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SPHERICAL FUNCTIONS ON ORDERED SYMMETRIC SPACES

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In the harmonic analysis on ordered symmetric spaces there are many similarities with the harmonic analysis of Riemannian symmetric spaces. In both cases there is a commutative algebra of biinvariant functions, and spherical functions play a central role. On Riemannian symmetric spaces one considers the spherical Fourier transform, and, on ordered symmetric spaces, a spherical Laplace transform. In this talk we give a survey of the spherical Laplace analysis on ordered symmetric spaces, and present some recent results and open questions.

1. Ordered symmetric spaces. — A causal structure on a manifold $\mathcal{M}$ is the data of a cone field $\{C_x\}$ on $\mathcal{M}$. For each point $x \in \mathcal{M}$, C_x is a regular cone in the tangent space $T_x(\mathcal{M})$ (regular means convex, closed, pointed, and with non empty interior). A curve $\gamma : [\alpha, \beta] \to \mathcal{M}$ is said to be causal if, for all t, $\gamma'(t) \in C_{\gamma(t)}$. If there is no non trivial closed causal curve the causal structure is said to be global. It then defines an ordering on $\mathcal{M}$: $x \leq y$ means that there is a causal curve starting at x and ending at y. If $\varphi : \mathcal{M} \to \mathcal{M}$ is a diffeomorphism, the causal structure is said to be φ-invariant if

$$C_{\varphi(x)} = d\varphi_x(C_x) \quad (x \in \mathcal{M}).$$

If $\mathcal{M}$ is a homogeneous space, $\mathcal{M} = G/H$, where G is a connected Lie group and H a closed subgroup, an invariant causal structure on $\mathcal{M}$ is determined by a regular cone C in the tangent space $T_{x_0}(\mathcal{M})$ which is H-invariant (x_0 denotes the base point $x_0 = eH$).

Assume further that $\mathcal{M} = G/H$ is a symmetric space: there is an involutive automorphism σ of G such that

$$(G^\sigma)_0 \subset H \subset G^\sigma,$$

where $G^\sigma = \{g \in G \mid \sigma(g) = g\}$, and $(G^\sigma)_0$ is the identity component of G^σ. Let $\mathfrak{g}$ and $\mathfrak{h}$ be the Lie algebras of G and H,

$$\mathfrak{h} = \{X \in \mathfrak{g} \mid d\sigma(X) = X\}.$$

The subspace

$$\mathfrak{q} = \{X \in \mathfrak{g} \mid d\sigma(X) = -X\}$$

can be identified with $T_{x_0}(\mathcal{M})$. Therefore a G-invariant causal structure on $\mathcal{M}$ is determined by the data of a regular cone C in $\mathfrak{q}$ which is $\mathrm{Ad}(H)$-invariant. Assume also that G is semi-simple with finite center, and that

$(\mathfrak{g}, \mathfrak{h})$ is an irreducible symmetric pair. There exists a Cartan involution θ which commutes with σ, $\theta \circ \sigma = \sigma \circ \theta$. Let us consider the corresponding Cartan decomposition of $\mathfrak{g}$,

$$\mathfrak{g} = \mathfrak{k} + \mathfrak{p}.$$

If $C \cap \mathfrak{k} = \{0\}$, as we will assume from now, the corresponding invariant causal structure is global and defines an ordering on $\mathcal{M}$. Then one says that $\mathcal{M}$ is an *ordered symmetric space.* One shows that the order intervals, i.e. the sets

$$D(x, y) = \{z \in \mathcal{M} \mid x \leq z \leq y\},$$

are compact ([Faraut,1991] théorème 4, [Hilgert-Ólafsson,1997], Theorem 5.3.5).

More general ordered symmetric spaces have been considered in [Krötz-Neeb-Ólafsson,1998].

2. Spherical functions. — The set

$$S = \{g \in G \mid g \cdot x_0 \geq x_0\}$$

is a closed semi-group in G, and one shows that $S = \exp C.H$. A *spherical function* is a continuous function φ on the interior S^0 of S such that, for $x, y \in S^0$, the function $h \mapsto \varphi(xhy)$ is integrable on H, and which satisfies the functional equation

$$\int_H \varphi(xhy)dh = \varphi(x)\varphi(y).$$

(Notice that this definition depends on the choice of the Haar measure dh on H.)

A Volterra kernel is a function on $\mathcal{M} \times \mathcal{M}$, continuous on $\{(x, y) \mid x \geq y\}$, vanishing out of this set. The composition product of two such kernels F_1 and F_2 is defined by

$$F_1 \diamond F_2(x, y) = \int_{\mathcal{M}} F_1(x, z)F_2(z, y)dz,$$

where dz is an invariant measure on $\mathcal{M}$. This integral is well defined since the function $z \mapsto F_1(x, z)F_2(z, y)$ vanishes out of the order interval $D(y, x) = \{z \in \mathcal{M} \mid y \leq z \leq x\}$, which is compact. So the space $V(\mathcal{M})$ of Volterra kernels is an algebra. The subspace $V(\mathcal{M})^\#$ of invariant kernels, i.e. such that

$$F(g \cdot x, g \cdot y) = F(x, y) \quad (g \in G),$$

is a commutative subalgebra ([Faraut,1987], théorème 1). If F is an invariant Volterra kernel, the function f, $f(x) = F(x, x_0)$, is H-invariant and supported by $\mathcal{M}^+ = \{x \geq x_0\}$, and this gives an isomorphism of $V(\mathcal{M})^\#$ with $\mathcal{C}(\mathcal{M}^+)$. We will see that the spherical functions appear as characters of the Volterra algebra $V(\mathcal{M})^\#$.

There exists in $\mathfrak{p} \cap \mathfrak{q}$ an Abelian subspace $\mathfrak{a}$ which is maximal Abelian in $\mathfrak{p}$. Let $\Delta(\mathfrak{g}, \mathfrak{a})$ be the root system of the pair $(\mathfrak{g}, \mathfrak{a})$. There is in $\mathfrak{a}$ an element X_0 which is $\mathrm{Ad}(K \cap H)$-invariant, such that the eigenvalues of $\mathrm{ad}\, X_0$ are $-1, 0, 1$. We put

$$\Delta_\varepsilon = \{\alpha \in \Delta \mid \alpha(X_0) = \varepsilon\} \quad (\varepsilon = -1, 0, 1).$$

We choose a positive system Δ_0^+ in Δ_0, then $\Delta^+ = \Delta_0^+ \cup \Delta_1$ is a positive system in Δ. We denote the corresponding positive Weyl chamber by $\mathfrak{a}_+$, and $\mathfrak{a}_- = -\mathfrak{a}_+$. Let $\mathfrak{g}^\alpha$ be the root space associated with the root α, m_α its dimension,

$$\mathfrak{n} = \bigoplus_{\alpha \in \Delta^+} \mathfrak{g}^\alpha, \quad \rho = \tfrac{1}{2} \sum_{\alpha \in \Delta^+} m_\alpha \alpha,$$
$$N = \exp \mathfrak{n}, \quad A = \exp \mathfrak{a}, \quad A_\pm = \exp \mathfrak{a}_\pm.$$

The map

$$N \times A \to \mathcal{M}, \quad (n, a) \mapsto na \cdot x_0,$$

is a difeomorphism from $N \times A$ onto its image $NA \cdot x_0$. For x in this set, $x = n \exp X \cdot x_0$ $(X \in \mathfrak{a})$, we write $X = A(x)$. If one assumes that $-X_0 \in C$, then one shows that $\mathcal{M}^+ \subset NA \cdot x_0$.

The spherical Laplace transform of an invariant Volterra kernel is defined by

$$\mathcal{L}F(\lambda) = \int_{\mathcal{M}^+} F(x, x_0) e^{(\rho - \lambda, A(x))} dx \quad (\lambda \in \mathfrak{a}_{\mathbb{C}}^*),$$

whenever the integral converges. If the spherical Laplace transform of the invariant Volterra kernels F_1 and F_2 are defined at λ, then the spherical Laplace transform of $F_1 \diamond F_2$ is defined at λ too, and

$$\mathcal{L}(F_1 \diamond F_2)(\lambda) = \mathcal{L}F_1(\lambda) \mathcal{L}F_2(\lambda).$$

By using the integration formula

$$\int_{\mathcal{M}^+} f(x) dx = c \int_{\mathfrak{a}_-} \int_H f(h \exp X \cdot x_0) dh\, \delta(X) dX,$$

where

$$\delta(X) = \prod_{\alpha\in-\Delta+} \big(\sinh\langle\alpha, X\rangle\big)^{m_\alpha},$$

the spherical Laplace transform can be written

$$\mathcal{L}F(\lambda) = c\int_{\mathfrak{a}_-} F(\exp X\cdot x_0, x_0)\varphi_\lambda(\exp X)\delta(X)dX,$$

where

$$\varphi_\lambda(x) = \int_H e^{\langle\rho-\lambda, A(h\cdot x)\rangle}dh.$$

This integral converges for $x \in S^0$, and if

$$\forall\alpha\in\Delta_1,\ \Re\langle\lambda+\rho,\alpha\rangle < 0.$$

Furthermore φ_λ is a spherical function. For the proof of the preceding results see [Faraut-Hilgert-Ólafsson,1994]. An inversion formula for the spherical Laplace transform has been established by G. Ólafsson [1997], and, in the rank one case, a Paley-Wiener theorem by N. Andersen and G. Ólafsson [1998].

3. Asymptotics, the c function. — Assume that

$$\forall\alpha\in\Delta_0^+,\ \Re\langle\lambda,\alpha\rangle > 0,$$
$$\forall\alpha\in\Delta_1,\ \Re\langle\lambda+\rho,\alpha\rangle < 0,$$

then, for $X \in \mathfrak{a}_-$,

$$\varphi_\lambda(\exp tX) \simeq c(\lambda)e^{t\langle\rho-\lambda, X\rangle} \quad (t\to\infty),$$

where

$$c(\lambda) = \int_{\bar{N}\cap NAH} e^{\langle\lambda+\rho, A(\bar{n})\rangle}d\bar{n}.$$

This function can be decomposed as

$$c(\lambda) = c_0(\lambda)c_1(\lambda),$$

where c_0 is the c-function of the Riemannian symmetric space G_0/K_0,

$$G_0 = \{g\in G \mid \mathrm{Ad}(g)X_0 = X_0\},\quad K_0 = G_0\cap K = H\cap K,$$

and

$$c_1(\lambda) = \int_D e^{\langle\rho+\lambda, A(\bar{n}_1)\rangle}d\bar{n}_1,$$

with $D = N_{-1} \cap NAH$. The set D is a bounded domain in N_{-1}. The function c_0 is given by the following formula due to Gindikin and Karpelevich ([1962]),

$$c_0(\lambda) = c \prod_{\alpha\in\Delta_0^+} B\Big(\frac{(\alpha|\lambda)}{(\alpha|\alpha)}, \frac{m_\alpha}{2}\Big).$$

We have conjectured that the function c_1 is given by

$$c_1(\lambda) = c \prod_{\alpha\in-\Delta_1} B\Big(\frac{(\alpha|\lambda)}{(\alpha|\alpha)} - \frac{m_\alpha}{2} + 1, \frac{m_\alpha}{2}\Big).$$

This formula has been proved in the case of rank one ordered symmetric spaces $SO_0(1,n)/SO_0(1,n-1)$, of symmetric spaces of Olshanski type, and of symmetric spaces of Cayley type ([Faraut,1995]). It has been proved also for $SL(n,\mathbb{R})/SO(p,q)$ by P. Graczyk ([1997]). His method applies for $SU^*(2n)/Sp(p,q)$ too, and for the ordered symmetric space associated with the exceptionnal pair $(\mathfrak{e}_{6(-26)}, \mathfrak{f}_{4(-20)})$.

A spherical function is a H-invariant solution of the differential system

$$D\varphi = \chi_D(\lambda), \varphi \quad D \in \mathbb{D}(G/H),$$

where $\mathbb{D}(G/H)$ is the algebra of invariant differential operators on G/H. The restriction to A_- of a solution of this system is a linear combination of the Harish-Chandra series $\Phi_{w\lambda}$ (see [Helgason,1984], Ch. IV, §5),

$$\Phi_\lambda(\exp X) = e^{\langle\rho-\lambda,X\rangle} \sum_{\mu\in\Lambda} \Gamma_\mu(\lambda)e^{\langle\mu,X\rangle},$$

where Λ is the positive lattice generated by the simple roots $\alpha_1, \dots, \alpha_\ell$,

$$\Lambda = \{\mu = n_1\alpha_1 + \cdots + n_\ell\alpha_\ell \mid n_j \in \mathbb{N}\}.$$

It has been proved by G. Ólafsson ([1997]) that

$$\varphi_\lambda(a) = c_1(\lambda) \sum_{w\in W_0} c_0(w\lambda)\Phi_{w\lambda}(a).$$

4. Exemples. — In some cases explicit formulas for the spherical functions φ_λ are known.

1) $SO_0(1,n)/SO_0(1,n-1)$

In this case

$$\sigma(g) = I_{n,1} g I_{n,1}, \quad \theta(g) = I_{1,n} g I_{1,n},$$

$$\mathfrak{a} = \mathfrak{p} \cap \mathfrak{q} = \mathbb{R} X_0, \text{ with } X_0 = -\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix},$$

and we put

$$a_t = \exp(-tX_0).$$

The integral representation of φ_λ reduces to

$$\begin{aligned} \varphi_\lambda(a_t) &= \int_0^\infty (\operatorname{ch} t + \operatorname{sh} t \operatorname{ch} \theta)^{-\lambda - \frac{n-1}{2}} (\operatorname{sh} \theta)^{n-2} d\theta \\ &= \gamma_n(\lambda) \frac{1}{(\operatorname{sh} t)^{\frac{n}{2}-1}} Q^{\frac{n}{2}-1}_{\lambda - \frac{1}{2}}(\operatorname{ch} t), \end{aligned}$$

where Q^μ_ν is a Legendre function of the second kind. In particular, for $n = 2$,

$$\varphi_\lambda(a_t) = Q_{\lambda - \frac{1}{2}}(\operatorname{ch} t),$$

and, for $n = 3$,

$$\varphi_\lambda(a_t) = \frac{1}{\lambda \operatorname{sh} t} e^{-\lambda t}.$$

In this case $m_\alpha = n - 1$, the domain D is the unit ball in $\mathbb{R}^{n-1}$, and

$$\begin{aligned} c_1(\lambda) &= c \int_D (1 - \|x\|^2)^{\lambda - \frac{n-1}{2}} dx \\ &= cB\Big(\lambda - \frac{n-1}{2} + 1, \frac{n-1}{2}\Big). \end{aligned}$$

2) *Symmetric spaces of Olshanski type*

In this case $\mathfrak{h}$ is a simple Hermitian Lie algebra, and $\mathfrak{g} = \mathfrak{h}_{\mathbb{C}}$ its complexification. The invariant causal structure of these spaces have been considered for the first time by G. Olshanski ([1981],[1982]).

$\mathfrak{g}$	$\mathfrak{sp}(n,\mathbb{C})$	$\mathfrak{sl}(n,\mathbb{C})$	$\mathfrak{so}(2n,\mathbb{C})$	$\mathfrak{so}(n+2,\mathbb{C})$	$\mathfrak{e}_6(\mathbb{C})$	$\mathfrak{e}_7(\mathbb{C})$
$\mathfrak{h}$	$\mathfrak{sp}(n,\mathbb{R})$	$\mathfrak{su}(p,q)$	$\mathfrak{so}^*(2n)$	$\mathfrak{so}(2,n)$	$\mathfrak{e}_{6(-14)}$	$\mathfrak{e}_{7(-25)}$

Let $\mathfrak{t}$ be a compact Cartan subalgebra of $\mathfrak{h}$, and $\mathfrak{a} = i\mathfrak{t}$. Then

$$\varphi_\lambda(\exp X) = \gamma \frac{1}{\prod_{\alpha \in \Delta^+} \langle \lambda, \alpha \rangle} \frac{\sum_{w \in W_0} \varepsilon(w) e^{-\langle w\lambda, X \rangle}}{\prod_{\alpha \in -\Delta^+} \operatorname{sh}\langle \alpha, X \rangle}.$$

In this case $m_\alpha = 2$, and, since

$$B\Big(z - \frac{m_\alpha}{2} + 1, \frac{m_\alpha}{2}\Big) = B(z,1) = \frac{1}{z},$$

we obtain

$$c_1(\lambda) = \frac{c}{\prod_{\alpha\in-\Delta_1}\langle\lambda,\alpha\rangle}.$$

3) $SU(n,n)/SL(n,\mathbb{C}) \times \mathbb{R}_+$

In that case J. Unterberger obtained explicit formulas *à la Berezin-Karpelevic* ([1998]). In fact Berezin and Karpelevic obtained explicit formulas in the case of the Riemannian symmetric space $SU(p,q)/S\big(U(p)\times U(q)\big)$ ([1958], see also [Takahashi,1977], and [Hoogenboom,1982]). Notice that the ordered symmetric space $SU(n,n)/SL(n,\mathbb{C}) \times \mathbb{R}_+$ and the Riemannian symmetric space $SU(n,n)/S\big(U(n)\times U(n)\big)$ are two real forms of the same complex symmetric space $SL(2n,\mathbb{C})/S\big(GL(n,\mathbb{C})\times GL(n,\mathbb{C})\big)$, and this explains the similarity of the formulas. Let $\mathfrak{a}$ be the space of the matrices

$$T = \frac{1}{2}\begin{pmatrix} & & & t_1 & & \\ & & & & \ddots & \\ & & & & & t_n \\ t_1 & & & & & \\ & \ddots & & & & \\ & & t_n & & & \end{pmatrix},$$

and let $\mathfrak{a}_-$ be defined by $0 < t_1 < \cdots < t_n$. Then, for $T \in \mathfrak{a}_-$,

$$\varphi_\lambda(\exp T) = \gamma\frac{\det\Big(Q_{\lambda_i - \frac{1}{2}}(\operatorname{ch} t_j)\Big)}{\prod_{i<j}(\lambda_j^2 - \lambda_i^2)\prod_{i<j}(\operatorname{ch} t_j - \operatorname{ch} t_i)}.$$

Let us mention that S. Aoki and S. Kato obtained formulas *à la Berezin-Karpelevic* for the spherical distributions on the symmetric spaces $U(p,q)/U(r)\times U(p-r,q)$, $U(n,n)/GL(n,\mathbb{C})$ ([1991]).

5. Asymptotics at the origin. — It is an open problem to determine the asymptotic behaviour of the spherical function $\varphi_\lambda(x)$ as x goes to the base point x_0. If $\mathcal{M}$ is the rank one ordered symmetric space $SO_0(1,n)/SO_0(1,n-1)$ one can show that $\varphi_\lambda(\exp -tX_0)$ has an asymptotic expansion as $t \to 0_+$: if n is odd,

$$\varphi_\lambda(\exp -tX_0) \simeq t^{-n+2}\sum_{k=0}^{\infty}\alpha_k(\lambda)t^{2k} + \sum_{k=0}^{\infty}\beta_k(\lambda)t^{2k},$$

and, if n is even,

$$\varphi_\lambda(\exp -tX_0) \simeq t^{-n+2}\sum_{k=0}^{\infty}\alpha_k(\lambda)t^{2k} + \log t\sum_{k=0}^{\infty}\beta_k(\lambda)t^{2k}.$$

There is a remarkable fact: the coefficient $\beta_0(\lambda) = \beta_0$ is a constant which does not depend on λ.

Let $\mathcal{M} = G/H$ be a symmetric space of Olshanski type, and put

$$J(X) = \prod_{\alpha\in-\Delta^+} \operatorname{sh}\langle\alpha, X\rangle, \quad \varpi(\lambda) = \prod_{\alpha\in\Delta^+}\langle\alpha, X\rangle.$$

It follows from the explicit formulas for φ_λ that

$$\varpi\Big(\frac{\partial}{\partial X}\Big)\big(J(X)\varphi_\lambda(\exp X)\big) = \beta_0,$$

where β_0 is a non zero constant which does not depend on λ.

For the general cas we don't know whether there is an asymptotic expansion, and we wonder whether there is in general such a phenomenon.

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