(E, \{v_{x,i}\}_{x\in S, 1\leq i\leq r})$ consists of a rank r locally free sheaf E on X, and a basis $\{v_{x,i}\}_{1\leq i\leq r}$ of $E\otimes k_x$ for each $x\in S$.*

Let $\mathcal{M}^{level}_{S,r,d}$ be the moduli stack of the degree d rank r bundles with level structures at S. Let $\mathcal{M}_{S,r,d}$ be the moduli stack of the degree d rank r bundles with parabolic structures. Then we have a natural homomorphism ϑ by making flags, i.e.

$$\{v_{x,i}\}_{1\leq i\leq r} \mapsto L.h < v_{x,k} >_{k\geq i} =: \{F_{x,i}E\otimes k_x\}.$$

Here we denote linear hull using basis by $L.h$. It is known that

$$T^*_{(E,\{v_{x,i}\}_{x\in S,1\leq i\leq r})}\mathcal{M}^{level}_{S,r,d} \cong Hom\,(E, E\otimes\omega(D)).$$

Infinitesimal symmetry of a fiber of ϑ is

$$\mathcal{I} := \{\phi\in Hom\,(E,E) \mid \phi\,(F_{x,i}E_x) \subseteq F_{x,i+r}E_x\}.$$

Therefore, we have

$$T_E\mathcal{M}_{S,r,d} \cong Coker\left(H^1\,(X,\mathcal{I}) \hookrightarrow H^1\,(X, Hom\,(E, E\otimes O(-D)))\right)$$

By the Serre duality and using the killing form, we have

Lemma 2 ([3]) *We have the following isomorphism*

$$T^*_E\mathcal{M}_{S,r,d} \cong \{\phi\in Hom\,(E, E\otimes\omega(D)) \mid \phi\,(F_{x,i}E_x) \subset F_{x,i+1}E_x\otimes\omega(D) \text{ for all } i\}$$

Above identification allows us to define as follows.

Definition 5 ([3]) *The global nilpotent variety Nilp of $\mathcal{M}_{S,r,d}$ is a substack of $T^*\mathcal{M}_{S,r,d}$ defined fiberwise as follows.*

$$Nilp\cap T^*_E\mathcal{M}_{S,r,d} \cong \{\phi\in Hom\,(E, E\otimes\omega(D)) \mid \phi\in T^*_E\mathcal{M}_{S,r,d}, \phi^r = 0\}$$

Remark 2 *In [7], it is shown that the stratification induced by irreducible components of Nilp for $S\neq\emptyset$ is the Shatz stratification. Therefore, we are dealing with an analogue of the Shatz strata in the parabolic setting.*

Example 1 (Well-known, See[2]) *If $X = \mathbf{P}^1$, then we have a smooth local open set F^N of $\mathcal{M}_{r,0}$ identical to N-fold product of flag varieties of GL_r. Let $S = \{z_1, \cdots, z_N\}$. Then using moment map, we can define an analogue of the Hitchin fibration as follows. Let $\mu : T^*F \to \mathcal{N}$ be a moment map. Then, let*

$$\mathcal{S} := \{(n_i)\in\mathcal{N}^N \mid \rho = \sum \frac{n_i}{u-z_i}du \text{ is holomorphic at } \mathbf{P}^1\backslash S,\ \rho^N = 0\}$$

This is a Poisson submanifold of $\mathcal{N}^N$. Its pullback to T^*F^N is a Lagrangian subvariety. This is a description of an open part of even degree case.

Definition 6 *Let $(E, \{F_{x,i}\}_{x\in S, 1\le i\le r})$ be a quasi-parabolic bundle. Let T be a finite subset(including empty set) of S. Then we define a vector bundle E_T by*

$$E_{T,x} := \begin{cases} E_x & \text{if } x \notin T \\ \{\phi \in \xi^{-r+1}E_x \mid Res_x \xi^m \phi \in F_{x,r-m}E \otimes k_x \ 0 \ge m \ge r-1\} & \text{if } x \in T \end{cases}.$$

Definition 7 *Let the situations as above definition. Define a vector bundle E^i_T by*

$$E^i_{T,x} := \begin{cases} E_x & \text{if } x \notin T \\ \{\phi \in \xi^{-i}E_x \mid Res_x \xi^{i-1}\phi \in F_{x,i}E \otimes k_x\} & \text{if } x \in T \end{cases}.$$

Definition and Lemma 1 *Let $(E, \{F_{x,i}\}_{x\in S, 1\le i\le r}, \{\alpha_{x,i}\}_{x\in S,1\le i\le r})$ be a parabolic bundle. Let $E \to L$ be a surjective homomorphism of vector bundles. Then there exist a unique quotient bundle L^i of E^i_S such that $\mathrm{rk}L = \mathrm{rk}L^i$, and $L \hookrightarrow L^i$ via natural homomorphism. Similarly, let $M \hookrightarrow E$ be an injective homomorphism of vector bundle. Then there exist a unique a unique subbundle M^i of E^i_S such that $\mathrm{rk}M = \mathrm{rk}M^i$, and $M \hookrightarrow M^i$ via natural homomorphism. We call this type of operation for M by the upper modification.*

From now on, we assume $X = \mathbf{P}^1_{\mathbf{C}}$. By Grothandieck's theorem, we can assume that every E_T is direct summand of $\mathcal{O}(m)$. Here $\mathcal{O}(m)$ is the line bundle of degree m on $\mathbf{P}^1$.

Definition 8 *A vector bundle E on $\mathbf{P}^1_{\mathbf{C}}$ is called general iff $Hom(E, E\otimes\omega) \cong \{0\}$.*

Definition 9 *The fine stratification of $\mathcal{M}_{S,r,d}$ is the stratification defined via the decomposition type(i.e. direct summands of E_T).*

Lemma 3 *In product of two flag varieties of GL_r, the fine stratification induced on it is union of Schubert cells. Moreover if we take an irreducible component, we can get the Schubert decomposition.*

Since parameters are discrete, we can assume that there exists an open stratum of $\mathcal{M}_{S,r,d}$.

Theorem 1 *Let $(E, \{F_{x,i}\}_{x\in S,1\le i\le r})$ be a parabolic bundle. Then the following two conditions are equivalent*

- *$Nilp \cap T^*_E\mathcal{M}_{S,r,d}$ is nontrivial.*
- *There exists $T \subseteq S$ such that E_T is non-generic.*

3 Construction of Automorphic sheaves

Every construction in this, and the next section is a generalization of Laumon's paper[8] to the parabolic case.
Let be a rank r irreducible local system on C. We assume that our local system is algebraic, i.e. it comes from an ODE on X which has at most regular singularity along S. We assume that it has unipotent monodromy along S. Then we mimic the original construction of Laumon to construct a Whittaker sheaf on the moduli stack of torsion $\mathcal{O}_X$ sheaves of width $\leq r$ as follows. First, take an m-fold external product on C^m and take an $\mathfrak{S}_m$-invariant part which is induced by switching factors. Taking a minimal extension via embedding

$$C^{(m)} \hookrightarrow Coh^{0,m,r}X,$$

we get a Whittaker sheaf W . Here this morphism is constructed by the composition of two morphisms defined below.

$$C^{(m)} \ni (x_1, \cdots, x_m) \mapsto \sum_{i=1}^{i=m} [x_i] \in Div^m C$$

$$Div^m C \ni D \mapsto \mathcal{O}_D \in Coh^{0,m,r} X$$

Here we denote $\mathcal{O}_X/\mathcal{O}_X(-D)$ by $\mathcal{O}_D$. Since we have $T^*_{E_0} Coh^{0,m,r}X \cong Hom\,(E_0, E_0 \otimes \omega)$, we can define the r-th nilpotent variety $\Lambda_{0,r}$ of $Coh^{0,m,r}X$ via the following fiberwise description

$$T^*_{E_0} Coh^{0,m,r} X \cap \Lambda_{0,r} := \{\phi \in Hom\,(E_0, E_0 \otimes \omega) \mid \phi^r = 0\}$$

From a result of Laumon,[6] we knows that characteristic variety of W coincides with $\Lambda_{0,r}$ if $S = \emptyset$.

Definition 10 *Let $\phi : L \to M$ be a morphism between torsion $\mathcal{O}_X$ modules. Let $M = \oplus M_x$ be its decomposition to stalk. Let us choose a local uniformizer ξ_x at each $x \in \mathrm{Supp}M$. Let $D = \sum_{x \in \mathrm{Supp}M} [x]$ We define a D shift of ϕ to be the composition of ϕ and $\sum \xi_x$, and denote it by $\phi(D)$.*

It is obvious that D shifts are compatible with the composition of morphisms. Moreover, if a D shift of a morphism ψ vanishes, then any D shifts of ψ vanishes.

Definition 11 *Let Ψ be a subvariety of $T^* Coh^{0,m,r} X$ defined fiberwise as follows.*

$$T^*_{E_0} Coh^{0,m,r} X \cap \Psi := \{\phi \in Hom\,(E_0, E_0 \otimes \omega) \mid \phi \mid_C = 0 \;\; \phi^r\,((r-1)D) = 0\}$$

Proposition 1 *$ChW \;= \Lambda_{0,r} + \Psi$, without multiplicity.*

Let $\mathcal{M}_{S,r,d,coh}$ be the moduli stack of coherent $\mathcal{O}_X$ module of generic rank r, degree i, equipped with parabolic structure for locally free part $E_{i,lf}$ of E_i along S. Let $\mathcal{C}_{i,d}(1 \leq i \leq r-1)$ be an open substack of $\mathcal{M}_{S,r,d,coh}$, defined as follows.

$$\mathcal{C}_{i,d} = \left\{\left(B_i, B_{i,lf}, \{F_*\}_{S,[1,r]}\right) \in \mathcal{M}_{S,r,d,coh} \mid Hom\left(B_i, \omega^{-r+i+1}\right) = \{0\}\right\}$$

We restrict locally free part for $i = r$. Let us define two fiber bundles on $\mathcal{C}_{i,d}$, $\mathcal{E}_i$, $\mathcal{E}_i^\vee$ fiberwise as follows.

$$\mathcal{E}_{i,\left(B_i,B_{i,lf},\{F_*\}_{S,[1,r]}\right)} := \mathbf{P}Hom\left(\omega^{-r+i}, B_{S,i}\right) \tag{3}$$

$$\mathcal{E}^\vee_{i,\left(B_i,B_{i,lf},\{F_*\}_{S,[1,r]}\right)} := \mathbf{P}Ext^1\left(B_{S,i}, \omega^{-r+i+1}\right) \tag{4}$$

By the Serre duality, these vector spaces are dual to each other before projectification. Here we denote locally free part of a coherent $\mathcal{O}_X$-module E by E_{lf}. This is an direct component of E. Let $\mathcal{E}^\circ_{i+1}$ be an locally closed substack of $\mathcal{E}_{i+1}$ defined fiberwise by

- $\phi \in \mathcal{E}^\circ_{i+1,E}$ is injective.
- $\phi \in \mathcal{E}^\circ_{i+1,E}$ comes from $Hom\left(\omega^{-r+i+1}, B_{i+1}\right)$.
- The maximal upper modification L^{i+1} of ω^{-r+i+1} does not meet $F_{x,1}B_{i+1,lf}$ for every $x \in S$.

Then we can define a homomorphism $\mathcal{E}^\circ_{i+1} \xrightarrow{j_i^\vee} \mathcal{E}^\vee_{i+1}$ just by sending quotient according to the diagram

$$0 \to \omega^{-r+i+1} \to B_{S,i+1} \to B_{S,i} \to 0.$$

Here $B_{S,i} := \{B_{i,lf}\}_S \oplus B_{i,tor}$. Let us denote the inclusion $\mathcal{E}^\circ_{i+1} \hookrightarrow \mathcal{E}_{i+1}$ by j_{i+1}. Let $\mathcal{P}_i$ be a fiber bundle on $C_{i,d}$ defined fiberwise as follows.

$$\mathcal{P}_{i,\left(B_i,B_{i,lf},\{F_*\}_{S,[1,r]}\right)} := \{(\phi, \xi) \in \mathcal{E}_{i,E} \times \mathcal{E}^\vee_{i,E} \mid <\phi, \xi>= 0\}$$

Then we can define the first, the second projections $\pi_i : \mathcal{R}_i \to \mathcal{E}_i$, $\check{\pi}_i : \mathcal{R}_i \to \mathcal{E}_i^\vee$. The Radon transform $\mathcal{R}_i$ on $\mathcal{E}_i$ is defined by $\mathcal{R}_i := \check{\pi}_{i!}\pi_i^*[-dim_{C_i}\mathcal{E}_i]$. We think $\check{\pi}_{i!}$ be its derived functor because of base change theorem. Let $W^{\circ 0}$ be a pullback of W . Then we put $W^{\circ i+1} := j_{i+1,!}j_i^{\vee *}\mathcal{R}_i W^{\circ i}$. In the original setting, it is conjectured to be isomorphic to $W^{\circ i+1} := j_{i+1,!+}j_i^{\vee *}\mathcal{R}_i W_L^{\circ i}$. Let $\mathcal{H}^n_{i,m}(0 \leq i \leq r-1)$ be a stack which classifies the triple (B_i, B_i', ϕ). Here $B_i \in \mathcal{C}_{i,m}$, $B_i' \in \mathcal{C}_{i,m+i}$,

$$0 \to B_i \xrightarrow{\phi} B_i' \to k_x^{\oplus i} \to 0 \text{ (exact)}$$

. Let $\rho_i^1 : \mathcal{H}^n_{i,m} \to \mathcal{C}_{0,n} \times \mathcal{C}_{i,m}$ be the morphism induced by the first projection and the cokernel of the third projection. Let $\rho_i^2 : \mathcal{H}^n_{i,m} \to \mathcal{C}_{i,m+n}$ be the second projection. Let $T^i := \rho^1_{i!}\rho_i^{2*}$ be the Hecke operators. In this case Hecke eigenvalue property is formulated by restriction to $C \ni x \mapsto k_x^n \in \mathcal{C}_{0,n}$. All of Laumon's arguments were local, so we have

Lemma 4 *W satisfies the Hecke eigenvalue property at $C = X\backslash S$.*

We can pullback the Hecke operators by taking fiber product and looking only at the effective part.

Proposition 2 *The property being a Hecke eigensheaf is preserved by*

- *Radon transform $\mathcal{R}_i$.*
- *pullback $j_i^{\vee,*}$.*

Lemma 5 *W^{or} satisfies the Hecke eigenvalue property for $x \in C$.*

4 Characteristic Varieties

Every construction in this, and the previous section is a generalization of Laumon's paper[8] to the parabolic case.
Let $\widetilde{Nilp}$ be a pullback of global nilpotent variety on $T^*\mathcal{E}_r^\circ$. Radon transform for characteristic varieties is projectified version of isomorphism

$$T^*\mathbf{C}^N \cong \mathbf{C}^N \times \mathbf{C}^{*N} \xrightarrow{\cong} \mathbf{C}^{*N} \times \mathbf{C}^N \cong T^*\mathbf{C}^{*N}.$$

Here the second isomorphism is switching factor to factor. This yields $\mathbf{P}T^*\mathcal{E}_i \cong \mathbf{P}T^*\mathcal{E}_i^\vee$. Therefore, what we have to do is to describe $j_i^{-1}T^*\mathcal{E}_i^\circ \to T^*\mathcal{E}_i$. This can be done as follows. First, since $\mathcal{E}_i$ and $\mathcal{E}_i^\vee$ are fiber bundles, we can think $T^*\mathcal{E}_i$ and $T^*\mathcal{E}_i^\vee$ are stacks representing $(B_i, \rho_i, \phi, v) \in T^*\mathcal{C}_{i,m} \times \mathcal{E}_{i,B_i} \times T_\phi^*\mathcal{E}_{i,B_i}$, $(B_i, \rho_i, \xi, v) \in \mathcal{C}_{i,m} \times \mathcal{E}_{i,B_i}^\vee \times T_\xi^*\mathcal{E}_{i,B_i}^\vee$.

Lemma 6 *Under the situations as above, we have*

$$T_\phi^*\mathcal{E}_{i,B_i} \cong ker\left(Ext^1\left(B_{S,i}, \omega^{-r+i+1}\right) \xrightarrow{\phi^*} Ext^1\left(\omega^{-r+i}, \omega^{-r+i+1}\right)\right) \tag{5}$$

$$T_\xi^*\mathcal{E}_{i,B_i}^\vee \cong \{\phi \in Hom\left(\omega^{-r+i}, B_{S,i}\right) \mid \tag{6}$$

$$\left.\begin{array}{ccccccccc} 0 & \to & B_{S,i} & \to & B_{S,i+1} & \to & \omega^{r+i+1} & \to & 0 \\ & & \uparrow \phi & & \uparrow & & \uparrow id & & \\ 0 & \to & \omega^{-r+i} & \to & \omega^{-r+i} \bigoplus \omega^{-r+i+1} & \to & \omega^{-r+i+1} & \to & 0 \end{array}\right\} \tag{7}$$

Let $(B_{i+1}, \rho_{i+1}, \phi_{i+1}, v) \in T^*\mathcal{E}_{i+1}$. Then the inverse image via $j_i^\vee$ is constructed as follows. By the arguments at above, we can think $\rho_{i+1} \in Hom\,(B_{i+1}, B_{i+1} \otimes \omega(D))$ as

$$\rho_{i+1} : B_{i+1,S,x}^i \to B_{i+1,S,x}^{i+1} \otimes \omega \text{ for all } i.$$

Lemma 7 *Let the situations as above. Then the following assignment gives the restriction $T_{\mathcal{E}_i^\circ}^*\mathcal{E}_i \to T^*\mathcal{E}_{i-1}^\vee$.*

- B_i *comes from*

$$0 \to \omega^{-r+i+1} \xrightarrow{\phi} B_{S,i+1} \to B_{S,i} \to 0.$$

- ρ_i *is a unique vector in* $Hom\,(B_i, B_i \otimes \omega(D))$ *which comes from*

$$\begin{array}{ccccccc}
0 & \to & Hom\,(B_{i+1}, \omega^{-r+i+1}) & \to & Hom\left(B_{i+1}, B^1_{i+1} \otimes \omega\right) & \to & Hom\,(B_{i+1}, B^1_i \otimes \omega) \\
 & & \downarrow & & \downarrow & & \downarrow \\
0 & \to & Hom\,(B_i, \omega^{-r+i+1}) & \to & Hom\left(B_i, B^1_{i+1} \otimes \omega\right) & \to & Hom\,(B_i, B^1_i \otimes \omega) \\
 & & \downarrow & & \downarrow & & \downarrow \\
0 & \to & Ext^1\,(\omega^{-r+i}, \omega^{-r+i+1}) & \to & Ext^1\left(\omega^{-r+i}, B^1_{i+1} \otimes \omega\right) & \to & Ext^1\,(\omega^{-r+i}, B^1_i \otimes \omega)
\end{array}$$

by using $\rho_{i+1} \in Hom\left(B_{i+1}, B^1_{i+1} \otimes \omega\right)$, $\bar{v} \in Im\,(Ext^1\,(B_{S,i+1}, \omega^{-r+i+2}) \to Ext^1\,(B_{i+1}, \omega^{-r+i}$ *and* ϕ_{i+1}. *Here the diagram above is exact for all lines.*

- ϕ_i *is uniquely determined via the composition*

$$0 \to \omega^{-r+i+1} \xrightarrow{\phi_{i+1}} B_{i+1} \xrightarrow{\rho_{i+1}} B^1_{i+1} \otimes \omega \to B_i \otimes \omega.$$

- *Extension is the short exact sequence above.*

Let $[\rho]_i : B_i \to B_{i-1} \otimes \omega$ be the composition of ρ_i and $B^1_i \to B_{i-1}$. Then we have

Corollary 1 *The following diagram is commutative.*

$$\begin{array}{ccccc}
\omega^{-r+i} & \hookrightarrow & B_i & \to & B_{i-1}(-D) \\
\downarrow id & & \downarrow [\rho]_i & & \downarrow [\rho]_{i-1} \\
\omega^{-r+i} & \hookrightarrow & B_{i-1} \otimes \omega & \to & B_{i-2} \otimes \omega(-D)
\end{array}$$

Using this corollary repeatedly, we have the reverse composition.

Proposition 3 *The image of composition of Radon transforms and restrictions from* $\mathcal{E}^\circ_r$ *to* $\mathcal{E}_0$ *carries* $\widetilde{Nilp} \subset T^*\mathcal{E}^\circ_r$ *into pullback of* $\Lambda_{0,r} \cup \Psi$ *in* $T^*\mathcal{E}_0$.

To prove this correspondence more rigidly, we need some extra lemmas.

Lemma 8 *Let* R *be a commutative ring.*

$$0 \to A \to B \to C \to 0$$

be a short exact sequence of R*-modules. Let* L *be a* R*-module. Then the morphism* ι *in*

$$Hom_R\,(L, B) \to Hom_R\,(L, C) \xrightarrow{\iota} Ext^1_R\,(L, A)$$

is given by

$$Hom_R\,(L, C) \ni \phi \quad \mapsto \quad \begin{array}{ccccccccc}
0 & \to & A & \to & B & \to & C & \to & 0 \\
 & & \uparrow id & & \uparrow & & \uparrow \phi & & \\
0 & \to & A & \to & B^- & \to & L & \to & 0
\end{array}$$

Here B^- *is a* R*-module generated by the preimage of* L *in* $B \oplus ker\phi$.

Using this result, we get the following.

Proposition 4 *If the corresponding cotangent vector defines locally free sheaf on $\mathcal{C}_r$, we have*

- $ker\rho_0^i \cong \mathcal{O}_{D_1} \oplus \cdots \oplus \mathcal{O}_{D_i}$ *for all* i.
- $D_1 \geq D_2 \geq \cdots \geq D_r$.

Theorem 2 *Under the identification above the preimage of inverse image of $\Lambda_{0,r} \cup \Psi$ in $T^*\mathcal{E}_0$ to $T^*\mathcal{E}_r^\circ$ is $\widetilde{Nilp}$.*

Corollary 2 ([3]) *$\widetilde{Nilp}$ is a Lagrangian substack of $T^*\mathcal{E}_r^\circ$.*

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