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Chevalley groups associated to elliptic Lie algebras of type $A_l^{(1,1)}, B_l^{(1,1)}, C_l^{(1,1)}, D_l^{(1,1)}$ *

Tadayoshi Takebayashi

1 Introduction

A toroidal Lie algebra is the universal central extension of a Lie algebra $\mathfrak{g}\otimes\mathbb{C}[t_1^{\pm1}, t_2^{\pm1}, \cdots, t_m^{\pm1}]$, where $\mathfrak{g}$ is one of the finite dimensional simple Lie algebras over $\mathbb{C}$ and $\mathbb{C}[t_1^{\pm1}, t_2^{\pm1}, \cdots, t_m^{\pm1}]$ is the ring of Laurent polynomials in m variables $t_1, \cdots, t_m$ over $\mathbb{C}$. Let $A := \mathbb{C}[t_1^{\pm1}, t_2^{\pm1}, \cdots, t_m^{\pm1}]$, and $\Omega_A := \oplus_{i=1}^m A dt_i$ be the A-module with generators da for $\forall a \in A$, and the relation $d(ab) = ad(b) + bd(a)$. Let $\bar{\ } : \Omega_A \to \Omega_A/dA$ be the canonical projection, in which there holds the relation, $0 = \overline{d(ab)} = \overline{ad(b)} + \overline{bd(a)}$, then a Lie algebra $\mathfrak{u} = A \otimes \mathfrak{g} \oplus (\Omega_A/dA)$, with Lie bracket $[a \otimes X, b \otimes Y] = ab \otimes [X, Y] + \overline{(da)b}\,(X \mid Y)$, $[c, \mathfrak{u}] = 0$, $\forall c \in \Omega_A/dA$, is the universal central extension of $A \otimes \mathfrak{g}$. The same algebras have been given by Slodowy ([Slo]). Let $A = (< \alpha_i, \alpha_j >)_{1\le i,j\le l}$ be any simply-laced finite Cartan matrix of rank $l \ge 2$, and $A^{[m]} = (< \alpha_i, \alpha_j >)_{1\le i,j\le l+m}$ be any m-fold affinization of A, then Slodowy introduced intersection matrix algebra, $im(A^{[m]}) := gim(A^{[m]})/r(A^{[m]})$, and there holds $\mathfrak{u} \simeq im(A^{[m]})$. In Vertex operator's method, Saito and Yoshii ([S-Y]) constructed a Lie algebra $\mathfrak{g}(\Phi)$ attached to any m-extended homogeneous root system Φ as certain subalgebra of $V_{Q(\Phi)}/DV_{Q(\Phi)}$, here V_Q is the lattice Vertex algebra attached to a lattice Q and D is the derivation, (studied by Borchers ([Bo 1,2])). Then there holds $\mathfrak{u} \simeq \mathfrak{g}(\Phi)$. Saito and Yoshii also defined a Lie algebra $\tilde{e}(\Gamma(\Phi, G))$ by Chevalley generators and generalized Serre relations attached to $\Gamma(\Phi, G)$, where $\Gamma(\Phi, G)$ is the simply-laced elliptic Dynkin diagram and (Φ, G) is a pair consisting of an elliptic root system Φ (i.e. 2-extended affine root system) with a marking G. Then there holds $\tilde{\mathfrak{g}}(\Phi) \simeq \tilde{e}(\Gamma(\Phi, G))$, where $\tilde{\mathfrak{g}}(\Phi)$ is generated by $\mathfrak{g}(\Phi)$ and nondegenerate $\tilde{\mathfrak{h}}$ extended from Cartan subalgebra $\mathfrak{h}$. In the toroidal Lie algebra $\mathfrak{u}$, one can consider the algebra $\mathfrak{t}$ which has only degree 0 elements as the center i.e. $c_i := \overline{t_i^{-1}dt_i} \in \Omega_A/dA$, and add to it the degree derivation d_i, thus $\mathfrak{t} = \mathfrak{g} \otimes \mathbb{C}[t_1^{\pm1}, \cdots, t_m^{\pm1}] \oplus (\oplus_{i=1}^m \mathbb{C}c_i) \oplus (\oplus_{i=1}^m \mathbb{C}d_i)$, with $[d_i, a \otimes t_1^{n_1} \cdots t_m^{n_m}] = n_i(a \otimes t_1^{n_1} \cdots t_m^{n_m})$, and $[d_i, c_j] = 0$ for $1 \le i, j \le m$, which is also called toroidal Lie algebra, or Quasi-simple Lie algebra. In the above Lie algebras, in the case of $m = 2$, we call elliptic Lie algebra, because its root system is associated to elliptic root system ([Sa]). In the sequel, we denote an elliptic Lie algebra by $\hat{\mathfrak{g}}$ as the universal

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central extension of 2-fold affinization $k[T^{\pm1},\ S^{\pm1}]\otimes\mathfrak{g}$, where $k[T^{\pm1},S^{\pm1}]$ is the ring of two variable Laurent polynomials with coefficients in a field k . Moody, Rao and Yokonuma ([M-R-Y]) constructed the integrable representations of $\hat{\mathfrak{g}}$, called vertex representations, and after that, using them , Shi ([Sh]) constructed a group (*toroidal group*) associated to $\hat{\mathfrak{g}}$. In this article, we construct a group associated to $\hat{\mathfrak{g}}$, in the following way. Let ρ be a faithful representation of the Lie algebra $\mathfrak{g}$ on a finite dimensional complex vector space. Using ρ and a Chevalley basis of $\mathfrak{g}$, Chevalley ([Ch1]), [Ch2]) and Demazure constructed an affine group scheme $G_\rho(\Delta,\ )$ over $\mathbb{Z}$, where Δ is the root system of $\mathfrak{g}$ with respect to a (fixed) Cartan subalgebra, and $G_\rho(\Delta,\ )$ is called the Chevalley-Demazure group scheme associated to $\mathfrak{g}$ and ρ. Since $G_\rho(\Delta,\ )$ is a representable covariant functor from the category of commutative rings with 1 to the category of groups, one can get a group $G_\rho(\Delta,\ )$ of the points of a commutative ring R with 1. It is written simply $G(R)$ when Δ is arbitrary or fixed, and $G(R)$ is called a Chevalley group over R. For each root system $\alpha\in\Delta$, there is a group isomorphism of the additive group R^+ of R onto a subgroup X_α of $G(R)$. The subgroup of $G(R)$ generated by all $X_\alpha,\alpha\in\Delta$, is denoted by $E(\Phi)$ and called the elementary subgroup of $G(R)$. Morita ([Mo]) showed that the elementry subgroup $E(k[T,T^{-1}])$ of a Chevalley group $G(k[T,T^{-1}])$ has the structure of a Tits system with an affine Weyl group. As an extension of above results, we examine the algebraic structure of the elementary subgroup $E(k[T^{\pm1},S^{\pm1}])$ of a Chevalley group $G(k[T^{\pm1},S^{\pm1}])$, where $G(k[T^{\pm1},S^{\pm1}])$ is considered as a Chevalley group associated to the elliptic Lie algebra $\hat{\mathfrak{g}}$. The elliptic Weyl group defined from the elliptic root system ([Sa]), is not a Coxeter group, so we see that $E(k[T^{\pm1},S^{\pm1}])$ does not have a Tits system associated to its Weyl group (see [Bo]) . We write down some relations of the generators of the Weyl group defined from $E(R_2)$, where we set $R_2:=k[T^{\pm1},S^{\pm1}]$, they are a little different from those of the elliptic Weyl group ([S-T]). In the case of affine Lie algebra , Garland ([G]) , Iwahori and Matsumoto ([I-M]), showed the following result. Let $k((T))(=k[[T,T^{-1}]])$ denote the T-adic completion of $k[T,T^{-1}]$, i.e. $k((T))$ is the ring of all formal Laurent series $\sigma=\sum_{i\geq i_0}q_iT^i,\ q_i\in k$, where the sum on the right is allowed to be infinite, and it is called a local field. Then $E(k((T)))$ has the structure of a Tits system associated with the affine Weyl group of Δ. As an extension to 2-dimension of $k((T))$, one can consider the iterated power series $K=k((T))((S))$, (see [P1], [P2]), which is a 2-dimensional local field with a discrete valuation ring $O_K=k((T))[[S]]$, i.e. K is the quotient field of O_K whose residue field is a 1-dimensional local field $\bar{K}=k((T))$ with residue field k. We define the elementary subgroup $E(K)$, in this case, using the result by Abe ([A]), we see that $SL(n,K)=E(K)$. In a similar way as in the case of SL_2 over p-adic field (1-dimensional local fild) ([H2]), we have the decomposition $SL(2,O'_K)=B\cup Bw_1B$, where $O'_K=k[[T]]\oplus Sk((T))[[S]]$ and the group B equal to the inverse image of the $\bar{B}$ from the $SL(2,\bar{K})$ and $\bar{B}$ is the inverse image of the Borel subgroup of $SL(2,k)$, and $w_1=\begin{pmatrix}0&1\\-1&0\end{pmatrix}$ is an element of the Weyl group of $SL(2,K)$.

2 Definition of Chevalley groups and some results

Let $\mathfrak{g}$ be a finite dimensional complex simple Lie algebra, Δ be a root system of $\mathfrak{g}$ with respect to a (fixed) Cartan subalgebra $\mathfrak{h}$, and $\Pi = \{\alpha_1, \cdots, \alpha_l\}$ be a set of simple roots relative to some fixed ordering. Let $W(\Delta)$ be the Weyl group of Δ, $\widetilde{W}(\Delta)$ affine Weyl group of Δ. For $\alpha, \beta \in \Delta$, we set $< \beta, \alpha >:= 2(\beta, \alpha)/(\alpha, \alpha)$, where (,) is a scalar product, which is invariant under $W(\Delta)$. For each $\alpha \in \Delta, w_\alpha$ denotes the reflection with respect to α , defined by $w_\alpha(\beta) := \beta - < \beta, \alpha >$. Set $\widehat{\Delta} := \Delta \times \mathbb{Z} \times \mathbb{Z}$, then an element of $\widehat{\Delta}$ is represented by $\alpha^{(n,p)}$, where $\alpha \in \Delta$, and $n, p \in \mathbb{Z}$, and $\widehat{\Delta}$ is identified with an elliptic root system introduced by Saito ([Sa]). For each $\alpha^{(n,p)} \in \widehat{\Delta}$, let $w_\alpha^{(n,p)}$ be the reflection with respect to $\alpha^{(n,p)}$, defined by $w_\alpha^{(n,p)}\beta^{(m,q)} = (w_\alpha\beta)^{(m-<\beta,\alpha>n, q-<\beta,\alpha>p)}$ for any $\beta^{(m,q)} \in \widehat{\Delta}$. Let $W(\widehat{\Delta})$ be the group generated by $w_\alpha^{(n,p)}$ for all $\alpha^{(n,p)} \in \widehat{\Delta}$, then $W(\widehat{\Delta})$ is the elliptic Weyl group ([S-T]). We note that $W(\Delta)$ is identified with the subgroup of $W(\widehat{\Delta})$ generated by $w_\alpha^{(0,0)}$ for all $\alpha \in \Delta$, and $\widetilde{W}(\Delta) \cong \{w_\alpha^{(n,0)}, \alpha \in \Delta, n \in \mathbb{Z}\} \cong \{w_\alpha^{(0,p)}, \alpha \in \Delta,\ p \in \mathbb{Z}\}$. Set $h_\alpha^{(n,p)} = w_\alpha^{(n,p)} w_\alpha^{(0,0)^{-1}}$ and $H(\widehat{\Delta})$ be the subgroup of $W(\widehat{\Delta})$ generated by $h_\alpha^{(n,p)}$ for all $\alpha^{(n,p)} \in \widehat{\Delta}$. In this paper, we write $G = G_1 \cdot G_2$ when a group G is a semidirect product of two groups G_1 and G_2, and G_2 normalizes G_1. Then in a similar way to [Mo], we have the following.

Lemma 2.1 *(1) Let $\alpha^{(n,p)}$ and $\beta^{(m,q)}$ be in $\widehat{\Delta}$, then*
$h_\alpha^{(n,p)}\beta^{(m,q)} = \beta^{(m+<\beta,\alpha>n,\ q+<\beta,\alpha>p)}$.

(2) $H(\widehat{\Delta})$ is a free abelian group generated by $h_{\alpha_i}^{(1,0)}$ and $h_{\alpha_i}^{(0,1)}$ for all $\alpha_i \in \Pi$.

(3) Let $\alpha^{(n,p)}$ and $\beta^{(m,q)}$ be in $\widehat{\Delta}$, and $\gamma = w_\alpha\beta$, then $w_\alpha^{(n,p)} h_\beta^{(m,q)} w_\alpha^{(n,p)^{-1}} = h_\gamma^{(m,q)}$.

(4) $W(\widehat{\Delta}) = H(\widehat{\Delta}) \cdot W(\Delta)$.

(5) Let $\alpha^{(n,p)}$ be in $\widehat{\Delta}$ and w in $W(\widehat{\Delta})$ and set $\beta^{(m,q)} = w\alpha^{(n,p)}$, then
$w w_\alpha^{(n,p)} w^{-1} = w_\beta^{(m,q)}$.

Let $\widehat{\Delta}^+ = (\Delta \times \mathbb{Z} \times \mathbb{Z}_{>0}) \cup (\Delta \times \mathbb{Z}_{>0} \times 0) \cup (\Delta^+ \times 0 \times 0)$, then $\widehat{\Delta}^+$ is identified with the set of positive real roots of the elliptic root system defined in ([B-C]). Let $\{H_{\alpha_1}, \cdots, H_{\alpha_l},\ e_\alpha,\ \alpha \in \Delta\}$ be a Chevalley basis of $\mathfrak{g}$ ([C]). Let ρ be a faithful representation of the Lie algebra $\mathfrak{g}$ on a finite dimensional complex vector space, $G_\rho(\Delta\ ,\ \ )$ (we simply write G) be a Chevalley-Demazure group scheme associated with $\mathfrak{g}$ and ρ ([Ch1], [Ch2]). Let $\mathfrak{U}$ be a universal enveloping algebra of $\mathfrak{g}$, $\mathfrak{U}_\mathbb{Z}$ the subring of $\mathfrak{U}$ generated by 1 and $e_\alpha^k/k!$ for all $\alpha \in \Delta$ and $k \in \mathbb{Z}_{>0}$, and $\mathfrak{g}_\mathbb{Z} = \sum_{i=1}^{l} \mathbb{Z} H_{\alpha_i} + \sum_{\alpha \in \Delta} \mathbb{Z} e_\alpha$ a Chevalley lattice in $\mathfrak{g}$. Let V be the representation space of ρ , Λ the weights of V with respect to $\mathfrak{h}$, and $V = \amalg_{\mu \in \Lambda} V_\mu$ the weight decomposition of V. Let M be an admissible lattice in V , i.e. M is the $\mathbb{Z}$-span of a basis of V, invariant under $\mathfrak{U}_\mathbb{Z}$, and set $M_\mu = M \cap V_\mu$. Let $R_2 = k[T^{\pm 1}, S^{\pm 1}]$ be the ring of Laurent polynomials with coefficients in a field k , and set $\widehat{M} = R_2 \otimes_\mathbb{Z} M$ and $\widehat{M_\mu} = R_2 \otimes_\mathbb{Z} M_\mu$. For each $t \in k$, $n, p \in \mathbb{Z}$ and $\alpha \in \Delta$,

$$exp\ tT^nS^p\rho(e_\alpha) = 1 + tT^nS^p\rho(e_\alpha)/1! + t^2T^{2n}S^{2p}\rho(e_\alpha)^2/2! + \cdots$$

induces an automorphism of $\widehat{M}$ under the following action:

$$(t^kT^{nk}S^{pk}\rho(e_\alpha)^k/k!)(f\otimes v) = (t^kT^{nk}S^{pk}f)\otimes(\rho(e_\alpha)^k/k!)v, \quad \text{where } f\in R_2 \text{ and } v\in M.$$

Then $X_\alpha =< exp\ tT^nS^p\rho(e_\alpha); t\in k,\ n,p\in\mathbb{Z} >$ is a subgroup of $G(R_2)$ and isomorphic to the additive group of R_2. Let $E(R_2)$ denote the subgroup of $G(R_2)$ generated by X_α for all $\alpha\in\Delta$. We write $x_\alpha^{(n,p)}(t) = x_\alpha(T^nS^pt) = exp\ tT^nS^p\rho(e_\alpha)$, for each $\alpha\in\Delta$, $n,p\in\mathbb{Z}$ and $t\in k$. Let k^* be the multiplicative group of k. For each $\alpha\in\Delta, n,p\in\mathbb{Z}$ and $t\in k^*$, we set

$$w_\alpha^{(n,p)}(t) := x_\alpha^{(n,p)}(t)x_{-\alpha}^{(-n,-p)}(-t^{-1})x_\alpha^{(n,p)}(t),$$

$$h_\alpha^{(n,p)}(t) := w_\alpha^{(n,p)}(t)w_\alpha^{(0,0)}(1)^{-1}.$$

We note that $w_\alpha^{(n,p)}(t) = w_\alpha(T^nS^pt)$, $h_\alpha^{(n,p)}(t) = h_\alpha(T^nS^pt)$. Let $\widehat{U}$ the subgroup of $E(R_2)$ generated by $x_\alpha^{(n,p)}(t)$ for all $\alpha^{(n,p)}\in\widehat{\Delta}^+$ and $t\in k$, H the subgroup generated by $h_\alpha^{(0,0)}(t)$ for all $\alpha\in\Delta$ and $t\in k^*$, $\widehat{B}$ the subgroup generated by $\widehat{U}$ and H, and $\widehat{N}$ the subgroup generated by $w_\alpha^{(n,p)}(t)$ for all $\alpha^{(n,p)}\in\widehat{\Delta}$ and $t\in k^*$. Then we have the following three lemmas, whose proofs are similar to those found in ([Mo],[Ste]).

Lemma 2.2 *Let $\alpha^{(n,p)}$ and $\beta^{(m,q)}$ be in $\widehat{\Delta}$, and assume $\alpha+\beta\neq 0$, then*

$$[x_\alpha^{(n,p)}(t), x_\beta^{(m,q)}(u)] = \prod x_{i\alpha+j\beta}^{(in+jm,\ ip+jq)}(c_{ij}t^iu^j)$$

for all $t,u\in k$, where the product is taken over all roots of the form $i\alpha+j\beta$, $i,j\in\mathbb{Z}_{>0}$ in some fixed order, and c_{ij}'s are as in ([Ste], Lemma 15).

(Proof) Let ξ and η be indeterminates, and let α and β be in Δ such that $\alpha+\beta\neq 0$, then we have

$$[exp\ \xi e_\alpha, exp\ \eta e_\beta] = \Pi exp\ c_{ij}\ \xi^i\eta^j\ e_{i\alpha+j\beta}$$

in $\mathfrak{U}_{\mathbb{Z}}[[\xi,\eta]]$, where $c_{ij}\in\mathbb{Z}$ (cf. [Ste], Lemma15). The representation ρ induces a map, also denoted ρ of $\mathfrak{U}_{\mathbb{Z}}$ to $End(M)$ because M is admissible. The map $\rho\to id\otimes\rho$ of $End(M)$ to $End(\widehat{M})$ yields a map , again called ρ , of $\mathfrak{U}_{\mathbb{Z}}$ to $End(\widehat{M})$, and next map $\mathfrak{U}_{\mathbb{Z}}[[\xi,\eta]]$ to $End(\widehat{M})$ as follows: (for $t,u\in k$, and $u_{ij}\in\mathfrak{U}_{\mathbb{Z}}$)

$$\sum_{ij} u_{ij}\ \xi^i\eta^j \longrightarrow \sum_{ij} t^iu^j\ T^{in+jm}S^{ip+jq}\rho(u_{ij}),$$

where if $f\in k[T^{\pm1},S^{\pm1}]$, $g\in End(\widehat{M})$ then fg is the element in $End(\widehat{M})$ which is " first act by g and then left multiply by f " . Then the lemma is proved as in [Mo]. □

Lemma 2.3 *Let α be in Δ , m amd n in $\mathbb{Z}$, and t and u in k^*, then*

(1) $h_\alpha^{(n,p)}(t)$ *acts on $\widehat{M}_\mu$ as multiplication by* $t^{<\mu,\alpha>}\ T^{<\mu,\alpha>n}\ S^{<\mu,\alpha>p}$

(2) $h_\alpha^{(m,q)}(t)\ h_\alpha^{(n,p)}(u) = h_\alpha^{(m+n,q+p)}(tu)$.

Lemma 2.4 *Let $\alpha^{(n,p)}$ and $\beta^{(m,q)}$ be in $\widehat{\Delta}$, and set $\gamma = w_\alpha\beta$, then*

(1) $w_\alpha^{(n,p)}(t)x_\beta^{(m,q)}(u)w_\alpha^{(n,p)}(t)^{-1} = x_\gamma^{(m-<\beta,\alpha>n,\ q-<\beta,\alpha>p)}(ct^{-<\beta,\alpha>}u)$, *for any* $t \in k^*$, *and* $u \in k$, *where* c *is as in ([Ste], Lemma 19).*

(2) $w_\alpha^{(n,p)}(t)w_\beta^{(m,q)}(u)w_\alpha^{(n,p)}(t)^{-1} = w_\gamma^{(m-<\beta,\alpha>n,\ q-<\beta,\alpha>p)}(ct^{-<\beta,\alpha>}u)$.

(3) $w_\alpha^{(n,p)}(t) = w_{-\alpha}^{(-n,-p)}(-t^{-1})$.

(4) $w_\alpha^{(n,p)}(t)h_\beta^{(m,q)}(u)w_\alpha^{(n,p)}(t)^{-1} = h_\gamma^{(m-<\beta,\alpha>n,q-<\beta,\alpha>p)}(ct^{-<\beta,\alpha>}u)h_\gamma^{(-<\beta,\alpha>n,-<\beta,\alpha>p)}(ct^{-<\beta,\alpha>})^{-1}$, *for any* $t, u \in k^*$.

(5) $h_\alpha^{(n,p)}(t)x_\beta^{(m,q)}(u)h_\alpha^{(n,p)}(t)^{-1} = x_\beta^{(m+<\beta,\alpha>n,q+<\beta,\alpha>p)}(t^{<\beta,\alpha>}u)$.

(6) $h_\alpha^{(n,p)}(t)w_\beta^{(m,q)}(u)h_\alpha^{(n,p)}(t)^{-1} = w_\beta^{(m+<\beta,\alpha>n,q+<\beta,\alpha>p)}(t^{<\beta,\alpha>}u)$.

(7) $h_\alpha^{(n,p)}(t)h_\beta^{(m,q)}(u)h_\alpha^{(n,p)}(t)^{-1} = h_\beta^{(m+<\beta,\alpha>n,q+<\beta,\alpha>p)}(t^{<\beta,\alpha>}u)h_\beta^{(<\beta,\alpha>n,<\beta,\alpha>p)}(t^{<\beta,\alpha>})^{-1}$.

Let N be the subgroup of $E(R_2)$ generated by $w_\alpha^{(0,0)}(t)$ for all $\alpha \in \Delta$ and $t \in k^*$, and $\widehat{H}$ the subgroup generated by $h_\alpha^{(n,p)}(t)$ for all $\alpha^{(n,p)} \in \widehat{\Delta}$ and $t \in k^*$, then we have the following, whose proof is similar to that found in ([Mo]).

Lemma 2.5 *(1)* $\widehat{B} = \widehat{U} \cdot H$.

(2) H *and* $\widehat{H}$ *are normal subgroups of* $\widehat{N}$.

(3) $\widehat{N} = \widehat{H}N$ *and* $\widehat{H} \cap N = H$.

(4) $\widehat{N}/H \cong W(\widehat{\Delta})$.

In the sequel, we assume Δ is of rank 1, then $\widehat{\Delta} = \{\pm\alpha^{(n,p)}, n, p \in \mathbb{Z}, \pm\alpha \in \Delta\}$, and $\widehat{\Delta}^+ = \{\pm\alpha^{(n,p)},\ n \in \mathbb{Z},\ p \in \mathbb{Z}_{>0},\ \pm\alpha^{(n,0)},\ n \in \mathbb{Z}_{>0}$ and $\alpha^{(0,0)}\ \}$. Set $E = E(R_2)$ and for each $\alpha^{(n,p)} \in \widehat{\Delta}$, let $X_\alpha^{(n,p)}$ be the subgroup of E generated by $x_\alpha^{(n,p)}(t)$ for all $t \in k$. We identify $w_\alpha^{(0,0)}$, $w_{-\alpha}^{(1,0)}$, $w_\alpha^{(0,1)}$ and $w_{-\alpha}^{(1,1)}$ in $W(\widehat{\Delta})$ with $w_\alpha^{(0,0)}(1)$, $w_{-\alpha}^{(1,0)}(1)$, $w_\alpha^{(0,1)}(1)$ and $w_{-\alpha}^{(1,1)}(1)$ in $\widehat{N}$ respectively, and we simply write $w_1 := w_\alpha^{(0,0)}$, $w_0 := w_{-\alpha}^{(1,0)}$, $w_1^* := w_\alpha^{(0,1)}$ and $w_0^* := w_{-\alpha}^{(1,1)}$. Then the following statements hold.

Lemma 2.6 *(1)* $w_1 X_\alpha^{(n,p)} w_1^{-1} = X_{-\alpha}^{(n,p)}$.

(2) $w_1 X_{-\alpha}^{(n,p)} w_1^{-1} = X_\alpha^{(n,p)}$.

(3) $w_0 X_\alpha^{(n,p)} w_0^{-1} = X_{-\alpha}^{(n+2,p)}$.

(4) $w_0 X_{-\alpha}^{(n,p)} w_0^{-1} = X_\alpha^{(n-2,p)}$.

(5) $w_1^* X_\alpha^{(n,p)} w_1^{*-1} = X_{-\alpha}^{(n,p-2)}$.

(6) $w_1^* X_{-\alpha}^{(n,p)} w_1^{*-1} = X_\alpha^{(n,p+2)}$.

(7) $w_0^* X_\alpha^{(n,p)} w_0^{*-1} = X_{-\alpha}^{(n+2,p+2)}$.

(8) $w_0^* X_{-\alpha}^{(n,p)} w_0^{*-1} = X_\alpha^{(n-2,p-2)}$.

(Proof) From Lemma 2.4 (1) , we have the following relations .

$$w_\alpha(t)x_\alpha(u)w_\alpha(t)^{-1} = x_{-\alpha}(-t^{-2}u),$$
$$w_\alpha(t)x_{-\alpha}(u)w_\alpha(t)^{-1} = x_\alpha(-t^2u),$$
$$w_{-\alpha}(t)x_\alpha(u)w_{-\alpha}(t)^{-1} = x_{-\alpha}(-t^2u),$$
$$w_{-\alpha}(t)x_{-\alpha}(u)w_{-\alpha}(t)^{-1} = x_\alpha(-t^{-2}u).$$

From the above relations and the fact $w_\alpha^{(n,p)}(t) = w_\alpha(T^nS^pt)$, we have,

$$w_\alpha^{(n,p)}(t)x_\alpha^{(m,q)}(u)w_\alpha^{(n,p)}(t)^{-1} = x_{-\alpha}^{(m-2n,q-2p)}(-t^{-2}u),$$
$$w_\alpha^{(n,p)}(t)x_{-\alpha}^{(m,q)}(u)w_\alpha^{(n,p)}(t)^{-1} = x_\alpha^{(m+2n,q+2p)}(-t^2u),$$
$$w_{-\alpha}^{(n,p)}(t)x_\alpha^{(m,q)}(u)w_{-\alpha}^{(n,p)}(t)^{-1} = x_{-\alpha}^{(m+2n,q+2p)}(-t^2u),$$
$$w_{-\alpha}^{(n,p)}(t)x_{-\alpha}^{(m,q)}(u)w_{-\alpha}^{(n,p)}(t)^{-1} = x_\alpha^{(m-2n,q-2p)}(-t^{-2}u).$$

Using these, we can prove this lemma. □

Further we have the following statements.

Lemma 2.7 *(1)* $w_1\widehat{B}w_1^{-1} \subset \widehat{B} \cup \widehat{B}w_1\widehat{B}$.

(2) $w_0\widehat{B}w_0^{-1} \subset \widehat{B} \cup \widehat{B}w_0\widehat{B}$.

(3) $$w_1^*\widehat{B}w_1^{*-1} \subset \widehat{B} \cup \bigcup_{n\in\mathbb{Z}_{\leq 0}} \widehat{B}w_\alpha^{(n,2)}\widehat{B} \cup \bigcup_{n\in\mathbb{Z}} \widehat{B}w_\alpha^{(n,1)}\widehat{B} \cup \widehat{B}w_\alpha^{(0,0)}\widehat{B}.$$

(4) $$w_0^*\widehat{B}w_0^{*-1} \subset \widehat{B} \cup \bigcup_{n\in\mathbb{Z}_{\leq 1}} \widehat{B}w_{-\alpha}^{(n,2)}\widehat{B} \cup \bigcup_{n\in\mathbb{Z}} \widehat{B}w_{-\alpha}^{(n,1)}\widehat{B}.$$

(Proof) (1) From Lemma 2.5 (1), $\widehat{B} = \widehat{U}\cdot H$ and $\widehat{U}$ is generated by $X_\alpha^{(n,p)}$ and $X_{-\alpha}^{(n,p)}$ for $\alpha^{(n,p)}$ and $-\alpha^{(n,p)} \in \widehat{\Delta}^+$ respectively. By Lemma 2.6 (1) and (2), except for the element $X_\alpha^{(0,0)}$, all the elements $X_\alpha^{(n,p)}, X_{-\alpha}^{(n,p)} \in \widehat{U}$, satisfy $w_1X_\alpha^{(n,p)}w_1^{-1} \in \widehat{U}$, $w_1X_{-\alpha}^{(n,p)}w_1^{-1} \in \widehat{U}$, and actually $w_1X_\alpha^{(0,0)}w_1^{-1} = X_{-\alpha}^{(0,0)}$. However from the relation, $x_\alpha^{(0,0)}(t) = x_\alpha^{(0,0)}(t^{-1})w_\alpha^{(0,0)}(-t^{-1})x_\alpha^{(0,0)}(t^{-1}) \in \widehat{B}w_1\widehat{B}$, we see $w_1X_\alpha^{(0,0)}w_1^{-1} = X_{-\alpha}^{(0,0)} \in \widehat{B}w_1\widehat{B}$. Further, by Lemma 2.5 (2), $w_1Hw_1^{-1} \subset H$, so we get $w_1\widehat{B}w_1^{-1} \subset \widehat{B} \cup \widehat{B}w_1\widehat{B}$. The other relations are proved similarly □

3 Relations of the generators of the Weyl group defined from $E(R_2)$

In this section, we write down some relations of the generators of the Weyl groups of the elliptic root systems $A_l^{(1,1)}, B_l^{(1,1)}, C_l^{(1,1)}$ and $D_l^{(1,1)}$ defined from the group $E(R_2)$, the relations are inducd from the relations of $w_\alpha^{(n,p)}(t)$. Dynkin diagrams of $A_l^{(1,1)}, B_l^{(1,1)}, C_l^{(1,1)}$ and $D_l^{(1,1)}$ are given in the **Appendex**.

Let α, β, α^*, and $\beta^* \in \{\alpha_0, \alpha_0^*, \cdots, \alpha_l, \alpha_l^*\}$ be the roots corresponding to the vertices of the Dynkin diagrm, and a, b, a^*, and b^* denote the corresponding reflections, and A and B denote α, α^* and β, β^*, respectively, and further with the abuse of notation, they denote a, a^* and b, b^*, respectively. Then we have the following.

Theorem 3.1 *For any subdiagram of the Dynkin diagrams, the following relations hold.*

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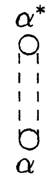
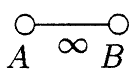


$\Longrightarrow a^2a^* = a^*a^2, \qquad a^{*2}a = aa^{*2}$

0_2

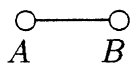
$\Longrightarrow AB = BA$

∞_2



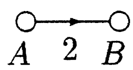
$\Longrightarrow A^2B = BA^2, \qquad B^2A = AB^2$

I_2



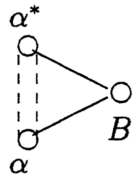
$\Longrightarrow ABA = BAB, \qquad AB^2A = B^2, \qquad BA^2B = A^2$

II_2



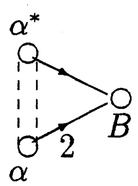
$\Longrightarrow (AB)^2 = (BA)^2, \qquad AB^2 = B^2A, \qquad BA^2B = A^2$

I_3



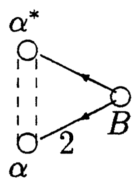
$$\Longrightarrow \quad \begin{aligned} &Ba^*B^{-1}\cdot a\cdot Ba^*B^{-1} = a\cdot Ba^*B^{-1}\cdot a \\ &BaB^{-1}\cdot a^*\cdot BaB^{-1} = a^*\cdot BaB^{-1}\cdot a^* \end{aligned}$$

II_3



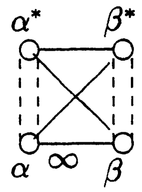
$\Longrightarrow a^*BaB = BaBa^*, \qquad aBa^*B = Ba^*Ba$

II_3^{-1}



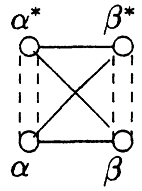
$\Longrightarrow a^*BaB = BaBa^*, \qquad aBa^*B = Ba^*Ba$

∞_4



$$\Longrightarrow \quad aa^*bb^* = b^*aa^*b = bb^*aa^* = a^*bb^*a$$

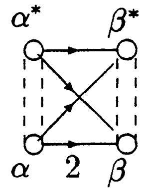
I$_4$



$$\Longrightarrow \quad aba^* = b^*ab, \quad ab^*a^* = b^*a^*b$$

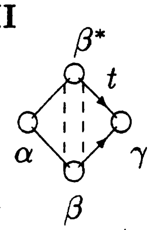
$$a^*ba = bab^*, \quad a^*b^*a = ba^*b^*$$

II$_4$



$$\Longrightarrow \quad a^*ba^{*-1} = ab^*a^{-1}$$

I + II



$$\Longrightarrow \quad ab^*a^{-1}bcb^{-1} = bcb^{-1}ab^*a^{-1}$$
$$aba^{-1}b^*cb^{*-1} = b^*cb^{*-1}aba^{-1}$$

$t = 1,\ 2^{\pm 1}$

(**Remark 1**) From the diagram $\mathbf{II}_3^{-1}$, we obtain $a^2 = a^{*2}$.

(Proof) We have the following relations : (i) $a^*BaB = BaBa^*$, (ii) $aBa^*B = Ba^*Ba$, (iii) $A^2B = BA^2$, (iv) $AB^2A = B^2$. From (i) and (ii), $aBa^* = B^{-1}a^*BaB$, $aBa^* = Ba^*BaB^{-1}$, then we get $B^{-1}a^*BaB = Ba^*BaB^{-1}$. We multiply by a^* and a in the above equation, and use (iii) and (iv) to get $a^2 = a^{*2}$.

(**Remark 2**) From the dagram $\mathbf{II}_2$, we odtain $B^4 = 1$.

(Proof) We have the relations : (i) $BA^2B = A^2$, (ii) $AB^2 = B^2A$.
We multiply by B in (i), so we get $B^2A^2B^2 = A^2$, and using (ii), we get $B^4 = 1$.

(Proof of Theorem 3.1) Let $\{e_\alpha, H_\alpha \mid \alpha \in \Delta\}$ be a Chevalley basis of $\mathfrak{g}$ satisfying

$$[e_\alpha, e_\beta] = N_{\alpha,\beta}e_{\alpha+\beta}, \quad \alpha + \beta \neq 0,$$

$$N_{\alpha,\beta} = -N_{\beta,\alpha} = -N_{-\alpha,-\beta},$$

$$N_{\alpha,\beta} = 0 \text{ if and only if } \alpha + \beta \neq 0,\ \alpha + \beta \notin \Delta,$$

$$N_{\alpha,\beta} = \pm(r+1) \text{ if } \alpha + \beta \in \Delta,$$

where r is the largest integer such that $\beta - r\alpha \in \Delta$.

If $\mathfrak{g}$ is of type A_l, we have the following relations .

Lemma 3.2 *([Ste], [St], [Ma])* *If* $\alpha, \beta, \alpha+\beta \in \Delta$ *(* Δ *is of type* A_l *), then*

(1) $w_\alpha(t)w_\beta(u)w_\alpha(t)^{-1} = w_{\alpha+\beta}(N_{\alpha,\beta}tu)$,

(2) $w_\alpha(t)w_{\alpha+\beta}(u)w_\alpha(t)^{-1} = w_\beta(N_{\alpha,\beta}(-t^{-1}u))$,

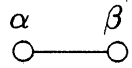
(3) $w_{\alpha+\beta}(t)w_\alpha(u)w_{\alpha+\beta}(t)^{-1} = w_{-\beta}(N_{\alpha,\beta}(-t^{-1}u))$,

(4) $w_\alpha(t)w_\alpha(u)w_\alpha(t)^{-1} = w_{-\alpha}(-t^{-2}u) = w_\alpha(t^2u^{-1})$.

We identify the elliptic roots α, α^* with $\alpha^{(0,0)}$, $\alpha^{(0,1)}$ in $\widehat{\Delta}$, respectively. Using $w_\alpha^{(n,p)}(t) = w_\alpha(T^nS^pt)$, we identify $a := w_\alpha = w_\alpha^{(0,0)}(1) = w_\alpha(1)$, $a^* := w_{\alpha^*} = w_\alpha^{(0,1)}(1) = w_\alpha(S)$. From Lemma 3.2 (4), we see that

$$w_\alpha(t)w_\alpha(u)w_\alpha(t)^{-1} = w_\alpha(t^2u^{-1}), \quad w_\alpha(t)^{-1}w_\alpha(u)w_\alpha(t) = w_\alpha(t^2u^{-1}),$$

from these, we get $\quad w_\alpha(t)^2w_\alpha(u) = w_\alpha(u)w_\alpha(t)^2 \quad (0)$

so, for the diagram **0** we obtain the relation $a^2a^* = a^*a^2$, $a^{*2}a = aa^{*2}$. The diagrams $\infty_\mathbf{2}$ and $\infty_\mathbf{4}$ appear only in $A_1^{(1,1)}$, so we set $a := w_{\alpha_0} = w_{-\alpha}^{(1,0)}(1) = w_{-\alpha}(T) = w_\alpha(-T^{-1})$, $a^* := w_{\alpha_0^*} = w_{-\alpha}^{(1,1)}(1) = w_{-\alpha}(TS) = w_\alpha(-T^{-1}S^{-1})$, $b := w_{\alpha_1} = w_\alpha^{(0,0)}(1) = w_\alpha(1)$, $b^* := w_{\alpha_1^*} = w_\alpha^{(0,1)}(1) = w_\alpha(S)$. From these, for the diagram $\infty_\mathbf{2}$, using (0), we obtain $A^2B = BA^2$, $AB^2 = B^2A$. For the diagram $\infty_\mathbf{4}$, we use the following fact $ABCD = BCDA \Longleftrightarrow BCDAD^{-1}C^{-1}B^{-1} = A \quad (*)$ and set $A = w_\alpha(t)$, $B = w_\alpha(s)$, $C = w_\alpha(u)$ and $D = w_\alpha(p)$, then $BCDAD^{-1}C^{-1}B^{-1} = w_\alpha(s^2p^2u^{-2}t^{-1})$, so from $(*)$ we get $s^2p^2 = t^2u^2$. Here s, p, t and u are either of ± 1, $\pm S$, $\pm T^{-1}S^{-1}$ and $\pm T^{-1}$ so we obtain the relation $aa^*bb^* = b^*aa^*b = bb^*aa^* = a^*bb^*a \quad (**)$ and corresponding relations where we arbitrarily replace a, a^*, b, and b^* by a^{-1}, a^{*-1}, b^{-1} and b^{*-1} respectively. However they are reduced to the relations $(**)$, by using the relation of $\infty_\mathbf{2}$. For the diagram $\mathbf{0_2}$, we obtain the relation $w_\alpha(t)w_\beta(u)w_\alpha(t)^{-1} = w_\beta(u)$, so $w_\alpha(t)w_\beta(u) = w_\beta(u)w_\alpha(t)$. Since $b = w_\beta(1)$, $b^* = w_\beta(S)$, for the diagram $\mathbf{0_2}$ we obtain $AB = BA$. For the diagram 

we obtain the relation

$$w_\alpha(t)w_\beta(u)w_\alpha(t)^{-1} = w_{\alpha+\beta}(N_{\alpha,\beta}tu), \qquad (1)$$

$$w_\beta(t)w_\alpha(u)w_\beta(t)^{-1} = w_{\alpha+\beta}(-N_{\alpha,\beta}tu). \qquad (2)$$

From (1), we obtain

$$w_\alpha(t)w_\beta(u)w_\alpha(t)^{-1} = w_\alpha(t)^{-1}w_\beta(u)^{-1}w_\alpha(t), \qquad (3)$$

$$w_\alpha(t)w_\beta(u)w_\alpha(t)^{-1} = w_\alpha(u)w_\beta(t)w_\alpha(u)^{-1}. \qquad (4)$$

From (2), we obtain

$$w_\beta(t)w_\alpha(u)w_\beta(t)^{-1} = w_\beta(t)^{-1}w_\alpha(u)^{-1}w_\beta(t), \qquad (5)$$

$$w_\beta(t)w_\alpha(u)w_\beta(t)^{-1} = w_\beta(u)w_\alpha(t)w_\beta(u)^{-1}. \qquad (6)$$

From (1) and (2), we obtain

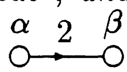
$$w_\alpha(t)w_\beta(u)w_\alpha(t)^{-1} = w_\beta(t)^{-1}w_\alpha(u)w_\beta(t), \qquad (7)$$

$$w_\alpha(t)w_\beta(u)w_\alpha(t)^{-1} = w_\beta(t)w_\alpha(u)^{-1}w_\beta(t)^{-1}, \qquad (8)$$
$$w_\alpha(t)w_\beta(u)w_\alpha(t)^{-1} = w_\beta(u)^{-1}w_\alpha(t)w_\beta(u), \qquad (9)$$
$$w_\alpha(t)w_\beta(u)w_\alpha(t)^{-1} = w_\beta(u)w_\alpha(t)^{-1}w_\beta(u)^{-1}. \qquad (10)$$

We find that all relations among a, a^*, b and b^* in (3) - (10), are reduced to the following relations, $ABA = BAB,\ AB^2A = B^2,\ BA^2B = A^2.$

For the diagram $\mathbf{I_3}$, from the relations

$$w_\alpha(s)w_{\alpha+\beta}(t)w_\alpha(s)^{-1} = w_\beta(N_{\alpha,\beta}(-s^{-1}t)),$$
$$w_{\alpha+\beta}(t)^{-1}w_\alpha(s)w_{\alpha+\beta}(t) = w_{-\beta}(N_{\alpha,\beta}(t^{-1}s)) = w_\beta(N_{\alpha,\beta}(-s^{-1}t)),$$

we obtain $w_{\alpha+\beta}(t)w_\alpha(s)w_{\alpha+\beta}(t) = w_\alpha(s)w_{\alpha+\beta}(t)w_\alpha(s)$, and noting that $w_\beta(t)w_\alpha(u)w_\beta(t)^{-1} = w_{\alpha+\beta}(N_{\alpha,\beta}tu)$, we obtain $w_\beta(t)w_\alpha(u)w_\beta(t)^{-1}\cdot w_\alpha(s)\cdot w_\beta(t)w_\alpha(u)w_\beta(t)^{-1} = w_\alpha(s)\cdot w_\beta(t)w_\alpha(u)$ $w_\beta(t)^{-1}\cdot w_\alpha(s)$, therefore we get $Ba^*B^{-1}\cdot a\cdot Ba^*B^{-1} = a\cdot Ba^*B^{-1}\cdot a$, *and* $BaB^{-1}\cdot a^*\cdot BaB^{-1} = a^*\cdot BaB^{-1}\cdot a^*$. For the diagram $\mathbf{I_4}$, all relations of a, a^*, b and b^* in (3) - (10) are reduced the follwing relations $aba^* = b^*ab$, $ab^*a^* = b^*a^*b$, $a^*ba = bab^*$, *and* $a^*b^*a = ba^*b^*$. Next we examine the relations associated to the diabgram 

. In this case α, β, $\alpha+\beta$ and $\alpha+2\beta$ are roots, and their scalar products are given by

$$(\alpha,\beta^\vee) = -2,\ (\beta,\alpha^\vee) = -1,\ (\alpha,\beta) = -2,\ (\alpha,\alpha) = 4,\ (\beta,\beta) = 2,\ (\alpha,\alpha^\vee) = 2,\ (\beta,\beta^\vee) = 2$$

(see [Bo]). Then the following relations hold ([Ma]).

$$w_\alpha(t)w_\beta(u)w_\alpha(t)^{-1} = w_{\alpha+\beta}(ctu), \qquad (1)$$
$$w_\alpha(t)w_{\alpha+\beta}(u)w_\alpha(t)^{-1} = w_\beta(c't^{-1}u), \qquad (2)$$
$$w_\alpha(t)w_{\alpha+2\beta}(u)w_\alpha(t)^{-1} = w_{\alpha+2\beta}(u), \qquad (3)$$
$$w_\beta(t)w_\alpha(u)w_\beta(t)^{-1} = w_{\alpha+2\beta}(c''t^2u), \qquad (4)$$
$$w_\beta(t)w_{\alpha+\beta}(u)w_\alpha(t)^{-1} = w_{\alpha+\beta}(-u), \qquad (5)$$
$$w_\beta(t)w_{\alpha+2\beta}(u)w_\beta(t)^{-1} = w_\alpha(c'''t^{-2}u), \qquad (6)$$

where $c = c(\alpha,\beta)$, $c' = c(\alpha,\alpha+\beta)$, $c'' = c(\beta,\alpha)$, $c''' = c(\beta,\alpha+2\beta)$ equal ± 1 .

From (1) and (5), we get $w_\beta(s)w_\alpha(u)w_\beta(t)w_\alpha(u)^{-1}w_\beta(s)^{-1} = w_{\alpha+\beta}(-ctu)$ and from (1), we see

$$w_\alpha(u)^{-1}w_\beta(t)w_\alpha(u) = w_{\alpha+\beta}(-ctu),$$
$$w_\alpha(u)w_\beta(t)^{-1}w_\alpha(u)^{-1} = w_{\alpha+\beta}(-ctu),$$

from above relations, we get

$$w_\alpha(t)w_\beta(u)w_\alpha(t)w_\beta(s) = w_\beta(s)w_\alpha(t)w_\beta(u)w_\alpha(t), \qquad (7)$$
$$w_\alpha(t)w_\beta(u)^{-1}w_\alpha(t)^{-1}w_\beta(s) = w_\beta(s)w_\alpha(t)w_\beta(u)w_\alpha(t)^{-1}. \qquad (8)$$

From (3) and (4), we get

$$w_\alpha(s)w_\beta(u)w_\alpha(t)w_\beta(u)^{-1}w_\alpha(s)^{-1} = w_{\alpha+2\beta}(u^2t)$$

and from (4), we see that

$$w_\beta(u)w_\alpha(t)w_\beta(u)^{-1} = w_{\alpha+2\beta}(u^2t),$$
$$w_\beta(u)^{-1}w_\alpha(t)w_\beta(u) = w_{\alpha+2\beta}(u^2t),$$

so from the above relations, we get

$$w_\alpha(t)w_\beta(u)w_\alpha(s)w_\beta(u) = w_\beta(u)w_\alpha(s)w_\beta(u)w_\alpha(t), \quad (9)$$
$$w_\alpha(t)w_\beta(u)w_\alpha(s)w_\beta(u)^{-1} = w_\beta(u)w_\alpha(s)w_\beta(u)^{-1}w_\alpha(t). \quad (10)$$

We examine all relations of a, a^*, b and b^* in (7) - (10), and they are reduced the relations in $\mathbf{II_2}$, $\mathbf{II_3}$, and $\mathbf{II_3^{-1}}$. Further from the relation (1), we obtain $w_\alpha(t)w_\beta(u)w_\alpha(t)^{-1} = w_\alpha(u)w_\beta(t)w_\alpha(u)^{-1}$, so we have the relation in $\mathbf{II_4}$. Next we prove $\mathbf{I}+\mathbf{II}$. In all cases $t = 1, 2^{\pm 1}$, we have $w_{\beta+\gamma}(t)w_{\alpha+\beta}(u)w_{\beta+\gamma}(t)^{-1} = w_{\alpha+\beta}(ct^{-<\alpha+\beta,\beta+\gamma>}u)$, where $c = c(\beta+\gamma, \alpha+\beta) = 1$, because $(\beta+\gamma)+(\alpha+\beta) \neq 0, (\beta+\gamma)+(\alpha+\beta) \notin \Delta$ (see [Ma]), and $<\alpha+\beta, \beta+\gamma> = 0$, so we get $w_{\beta+\gamma}(t)w_{\alpha+\beta}(u) = w_{\alpha+\beta}(u)w_{\beta+\gamma}(t)$. This means $w_\beta(s)w_\gamma(t)w_\beta(s)^{-1} \cdot w_\alpha(u)w_\beta(p)w_\alpha(u)^{-1} = w_\alpha(u)w_\beta(p)w_\alpha(u)^{-1} \cdot w_\beta(s)w_\gamma(t)w_\beta(s)^{-1}$, and which implies $\mathbf{I}+\mathbf{II}$. □

4 Chevalley groups over 2-dimensional local field

We recall the definition of 2-dimensional local field ([P1], [P2]). We say that K is a 2-dimensional local field with k as the last residue field if K is the quotient field of a (complete) discrete valuation ring O_K whose residue field is a local field of dimension 1 with residue field k. The first residue field is denoted by $\bar{K}$. As such an example, let $K = k((T))((S))$ be the field of iterated power series with $O_K = k((T))[[S]]$, and $\bar{K} = k((T))$. There exists the reduction map $\varphi : O_K \longrightarrow \bar{K}$ and denote by $\bar{m}$ the maximal ideal of the local ring $O_{\bar{K}} = k[[T]]$, then $\bar{m} = Tk[[T]]$. There also exists the canonical map $\phi : O_{\bar{K}} \longrightarrow O_{\bar{K}}/\bar{m} \cong k$. Let $O'_K = \varphi^{-1}(O_{\bar{K}})$ be a subring in K, and $m = \varphi^{-1}(\bar{m})$, then m is the maximal ideal of O'_K, and let $(O'_K)^*$ be the group of units in O'_K then

$$O'_K = k[[T]] \oplus Sk((T))[[S]],$$
$$m = Tk[[T]] \oplus Sk((T))[[S]],$$
$$(O'_K)^* = k^* \oplus Tk[[T]] \oplus Sk((T))[[S]],$$

with the obvious abuse of notation. The maps φ and ϕ induce the maps of the matrices

$$\varphi : SL(2, O_K) \longrightarrow SL(2, \bar{K}),$$
$$\phi : SL(2, O_{\bar{K}}) \longrightarrow SL(2, k),$$

and we let the group $\bar{B}$ be the inverse image of the upper triangular group, i.e. the Borel subgroup of $SL(2,k)$, and B be the inverse image of $\bar{B}$ from $SL(2, \bar{K})$, then

$$B = \begin{pmatrix} (O'_K)^* & O'_K \\ m & (O'_K)^* \end{pmatrix}, \quad \bar{B} = \begin{pmatrix} (O_{\bar{K}})^* & O_{\bar{K}} \\ \bar{m} & (O_{\bar{K}})^* \end{pmatrix},$$

where $(O_K)^* = k^* \oplus Tk[[T]]$. Let N equal the subgroup of monomial matrices and $W = N/T$, where $T = B \cap N$, then $T = \begin{pmatrix} (O'_K)^* & 0 \\ 0 & (O'_K)^* \end{pmatrix}$ and $W =< w_0, w_1, w_2 >$, $w_0 = \begin{pmatrix} 0 & -T^{-1} \\ T & 0 \end{pmatrix}$, $w_1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, and $w_2 = \begin{pmatrix} 0 & -S^{-1} \\ S & 0 \end{pmatrix}$. We set $P_1 = SL(2, O'_K)$, $P_0 = \left\{ \begin{pmatrix} a & T^{-1}b \\ Tc & d \end{pmatrix}, \text{with} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, O'_K) \right\}$, then P_1 and P_0 are subgroups of $SL(2, K)$ and we have the following.

Proposition 4.1 *For* $i = 0, 1, \quad P_i = B \cup Bw_iB$.

(Proof) There exists the canonical map $O'_K \to O'_K/m \cong k$ and which induces a homomorphism $\phi' : P_1 = SL(2, O'_K) \to SL(2, k)$. Clearly $ker\ \phi' \subset B$ (indeed, B is the inverse image of the upper triangular group). Note that ϕ' sends w_1 to $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \in SL(2, k)$, which represents the nontrivial generator for the Weyl group in $SL(2, k)$. Using the Bruhat decomposition in the (rank 1) group $SL(2, k)$ and the fact that $ker\ \phi' \subset B$, we get $P_1 = B \cup Bw_1B$ by lifting back to P_1. Next the matrix $g = \begin{bmatrix} 0 & 1 \\ T & 0 \end{bmatrix} \in GL(2, K) - SL(2, K)$ normalizes B, and $g^{-1}P_1g = P_0$, $g^{-1}w_1g = w_0$. So $P_1 = B \cup Bw_1B$ forces $P_0 = B \cup Bw_0B$, this is proved as in ([H2], § 15.3, Lemma 2). □

An element of $K = k((T))((S))$ can be written as

$$\sigma(T, S) = \sum_{i \geq i_0,\ j \geq j_0} q_{ij}T^iS^j, \quad q_{ij} \in k.$$

We set $x_\alpha(\sigma(T, S)) := \prod_{i \geq i_0,\ j \geq j_0} x_\alpha^{(i,j)}(q_{ij})$ and $\widetilde{M} := K \otimes_{\mathbb{Z}} M$, then $x_\alpha(\sigma(T, S)) \in Aut(\widetilde{M})$, because from the relation $x_\alpha(\sigma_1(T, S))x_\alpha(\sigma_2(T, S)) = x_\alpha(\sigma_1(T, S) + \sigma_2(T, S))$, we get $x_\alpha(\sigma(T, S))^{-1} = x_\alpha(-\sigma(T, S))$. We let $\widehat{E} \subset Aut(\widetilde{M})$ denote the subgroup generated by the elements $x_\alpha(\sigma(T, S))$, $\alpha \in \Delta$, $\sigma(T, S) \in K$. For $\alpha \in \Delta$, and $\sigma(T, S) \in K$ with $\sigma(T, S) \neq 0$, we set

$$w_\alpha(\sigma(T, S)) := x_\alpha(\sigma(T, S))x_{-\alpha}(-\sigma(T, S)^{-1})x_\alpha(\sigma(T, S)),$$
$$h_\alpha(\sigma(T, S)) := w_\alpha(\sigma(T, S))w_\alpha(1)^{-1}.$$

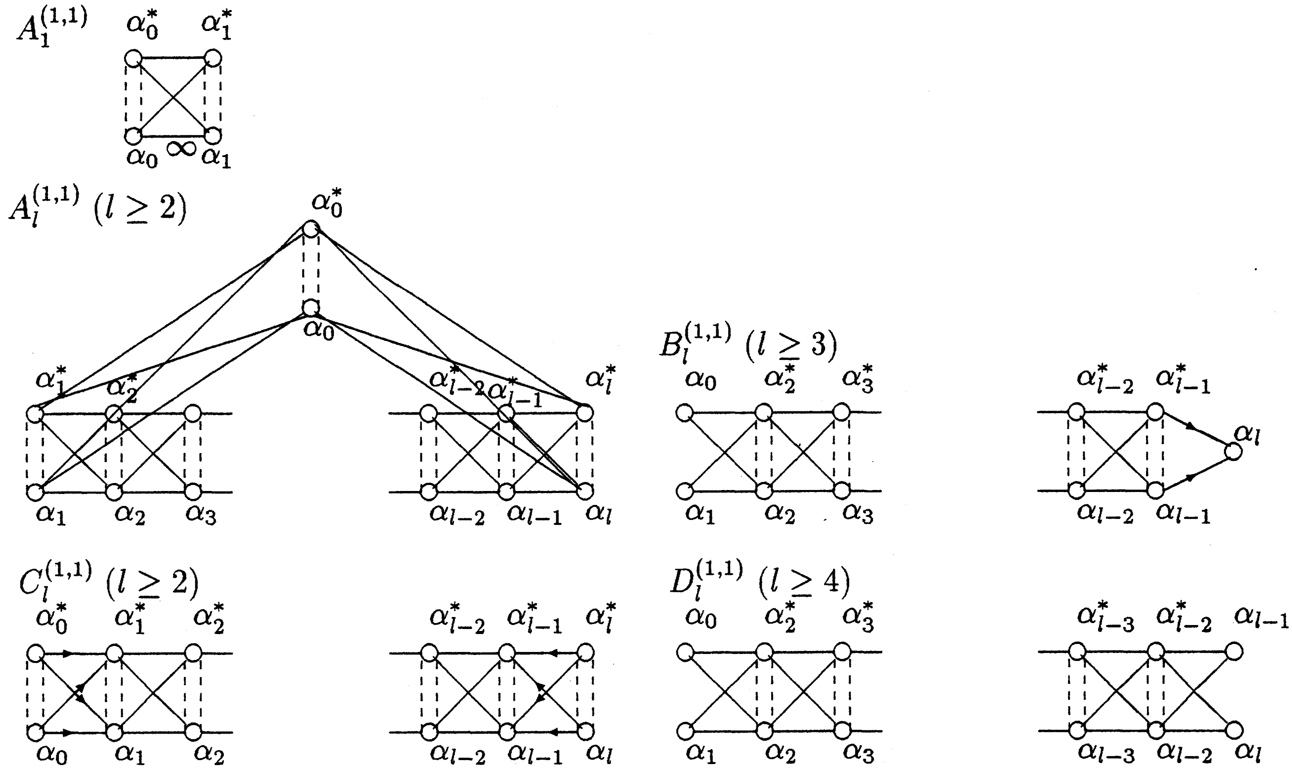
Similarly to the case of affine Lie algebra ([G]), we give the following definition.

Definition 4.2 *We let* $I \subset \widehat{E}$ *denote the subgroup generated by the elements* $x_\alpha(\sigma(T, S))$, *where either* $\alpha \in \Delta^+$, $\sigma(T, S) \in O'_K$, *or* $\alpha \in \Delta^-$, $\sigma(T, S) \in m$, *and by the elements* $h_\alpha(\sigma(T, S))$, $\sigma(T, S) \in (O'_K)^*$, $\alpha \in \Delta$. *We call* I *the Iwahori subgroup of* $\widehat{E}$.

We see the following .

(**Fact**) If Δ is of type A_1 , then $I \cong B$.

(**Appendex**) Dynkin diagrams of $A_l^{(1,1)}$, $B_l^{(1,1)}$, $C_l^{(1,1)}$ and $D_l^{(1,1)}$ are given by ;



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