表現論シンポジウム講演集, 1998
pp.87-89

Inductive limits of groups and algebras

R. S. Ismagilov

1). Let me begin with the simple example. Consider a linear real space $\mathcal{V}$ with a symplectic form τ and a family of operators $U(x), x \in \mathcal{V}$ such that

$$U(x+y) = e^{i\lambda\tau(x,y)}U(x)U(y). \tag{1}$$

and the mapping $s \rightarrow U(sx)U(sy)$,is weakly continious for any x, y from $\mathcal{U}$. These are called Canonical Commutation Relations (CCR for brevity). Define Weak CCR by the condition

$$U(x+y) = U(x)U(y), \quad \text{if } x \perp y. \tag{2}$$

($x \perp y$ means, that $\tau(x,y) = 0$).

The continuity Property is the same as above. We only consider irreducible families $U(x), x \in \mathcal{V}$. Clearly $1 \Longrightarrow 2$. Question: is the converse true? More presicely, suppose that the conditions (2) hold; is it true then that the conditions (1) hold (for some $\lambda \in \mathcal{R}$)?

The similar question can be posed for CCR in the infinitesemal form. Define weak CCR as a family of symmetrical operators $A(x), x \in \mathcal{V}$ (acting in a dense linear subset $D \subset H$) such that

$$[A(x), A(y)] = 0 \quad \text{and} \quad A(\alpha x + \beta y) = \alpha A(x) + \beta A(y)$$
$$\text{for any} \quad x \perp y, \quad x, y \in \mathcal{V}, \quad \alpha, \beta \in \mathcal{R}.$$

(So the linearity of the mapping $x \longmapsto A(x)$ is not supposed!). Question: is it true that the mapping $x \longmapsto A(x)$ is linear and the usual CCR (in the infinitesemal form) are satisfied?

Lastly consider CAR. Let W be a complex linear space with a symmetrical nondegerate 2-form θ and with an involution $x \longmapsto \overline{x}$ such that $\theta(x, \overline{x}) > 0$ for $x \neq 0$. Define weak CAR as a family of operators $\{C(x)\}$ in a Hilbert spase defined only for vectors x with $x \perp x$ (this means $\theta(x,x) = 0$) such that $\{C(x), C(y)\} = 0$ and $C(\alpha + \beta) = \alpha C(x) + \beta C(y)$ for any x, y with $x \perp x$, $y \perp y$, $x \perp y$ and for all $\alpha, \beta \in C$. Assume this family is irreducible. Question: is it true that the mapping $x \longmapsto C(x)$ is linear and satisfy the usual CAR $\{C(x), C(y)\} = \theta(x,y)$. We shall see later that the answers to these questions are positive. This will deduced from the explicite description of inductive limits for some families of groups and algebras.

2). Definitions ([1]). Let $\{G_\alpha\}$ be a family of groups and for any α, β a set (possibly void) of homomorphisms $\varphi_{\alpha\beta} : G_\alpha \rightarrow G_\beta$ is given. The inductive limit of this family is (by definition) a group G with homomorphisms $f_\alpha : G_\alpha \rightarrow G$ with the following properties:

1. $f_\beta = f_\beta \circ \varphi_{\alpha\beta}$
2. $\bigcup_\alpha f_\alpha(G_\alpha)$ generates G
3. for any group G' and homomorphisms f'_α with these properties there exists a homomorphism $p : G \longmapsto G'$ with $f'_\alpha = p \circ f_\alpha$ for all α.

The following particular case is of great importance: G_α are the subgroups of some group Γ such that $\bigcup_\alpha G_\alpha$ generates Γ and for any α, β the subgroup $G_\alpha \bigcap G_\beta$ belongs to our family. In this case the mappings $f_j : G_j \longmapsto G$, $f'_j : G_j \longmapsto G'$, give mappings $f : \bigcup_\alpha G_\alpha \longmapsto G$, $f' : \bigcup_\alpha G_\alpha \longmapsto G$. If Γ is a topological group define a restricted inductive limit supposing that G, G' in the above definition are topological and f, f' are continious. For algebras inductive limits are defined in the same way.

3). The Heisenberg group as an inductive limit of abelian groups. Concider a situation close to that in the Problem on weak CCR. Let $\mathcal{V}$ be a linear space over a field $\mathcal{K}$, char $\mathcal{K} \neq 2$, τ a nondegenerate antisymmetrical form. Denote by $Is\mathcal{V}$ the family of all isotropical subspaces of $\mathcal{V}$. We concider these subspaces as abelian groups with respect to addition of vectors.

We want to describe the inductive limit of $Is\mathcal{V}$. To do this consider the Heisenberg group H; recall that $H = \mathcal{V} \times \mathcal{K}$ with group operation $(v_1, k_1)(v_2, k_2) = (v_1 + v_2, k_1 + k_2 + \tau(v_1 + v_2)$. For any $\mathcal{V}_0 \in Is\mathcal{V}$ we have a homomorphism $\mathcal{V}_0 \longmapsto H$, $x \longrightarrow (x, 0)$. Suppose that $\dim \mathcal{V} > 2$

Theorem 1. The inductive limit of the family $Is\mathcal{V}$ is the Heisenberg group H.

The similar theorem is valid also for $Is\mathcal{V}$ considered as a family of abelian Lie algebras.

From these theorems it follows that the weak CCR and the ordinary CCR are the same thing.

4). Clifford algebras and the inductive limit of Grassman algebras. Consider now the situation close to that arose in the Problem on weak CAR.

Let W be a linear space over $\mathcal{K}$, char $\mathcal{K} \neq 2$ and θ asymmetrical nondegenerate form. Let $Cl(W)$ be the corresponding Clifford algebra and $Cl(W, \tau)$ the assosiative algebra generated by the space W and by the symbol T with relations $xy + yx = \theta(x, y)T$, $xT = Tx$, $(x, y \in W)$. Denote by IsW the family of all isotropical subspaces in W. Any $W_0 \in W$ generates a Grassman subalgebra $Gr(W_0)$ in $Cl(W)$; we have a homomorphism $Gr(W_0) \longmapsto Cl(W, T)$ given by $x \longmapsto xT$, $x \in W_0$. Suppose that $\dim W > 2$.

Theorem 2. The inductive limit of the family $\{Gr(W_0) \,|\, W_0 \in IsW\}$ is the algebra $Cl(W, \tau)$.

It follows from this Theorem that the weak CAR are the same thing as CAR.

5). The inductive limit of the family of abelian subalgebras of a simple compact Lie algebra g. Denote by Abg the family indicated in the title and by $T \circ g[T]$ the Lie algebra of polynomials $x_1 T + x_2 T^2 + \ldots + x_m T^m$ with the natural Lie bracket. We have homomorphism $L \longmapsto Tg[T]$ for any $L \in Abg$. We equipp g with the usual topology.

Theorem 3. Let $g = su(n)$, $n \geq 3$. Then the restricted inductive limit of Abg is $T \circ g[T]$ For other simple compact Lie algebras the corresponding result is not obtained yet.

5). Lie algebras of vector fields. Let (X, ω^2) be a connected symplectic manyfold $\dim X = 2n < \infty$ and Ham (X, ω^2) a Lie algebra of Hamiltonian vector fields with compact support. For any $Y \subset X$, $Y \simeq \mathcal{R}^{2n}$ we have a subalgebra Ham $(Y, \omega^2) \subset$ Ham (X, ω^2). If $Y_1 \subset Y_2$ then we have an inclusion Ham $(Y_1, \omega^2) \longmapsto$ Ham (Y_2, ω^2). Define also the subalgebra $\overset{\circ}{\text{Ham}}\,(Y, \omega^2)$ of all vector fields $\xi \in$ Ham (X, ω^2) having a hamiltonian f_ξ with $f_\xi = 0$ for an $x \notin Y$, $\int_X f_\xi \, \omega^{2n} = 0$. In the next theorem X is compact.

Theorem 4. The inductive limit of the family $\{\text{Ham}\,(Y, \omega^2) \mid Y \simeq \mathcal{R}^{2n}\}$ is Ham (X, ω^2)

(with obvious inclusions). The innductive limit of $\left\{\overset{\circ}{\text{Ham}}\,(Y,\omega^2)\; Y \simeq \mathcal{R}^{2n}\right\}$ is a central extension of $\text{Ham}\,(Y,\omega^2)$ by center $H^1\,(X,\mathcal{R})$

(We do not dwell on the description of this extension). The case of volume preserving vector fields was treated in [2]; the corresponding Problem for the group of volume preserving diffeomorphisms was also solved in this paper.

6). Applications to representations of the group $D\,(X,v^n)$. Let (X,v^n) be a connected manyfold with a volume form v^n and $D\,(X,v^n)$ the group of diffeomorphisms which can be connected with identity by a curve of diffeomorphisms supported in some compact subset. We have an inclusion $D(Y,v^n) \longmapsto D(X,v^n)$ for any open connected $Y \subset X$.

Consider unitary representations $g \longmapsto U(g)$, $g \in D(X,v^n)$ with the following Property: for any $Y \in X$ the restriction of this representation to $D(Y,v^n)$ is continious with respect to the weak topology on $D(Y,v^n)$ in the sense of measure theory and consequently can be prolonged to the group $M(Y,\mu)$ of measurable transformations of $Y \subset X$ preserving the corresponding measure μ. The representations $g \longmapsto U(g)$, $g \in D(X,v^n)$ with this Property are called locally weak continious. The Problem is to describe them. This Problem leads to the inductive limit of the family $D(X,v^n)$, $\{D(Y,v^n), Y \simeq \mathcal{R}^n\}$, $M(Y,\mu)$ (with the obvious inclusions). The explicite description of this inductive limit is obtained for X compact and $n \geq 5$. We describe it below.

Let $\Pi = \pi(X,x_0)$ be the fundamental group of X, $p : \tilde{X} \longmapsto X$ the universal covering and Γ the group of diffeomorphisms $\tilde{x} \to \gamma\tilde{x}$, $\tilde{x} \in \tilde{X}$ with $p \circ \gamma = p$. Denote by Γ_0 the subgroup of elements $\gamma \in \Gamma$ which can be connected with the identity by a curve of diffeomorphisms commuting with Γ. Clearily Γ_0 is central in Γ. We have an inclusion $\Gamma_0 \subset \Pi$ (for any $\gamma \in \Gamma_0$ connect the points $\tilde{x}_c$, $\gamma \circ \tilde{x}_0$, $(\tilde{x}_0 \in p^{-1}(x_0))$ by a curve and project it to X; this gives the desired element of $\Pi = \pi(X,x_0)$). Consider now the group Π^X of measurable mappings $f : X \longmapsto \Pi$; (identifying mappings which coinside almost everywhere). We have an inclusion $\Gamma_0 \longmapsto \Pi^X$ (considering elements of Γ_0 as constant mappings from X to $\Pi = \pi(X,x_0)$). The $M(X,\mu)$ acts on Π^X by translations; so we have a semidirect product $\Pi^X \circ M(X,\mu)$ and a group $\Pi^X \circ M(X,\mu)/\Gamma_0$. It turns out that $\Pi^X \circ M(X,\mu)/\Gamma_0$ is the desired inductive limit of the family above. The corresponding mappings $D(Y,v^n) \longmapsto \Pi^X \circ M(X,\mu)/\Gamma_0$, $Y \simeq \mathcal{R}^n$ are described in [5]. So the locally weak continious representations of $D(X,v^n)$ are the same thing as representations of the semidirect product $\Pi^X \circ M(X,\mu)/\Gamma_0$. (They must be weakly continious in measure theory sense). The beatiful theory of H.Araki allows to constract a family of representations of $\Pi^X \circ M(X,\mu)/\Gamma_0$ in boson Fock spase; see [2],[3]. In particular to any non zero element of $H'(X,\mathcal{R})$ gives an unitary irreducible representation of $D(X,v^n)$.

References

[1] Serre J.P.,Arbres, amalgame et 114, Soc.Math.France, Asterisque,1977,46,20-43.

[2] Araki H., Publ. Res. Inst. Math. Sci. Kyoto Univ.5, 1970, 311-402.

[3] A.M.Vershik, I.M.Gelfand, M.I.Graev. Uspechi Mat.Nauk,28,1 973, no 5, 83-128.

[4] R.S.Ismagilov, Izv. Akad. Nauk SSSR, Ser. Mat., 44, 1980, 831-867.

[5] The same Author, Mat. Sbornik, 113(155), 1(9), 1980.