表現論シンポジウム講演集, 1998
pp.1-17

Bernstein Degree and Associated Cycle of Harish-Chandra Modules *

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Abstract

Let $G = Sp(2n, \mathbb{R})$ be symplectic group and $\widetilde{G} = Mp(2n, \mathbb{R})$ its metaplectic double cover. By the theory of dual pair, irreducible unitary highest weight modules of $\widetilde{G}$ are parametrized by irreducible finite dimensional representations of orthogonal group $O(m)$ for various m (Howe correspondence, or theta correspondence). For $\sigma \in \widehat{O}(m)$, let us denote the corresponding unitary highest weight module of $\widetilde{G}$ by $L(\sigma)$. In this article, we give Bernstein degree and associated cycle of $L(\sigma)$ ($\sigma \in \widehat{O}(m)$). The similar method can be applicable to other non-compact simple groups of hermitian type.

1 Definition of Bernstein degree

Let G be a (linear) connected reductive group over $\mathbb{R}$. Take a maximal compact subgroup $K \subset G$ and denote a Cartan decomposition associated to K by $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$. We denote the complexification of the decomposition by $\mathfrak{g}_{\mathbb{C}} = \mathfrak{k}_{\mathbb{C}} + \mathfrak{p}_{\mathbb{C}}$.

1.1 Gelfand-Kirillov dimension and Bernstein degree

Let $\mathcal{H}$ be a Harish-Chandra $(\mathfrak{g}_{\mathbb{C}}, K)$-module (finitely generated), and take a finite dimensional K-stable generating subspace $\mathcal{H}_0 \subset \mathcal{H}$. The generating space $\mathcal{H}_0$ gives rise to a natural filtration $\{\mathcal{H}_n\}_{n\geq 0}$ of $\mathcal{H}$ as follows. Let us denote by $U_n(\mathfrak{g}_{\mathbb{C}})$ the standard filtration of the enveloping algebra $U(\mathfrak{g}_{\mathbb{C}})$ of $\mathfrak{g}_{\mathbb{C}}$. Then we put

$$\mathcal{H}_n = U_n(\mathfrak{g}_{\mathbb{C}})\mathcal{H}_0.$$

*表現論シンポジウム講演集 (休暇村加太 (和歌山市) 1998/11/17 - 11/20).

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Let us denote the associated graded module by $M = \oplus_n M_n$ tentatively, i.e.,

$$M = \operatorname{gr} \mathcal{H} = \bigoplus_{n \geq 0} \mathcal{H}_n / \mathcal{H}_{n-1}, \qquad M_n = \mathcal{H}_n / \mathcal{H}_{n-1}.$$

M is a graded $S(\mathfrak{g}_{\mathbb{C}}) = \operatorname{gr} U(\mathfrak{g}_{\mathbb{C}})$ module, and we write $A = S(\mathfrak{g}_{\mathbb{C}})$ for the time being.

As usual, we denote the Poincare series of $M = \operatorname{gr} \mathcal{H}$ by $P(M; t)$. Then, it is well-known by the commutative ring theory (see [16, §13] or [7] for example) that there exist a polynomial $Q(t)$ and a non-negative integer d such that

$$P(M; t) = \sum_{n \geq 0} \dim(M_n) t^n = \frac{Q(t)}{(1-t)^d}, \qquad Q(1) \neq 0.$$

It can be seen that $Q(1)$ becomes a positive integer, and d and $Q(1)$ do not depend on the choice of the generating subspace $\mathcal{H}_0$.

Definition 1.1 The integer d is denoted by $\operatorname{Dim} \mathcal{H}$ and called *Gelfand-Kirillov dimension* of $\mathcal{H}$. We call the positive integer $Q(1)$ *Bernstein degree* of $\mathcal{H}$ (it is often called *multiplicity* also). Denote by $\operatorname{Deg} \mathcal{H}$ the Bernstein degree of $\mathcal{H}$.

Remark 1.2 For sufficiently large n, $\dim M_n$ becomes a polynomial in n. It is written as

$$\dim M_n = \frac{Q(1)}{(d-1)!} n^{d-1} + (\text{lower terms of } n).$$

Therefore, we also conclude that $\dim \mathcal{H}_n$ is a polynomial in n for sufficiently large n and is given by

$$\dim \mathcal{H}_n = \sum_{k=0}^{n} \dim M_k = \frac{Q(1)}{d!} n^d + (\text{lower terms of } n).$$

Remark 1.3 In this article, we call $\operatorname{Deg} \mathcal{H}$ Bernstein degree because there are so many "multiplicities" in the representation theory. Moreover, at least in its spirit, Bernstein degree is a *characteristic* of the associated cycle of $\mathcal{H}$ (see Proposition 1.6 below). So we prefer the name to multiplicity.

1.2 Associated cycle

In the theory of commutative algebra, notion of associated variety is very important. We review briefly the results on the associated varieties of Harish-Chandra modules here, and clarify the relation between associated varieties, associated cycles, Gelfand-Kirillov dimensions and Bernstein degrees, nilpotent orbits, etc., etc. Our main references here are [24], [16], [7].

Put

$$\operatorname{Ann}(M) = \{a \in A \mid a \cdot M = 0\},$$

and

$$\mathcal{V}(\mathcal{H}) = \{x \in \mathfrak{g}_{\mathbb{C}} \mid a(x) = 0 \ (\forall a \in \mathrm{Ann}\,(M)\}.$$

Then $\mathcal{V}(\mathcal{H})$ does not depend on the choice of filtrations, and we call it the *associated variety* of $\mathcal{H}$. It is well-known that $\mathcal{V}(\mathcal{H})$ is a union of nilpotent $K_{\mathbb{C}}$-orbits in $\mathfrak{p}_{\mathbb{C}}$:

$$\mathcal{V}(\mathcal{H}) \subset \mathcal{N}_{\mathfrak{p}_{\mathbb{C}}}/\,\mathrm{Ad}\,K_{\mathbb{C}}.$$

Moreover, summarizing the results of many contributers (Joseph, Vogan, Borho-Brylinski, ...), we have

Theorem 1.4 *If $\mathcal{H}$ is an irreducible Harish-Chandra $(\mathfrak{g}_{\mathbb{C}}, K)$-module, then*

(1) *There exist nilpotent $K_{\mathbb{C}}$ orbits $\{\mathcal{O}_i\} \subset \mathcal{N}_{\mathfrak{p}_{\mathbb{C}}}/\,\mathrm{Ad}\,K_{\mathbb{C}}$ with dimension equal to* $\mathrm{Dim}\,\mathcal{H}$ *such that*

$$\mathcal{V}(\mathcal{H}) = \bigcup_{i=1}^{l} \overline{\mathcal{O}_i}.$$

(2) *Let $I = I_{\mathcal{H}} \subset U(\mathfrak{g}_{\mathbb{C}})$ be an associated primitive ideal of $\mathcal{H}$. Then $\mathcal{V}(U(\mathfrak{g}_{\mathbb{C}})/I)$ is the closure of a single nilpotent $G_{\mathbb{C}}$-orbit $\mathcal{O}_{\mathcal{H}}$, and for any i, the $G_{\mathbb{C}}$ orbit through $\mathcal{O}_i$ coincides with $\mathcal{O}_{\mathcal{H}}$. In fact, $\mathcal{O}_{\mathcal{H}} \cap \mathfrak{p}_{\mathbb{C}}$ decomposes into finite union of equidimensional nilpotent $K_{\mathbb{C}}$-orbits, and $\{\mathcal{O}_i\}$ is a subset of its components:*

$$\mathcal{O}_{\mathcal{H}} \cap \mathfrak{p}_{\mathbb{C}} \supset \mathcal{O}_1, \mathcal{O}_2, \cdots, \mathcal{O}_l.$$

Let us define the associated cycle of $\mathcal{H}$ after Vogan [24], which is a refinement of $\mathcal{V}(\mathcal{H})$. Since $\overline{\mathcal{O}_i}$ is an irreducible closed subvariety of $\mathfrak{g}_{\mathbb{C}}$, there is a prime ideals $\{\mathfrak{P}_i\}_i$ in A such that $\overline{\mathcal{O}_i} = V(\mathfrak{P}_i)$. Obviously we have

$$\sqrt{\mathrm{Ann}\,(M)} = \bigcap_{i=1}^{l} \mathfrak{P}_i \quad : \text{prime decomposition of } \sqrt{\mathrm{Ann}\,(M)}.$$

Definition 1.5 The *associated cycle* of $\mathcal{H}$ is defined to be

$$\mathcal{AC}\,(\mathcal{H}) = \sum_{i=1}^{l} m(\mathfrak{P}_i, M)\mathfrak{P}_i,$$

where $m(\mathfrak{P}, M)$ is the *multiplicity* of an associated prime ideal of M given by

$$m(\mathfrak{P}, M) = \mathrm{length}_{A_{\mathfrak{P}}}(M_{\mathfrak{P}}).$$

$\mathcal{AC}\,(\mathcal{H})$ does not depend on the choice of the filtration.

Proposition 1.6 (Taniguchi [23, Lemma 6.2.1]) *We have the following equality:*

$$\operatorname{Deg}\mathcal{H} = \sum_{i=1}^{l} m(\mathfrak{P}_i, M) \deg \overline{\mathcal{O}_i},$$

where $\deg \overline{\mathcal{O}_i}$ *means the degree of the projective variety* $\overline{\mathcal{O}_i}$ *in the sense of algebraic geometry.*

PROOF. By [16, Th. 6.4], there exists a finite filtration M^j of M, such that $M^j/M^{j-1} \simeq A/\mathfrak{Q}_j$ for some associated prime $\mathfrak{Q}_j$ of M. Clearly we have

$$P(M,t) = \sum_j P(M^j/M^{j-1}, t).$$

The multiplicity $m(\mathfrak{P}_i, M)$ is the number of the indices j for which $\mathfrak{Q}_j = \mathfrak{P}_i$ holds (see the arguments in [24, p. 322]; also see [16, Th. 14.7]). If $\mathfrak{Q}_j$ is not minimal among the associated prime ideals, it does not affect the coefficient of the pole at $t = 1$ because $\dim A/\mathfrak{Q}_j > d$. From this, it is clear that the Bernstein degree of the Poincare series of M is equal to that of

$$\sum_{i=1}^{l} m(\mathfrak{P}_i, M) P(\overline{\mathcal{O}_i}, t),$$

and this implies the proposition. (There should be some shift of grading for each term, but it does not affect the degree.) Q.E.D.

Remark 1.7 By Vogan [24, Def. 2.12 & Th. 2.13], it is known that the multiplicity $m(\mathfrak{P}_i, M)$ has a representation theoretic meaning. More precisely, take $\lambda \in \mathcal{O}_i$ and put $K_{\mathbb{C}}(\lambda) =$ (fixed subgroup of λ). Then $m(\mathfrak{P}_i, M)$ can be interpreted as the dimension of certain finite dimensional representation of $K_{\mathbb{C}}(\lambda)$.

2 Known results for Bernstein degree

In this section, we summarize known results and examples on Bernstein degrees.

2.1 Goldie rank polynomials and character polynomials

Let $\mu \in \mathfrak{h}_{\mathbb{C}}^*$ be an infinitesimal character of an irreducible Harish-Chandra module $\mathcal{H}$, and P be the weight lattice in $\mathfrak{h}_{\mathbb{C}}^*$. Take Θ a coherent family (of global characters or virtual representations) on $\mu + P$ such that

$$\begin{cases} \Theta(\mu) = \mathcal{H}, \\ \Theta(\lambda) \ (\lambda : \text{dominant regular}) \text{ is an irreducible character.} \end{cases}$$

If μ is singular, such a family is not unique but it exists. Let us consider the following map

$$(\mu + P)^{++} \ni \lambda \longmapsto \operatorname{Deg}\Theta(\lambda) =: c(\lambda),$$

where $(\mu + P)^{++}$ denotes the set of dominant regular elements.

Theorem 2.1 (Vogan, King, Joseph) *Let $\mathfrak{h} \subset \mathfrak{g}$ be an Iwasawa Cartan subalgebra (maximally split one).*

(1) *$c(\lambda)$ extends to a homogeneous harmonic polynomial on $\mathfrak{h}_{\mathbb{C}}^*$ of degree $|\Delta^+| - d$ ($d = \operatorname{Dim}\mathcal{H}$).*

(2) *$c(\lambda)$ is proportional to the Goldie rank polynomial determined by the coherent family Θ.*

(3) *Let $h \in \mathfrak{h}$ be dominant regular and put*

$$c_\Theta(\lambda) := \lim_{t \downarrow 0} t^d \Theta(\lambda)(\exp th) \qquad (\textit{the limit exists}).$$

Then $c_\Theta(\lambda)$ and $c(\lambda)$ are proportional. $c_\Theta(h, \lambda)$ is called character polynomial.

PROOF. See [25], [13, Th. 1.1 & Th. 1.2] and [11] for proof. Q.E.D.

It is well-known that Goldie rank polynomials generates particular representations of Weyl group which correspond to nilpotent orbits by Springer correspondence, and in fact, they form bases of such representations. Since Springer correspondence is well-understood in a sense, the polynomial $c(\lambda)$ (up to constant multiple) is not so difficult to calculate out in explicit cases. See for example [14] for discrete series of real rank one group, and [17] for complex semisimple Lie groups.

However, it is worth noting that to determine $\operatorname{Deg}\mathcal{H}$ is another thing. It requires finer investigation than the determination of the polynomial $c(\lambda)$ *up to constant.* If once $\operatorname{Deg}\mathcal{H}$ and the polynomial are known, we know Bernstein degree of any other member of the coherent family.

Example 2.2 (Finite dimensional representation) If $\mathcal{H}$ is a finite dimensional irreducible representation, then $\operatorname{Deg}\mathcal{H} = \dim\mathcal{H}$. This is immediate from the definition. In this case, $c(\lambda)$ is *Weyl's dimension function*:

$$c(\lambda) = \prod_{\alpha \in \Delta^+} \frac{\langle \lambda, \alpha \rangle}{\langle \rho, \alpha \rangle}.$$

Note that the highest weight of $\mathcal{H}$ is $\lambda - \rho$, since we denote the infinitesimal character by λ.

Example 2.3 (generalized Verma module) Let $\mathfrak{q}_{\mathbb{C}} = \mathfrak{l}_{\mathbb{C}} \oplus \mathfrak{n}_{\mathbb{C}}$ be a Levi decomposition of a parabolic subalgebra $\mathfrak{q}_{\mathbb{C}} \subset \mathfrak{g}_{\mathbb{C}}$. Take an "admissible" representation π of $\mathfrak{l}_{\mathbb{C}}$ and extend it to $\mathfrak{q}_{\mathbb{C}}$ putting the action of the nilradical $\mathfrak{n}_{\mathbb{C}}$ trivial. Then

$$\mathcal{H} = U(\mathfrak{g}_{\mathbb{C}}) \otimes_{\mathfrak{q}_{\mathbb{C}}} \pi$$

is called *generalized Verma module.* In general, $\mathcal{H}$ is not a Harish-Chandra $(\mathfrak{g}_{\mathbb{C}}, K)$-module, but we can still define Bernstein degree of $\mathcal{H}$. In this case, we get ([25, Lemma 2.3])

$$\operatorname{Deg} \mathcal{H} = \operatorname{Deg} \pi.$$

Let (G, K) be an irreducible Hermitian symmetric pair. Then G has holomorphic discrete series representations, whose properties are very well-known. Among all, the space of K-finite vectors of holomorphic discrete series is a generalized Verma module induced from a maximal parabolic subalgebra determined by the simple non-compact root (we take Borel-de Siebenthal positive system, i.e., positive system whose simple system contains the unique non-compact root). Such a maximal parabolic subalgebra can be written as

$$\mathfrak{q}_{\mathbb{C}} = \mathfrak{k}_{\mathbb{C}} \oplus \mathfrak{p}_{\mathbb{C}}^{\pm},$$

where $\mathfrak{p}_{\mathbb{C}} = \mathfrak{p}_{\mathbb{C}}^{+} \oplus \mathfrak{p}_{\mathbb{C}}^{-}$ is K-stable decomposition. Take an irreducible finite dimensional representation τ_Λ of K with highest weight Λ. Then

$$\mathcal{H} = U(\mathfrak{g}_{\mathbb{C}}) \otimes_{\mathfrak{q}_{\mathbb{C}}} \tau_\Lambda$$

is a holomorphic discrete series whose minimal K-type is τ_Λ. By the above comment we get Bernstein degree of the holomorphic discrete series $\mathcal{H}$ as

$$\operatorname{Deg} \mathcal{H} = \dim \tau_\Lambda.$$

In this case, $c(\lambda)$ is given by

$$c(\lambda) = \prod_{\alpha \in \Delta_c^+} \frac{\langle \lambda + \rho_n, \alpha \rangle}{\langle \rho_c, \alpha \rangle} \qquad (\lambda = \Lambda + \rho_c - \rho_n \ : \ \text{infinitesimal character}).$$

Remark 2.4 Let Q be a parabolic subgroup of G. It is well-known that there exists a non-degenerate pairing between a generalized Verma module induced from $\mathfrak{q}_{\mathbb{C}}$ and a principal series representation induced from Q. However, we cannot say much on the Bernstein degree of principal series representations as far as we know.

2.2 Large representations

If the Gelfand-Kirillov dimension of $\mathcal{H}$ is the largest possible, we call $\mathcal{H}$ *large* ([25]). Put $r = \#(\text{positive restricted roots counted with multiplicity}) = \dim \mathfrak{n}_m$. Then $\mathcal{H}$ is large iff $\operatorname{Dim} \mathcal{H} = r$. In this case, by result of Matumoto, Bernstein degree coincides with the dimension of Whittaker functions of $\mathcal{H}$. More precisely, we have

Theorem 2.5 (Matumoto) *Let $\mathfrak{n}_m$ be a maximal nilpotent subalgebra of $\mathfrak{g}$ (Iwasawa's $\mathfrak{n}$), and ψ be an admissible character of $\mathfrak{n}_m$. If* $\operatorname{Dim}\mathcal{H} = \dim \mathfrak{n}_m$, *i.e., $\mathcal{H}$ is large, then*

$$\dim \mathsf{Wh}^*_{\mathfrak{n}_m,\psi}(\mathcal{H}) = \operatorname{Deg}(\mathcal{H}).$$

PROOF. See [18, Cor. 2.2.2]. Q.E.D.

Theorem 2.6 *Let G be a quasi-split Lie group and $\mathcal{H}$ is a principal series representation induced from a minimal parabolic subgroup. Then* $\operatorname{Deg}\mathcal{H} = \#W(\mathfrak{g},\mathfrak{a})$, *where $W(\mathfrak{g},\mathfrak{a})$ is the little Weyl group.*

PROOF. By Kostant [15, Th. 5.5], we know $\dim \mathsf{Wh}^*_{\mathfrak{n}_m,\psi}(\mathcal{H})$ is equal to $\#W(\mathfrak{g},\mathfrak{a})$. Apply Matumoto's theorem. Q.E.D.

Remark 2.7 If G is quasi-split, then $\dim \mathfrak{n}_m$ is equal to the number of positive roots. So, in the quasi-split case, character polynomial is constant for large representations.

For large discrete series representations, there are explicit calculations of the Whittaker functions of low rank groups. From these calculations we know Bernstein degree for large representations in case of low rank groups.

group	representation	$\operatorname{Deg}\mathcal{H}$	contributers
$Sp(2,\mathbb{R})$	DS	4	Oda [21]
$SU(2,2)$	DS	4	Yamashita [26], Hayata-Oda [6]
	gen. PS	4	Hayata [5]
$SU(n,1)$	DS	$(*)$	Taniguchi [23]
$Spin(n,1)$	DS	$(**)$	Taniguchi [23]

$(*)$ and $(**)$: These values are given in Theorem A (1) and Theorem B (1) in [23] respectively.

2.3 Minimal representations

In this subsection, we will give Bernstein degrees of so-called minimal representations. Here we only consider *non-Hermitian* symmetric space G/K, though the arguments below equally works well for general situations.

If G/K is non-Hermitian, G has minimal representations if and only if G/K is in the following list.

- Classical case : $SO(p,q)/SO(p)\times SO(q)$ where $p \geq q \geq 3, p+q \in 2\mathbb{Z}$ or $p \in 2\mathbb{Z}, q = 3$.

• Exceptional case : The following 8 cases.

$$\begin{array}{ll} F_{4,4}/Sp(3)\times SU(2) & G_2/SO(4) \\ E_{6,4}/SU(2)\times SU(6) & E_{6,6}/Sp(4) \\ E_{7,4}/Spin(12)\times SU(2) & E_{7,7}/SU(8) \\ E_{8,4}/E_7\times SU(2) & E_{8,8}/Spin(16) \end{array}$$

Take the minimal nilpotent $G_{\mathbb{C}}$-orbit $\mathcal{O}_{\min} \subset \mathfrak{g}_{\mathbb{C}}$. Then in this case $\mathcal{O}_{\min} \cap \mathfrak{p}_{\mathbb{C}} =: Y$ is a single nilpotent $K_{\mathbb{C}}$-orbit.

Theorem 2.8 (Vogan) *Let $\pi_{\min}$ be the minimal representation of G. Then there exists some weight ν such that*

$$\pi_{\min}|_K \simeq \bigoplus_{m\geq 0} \tau_{m\psi+\nu},$$

where ψ is the highest weight of $\mathfrak{p}_{\mathbb{C}}$ (= the highest root), and $\tau_{m\psi+\nu}$ is the irreducible representation of K with highest weight $m\psi+\nu$.

Remark 2.9 The weight ν is the highest weight ν of the minimal K-type of $\pi_{\min}$ and can be explicitly given. But we do not need it here.

Put $\Delta_c^+(\psi) = \{\alpha \in \Delta_c^+ \mid \langle \psi, \alpha\rangle \neq 0\}$, where Δ_c^+ denotes the totality of positive compact roots. Note that $\langle \psi, \alpha\rangle \neq 0 \iff \langle \psi, \alpha\rangle = 1$ for $\alpha \in \Delta_c^+$.

Proposition 2.10 *With the above notation, we have*

$$\begin{aligned} \operatorname{Dim}(\pi_{\min}) &= \#\Delta_c^+(\psi)+1 = \dim_{\mathbb{C}} \mathcal{O}_{\min}/2 = \dim_{\mathbb{C}} Y, \\ \operatorname{Deg}(\pi_{\min}) &= \deg \overline{Y} = (\#\Delta_c^+(\psi))! \prod_{\alpha\in\Delta_c^+(\psi)} \frac{\langle \psi,\alpha\rangle}{\langle \rho_c,\alpha\rangle}. \end{aligned}$$

PROOF.

$$\begin{aligned} \dim \tau_{m\psi+\nu} &= \prod_{\alpha\in\Delta_c^+} \frac{\langle m\psi+\nu,\alpha\rangle}{\langle \rho_c,\alpha\rangle} \quad \text{(by Weyl's dimension formula)} \\ &= m^{\#\Delta_c^+(\psi)} \prod_{\alpha\in\Delta_c^+(\psi)} \frac{\langle \psi,\alpha\rangle}{\langle \rho_c,\alpha\rangle} + \text{(lower terms of } m) \\ &= \frac{\operatorname{Deg}(\pi_{\min})}{(d-1)!} m^{d-1} + \text{(lower terms of } m) \quad (d = \operatorname{Dim}(\pi_{\min})) \end{aligned}$$

Q.E.D.

3 Singular unitary highest weight modules of $G = Sp(2n, \mathbb{R})$

3.1 Weil representation and its tensor product

Let ω be Weil representation (or oscillator representation) of metaplectic double cover of $G = Sp(2n, \mathbb{R})$. Then the space of K-finite vectors of ω can be realized as a polynomial ring $\mathbb{C}[x_1, x_2, \cdots, x_n]$ in Fock model.

Action of a maximal compact subgroup $K \simeq U(n)$ is given by the symmetric tensor products of its defining representation $\mathbb{C}^n$ with shift by $\det^{1/2}$. Action of $\mathfrak{p}_{\mathbb{C}}^-$ is realized as second degree constant coefficient differential operators

$$\frac{\partial^2}{\partial x_i \partial x_j} \qquad (1 \leq i, j \leq n),$$

and the action of $\mathfrak{p}_{\mathbb{C}}^+$ is given by the multiplication by second degree polynomials

$$x_i x_j \qquad (1 \leq i, j \leq n).$$

Since these operators change the degrees by even integers, clearly ω is not irreducible. In fact it is the direct sum of two irreducible constituents:

$$\omega = \omega^+ \oplus \omega^- = (\text{even functions}) \oplus (\text{odd functions}).$$

These are minimal representations of the double cover of G (by an abuse of terminology, we also say they are minimal representations of G), and they are singular unitary highest weight modules (= SUHWMs). For full details of this, we refer to [12], [9] and [10].

Consider the m-fold tensor product of ω :

$$\omega^{\otimes m} = \mathbb{C}[x_i]^{\otimes m} \simeq \mathbb{C}[M_{n,m}].$$

From the last expression, one can see that there is an action of $O(m)$ by the right translation, and moreover, the action commutes with that of metaplectic group $\widetilde{G}$. In fact, they are commutants to each other in the larger Lie group $\widetilde{Sp}(2nm, \mathbb{R})$.

By the theory of reductive dual pair (cf. [9]), we get

Theorem 3.1 (Kashiwara-Vergne) *As $\widetilde{G} \times O(m)$-module, we have the multiplicity free decomposition*

$$\omega^{\otimes m} \simeq \bigoplus_{\sigma \in \widehat{O}(m)} L(\sigma) \otimes \sigma \qquad (\text{ possibly } L(\sigma) = \{0\}),$$

where $L(\sigma) \neq (0)$ is an irreducible UHWM (= unitary highest weight module) of $\widetilde{G}$.

(1) *If $m < n$, $\forall L(\sigma)$ is singular.*

(2) *If $m > 2n$, $\forall L(\sigma)$ is a holomorphic discrete series representation and hence regular.*

(3) *For $n \leq m \leq 2n$, both regular and singular UHWMs can occur in the decomposition.*

PROOF. See [12]. Q.E.D.

Due to many contributers (Kashiwara-Vergne, Parthasarathy, Jakobsen, Enright-Howe-Wallach, ...), we know these $\{L(\sigma)\}$ exhaust all the UHWMs.

Theorem 3.2 $\{L(\sigma) \neq (0) \mid \sigma \in \widehat{O}(m), m \geq 1\}$ *exhausts all the UHWMs of* $\widetilde{G}$ *except the trivial representation.*

PROOF. See [1] for example. Q.E.D.

Since Bernstein degree of holomorphic discrete series is well-known as we explained in Example 2.3, we are to consider Bernstein degrees of SUHWM $L(\sigma)$ for $\sigma \in \widehat{O}(m)$ $(m < n)$.

3.2 SUHWM with scalar K-type

First we consider $L(\sigma)$ for $\sigma =$ (trivial representation) $\in \widehat{O}(m)$. We denote by $\mathbf{1}_m \in \widehat{O}(m)$ the trivial representation and write $L(\mathbf{1}_m)$ for $\sigma = \mathbf{1}_m \in \widehat{O}(m)$. By Theorem 3.1 above, we know

$$L(\mathbf{1}_m)|_{\widetilde{K}} \simeq \mathbb{C}[M_{n,m}]^{O(m)} \otimes \det^{m/2} \quad (K = U(n)).$$

Theorem 3.3 (1) *As* $GL(n, \mathbb{C}) = K_{\mathbb{C}}$*-module, we have*

$$\mathbb{C}[M_{n,m}]^{O(m)} \simeq \bigoplus_{l(\lambda) \leq m} \rho_{2\lambda},$$

where $\rho_\mu \in \widehat{GL(n, \mathbb{C})}$ *is the irreducible representation with highest weight* μ.

(2) *Let* $\mathrm{Sym}_n(m) = \{X \in M_n(\mathbb{C}) \mid {}^tX = X, \mathrm{rank}\, X \leq m\}$. *This is called determinantal variety. Then there is a* GL_n*-equivariant isomorphism*

$$\mathbb{C}[\mathrm{Sym}_n(m)] \simeq \mathbb{C}[M_{n,m}]^{O(m)}$$

by the pull back of the following map of the base spaces:

$$\begin{array}{ccc} M_{n,m} & \longrightarrow & \mathrm{Sym}_n(m) \\ \Cup & & \Cup \\ Y & \longmapsto & Y{}^tY \end{array}$$

PROOF. Both facts are well-known in invariant theory. See [8] for example. Q.E.D.

The space $\mathfrak{p}_{\mathbb{C}}^+$ is naturally identified with the space of all the $n \times n$ symmetric matrices Sym_n. With this identification, there is a $K_{\mathbb{C}}$-equivariant embedding $\mathrm{Sym}_n(m) \hookrightarrow \mathfrak{p}_{\mathbb{C}}^+$, and the image is the closure of a single $K_{\mathbb{C}}$-orbit

$$\mathcal{O}_m = \mathrm{Sym}_n^\circ(m) = \{X \in \mathrm{Sym}_n \mid \mathrm{rank}\, X = m\}.$$

So Theorem 3.3 (2) says that, as $K_{\mathbb{C}}$-modules,

$$\mathbb{C}[M_{n,m}]^{O(m)} \simeq \mathbb{C}[\mathrm{Sym}_n(m)] = \mathcal{R}(\overline{\mathcal{O}_m}),$$

where $\mathcal{R}(Y)$ denotes the ring of (global) regular functions on a variety Y.

Corollary 3.4

$$\begin{aligned}
\mathrm{Dim}\, L(\mathbf{1}_m) &= \dim_{\mathbb{C}} \mathrm{Sym}_n(m) = nm - \frac{1}{2}m(m-1) \\
\mathrm{Deg}\, L(\mathbf{1}_m) &= \deg \mathrm{Sym}_n(m) = \deg \overline{\mathcal{O}_m} \\
\mathcal{AC}\, L(\mathbf{1}_m) &= \overline{\mathcal{O}_m}
\end{aligned}$$

Let $P(t)$ be the Poincare series of $\mathbb{C}[M_{n,m}]^{O(m)}$:

$$P(t) = \sum_{k\geq 0} \dim \left(\mathbb{C}[M_{n,m}]_k^{O(m)}\right) t^k = \sum_{l(\lambda)\leq m} \dim \rho_{2\lambda} t^{2|\lambda|}.$$

On the other hand, by Weyl's dimension formula, we have

$$P(t) = \sum_{l(\lambda)\leq m} \prod_{\alpha\in\Delta_c^+} \frac{\langle 2\lambda+\rho_c, \alpha\rangle}{\langle \rho_c, \alpha\rangle} t^{2|\lambda|}.$$

From this formula, we get $\mathrm{Deg}\, L(\mathbf{1}_m)$ as

$$\begin{aligned}
\mathrm{Deg}\, L(\mathbf{1}_m) &= \lim_{k\to\infty} \frac{d!}{k^d} \sum_{l(\lambda)\leq m, |\lambda|\leq k} \prod_{\alpha\in\Delta_c^+} \frac{\langle 2\lambda+\rho_c, \alpha\rangle}{\langle \rho_c, \alpha\rangle} \\
&= \frac{2^{d-m} d!}{m! \prod_{i=1}^m (n-i)!} \int_{\Omega_m} (x_1 x_2 \cdots x_m)^{n-m} \prod_{1\leq i<j\leq m} |x_i - x_j| dx_1 dx_2 \cdots dx_m
\end{aligned}$$

where the integral is taken over the region

$$\Omega_m = \{(x_1, x_2, \cdots, x_m) \in \mathbb{R}^m \mid x_i \geq 0, \sum_{i=1}^m x_i \leq 1\}. \tag{3.1}$$

Let us denote the integral as

$$I(r, m) = \int_{\Omega_m} (x_1 x_2 \cdots x_m)^r \prod_{1\leq i<j\leq m} |x_i - x_j| dx_1 dx_2 \cdots dx_m.$$

Theorem 3.5 *Bernstein degree of the singular unitary highest weight module $L(\mathbf{1}_m)$ corresponding to $\mathbf{1}_m \in \widehat{O}(m)$ is given by*

$$\operatorname{Deg} L(\mathbf{1}_m) = \frac{2^{d-m} d!}{m! \prod_{i=1}^{m} (n-i)!} I(n-m, m).$$

We shall give an exact formula of the integral and get the explicit value of Bernstein degree.

Theorem 3.6 (1) *The exact value of the integral $I(k,r)$ $(k > -1)$ is given by*

$$\begin{aligned} I(k,r) &= \int_{x_i \geq 0, \sum_{i=1}^{r} x_i \leq 1} (x_1 x_2 \cdots x_r)^k \prod_{1 \leq i < j \leq r} |x_i - x_j| dx \\ &= \frac{1}{2^{r(r-1)/2} \Gamma(rk + r(r+1)/2 + 1)} \prod_{j=1}^{r} \frac{\Gamma(j+1) \Gamma\left(k + \frac{j+1}{2}\right)}{\Gamma\left(\frac{j+1}{2}\right)} \end{aligned}$$

(2) *The Bernstein degree of $L(\mathbf{1}_m)$ is given by*

$$\operatorname{Deg} L(\mathbf{1}_m) = \deg \operatorname{Sym}_n(m) = \deg \overline{\mathcal{O}_m} = \prod_{l=0}^{m-1} \frac{l!}{l!!} \frac{(2n-2m+l)!!}{(n-m+l)!}.$$

PROOF. The statement (1) is proved in [19]. Then (2) immediately follows from Corollary 3.4 and Theorem 3.5. Q.E.D.

Note that the degree of determinantal varieties were already obtained by Giambelli in 1900's, and known as *Giambelli-Thom-Porteous formula*. See [2, Example 14.4.14] and [4] (for original reference, see [3], though we have not had a chance to glance over it). Our theorem above gives a new proof of the formula (in case of skew symmetric matrices).

4 Branching rule and Bernstein degree

We have determined Bernstein degree of $L(\mathbf{1}_m)$ $(1 \leq m \leq n)$. To obtain $\operatorname{Deg} L(\sigma)$ we use branching rule of the restriction $GL(m, \mathbb{C}) \downarrow O(m, \mathbb{C})$. However, to get explicit formula, we need informations on asymptotic behavior of the branching coefficients. Fortunately, we can use a formula by F. Sato [22] to get Bernstein degree $\operatorname{Deg} L(\sigma)$.

4.1 Sato's result

In this subsection, we collect Sato's results which we will need later. Note that we do not state all of his results but only for the case $(GL(m, \mathbb{C}), SO(m, \mathbb{C}))$.

We will mainly follow Sato's notation in [22]. In particular, λ is a highest weight of a general linear group $GL(m, \mathbb{C})$ and μ is that of special orthogonal group $SO(m, \mathbb{C})$. The multiplicity of $\sigma_\mu^{SO(m,\mathbb{C})}$ in $\tau_\lambda^{GL(m,\mathbb{C})}|_{SO(m,\mathbb{C})}$ is denoted by $m(\lambda, \mu)$.

Theorem 4.1 *If* $\min\{\lambda_i - \lambda_{i+1} \mid 1 \leq i \leq m-1\}$ *is sufficiently large (depending on* μ*), the branching coefficient* $m(\lambda, \mu)$ *does not depend on individual* λ *but only depends on the class* $[\lambda]$ *which is determined by* $\lambda_i \mod 2$ $(1 \leq i \leq m)$.

We denote the branching coefficient for sufficiently large λ by $m([\lambda], \mu)$ and call it *stable branching coefficient.* The individual stable branching coefficient $m([\lambda], \mu)$ has somewhat complicated expression, but the summation of them has very simple interpretation in most cases. Let us explain it.

Theorem 4.2 (Sato) *If* $\lambda_i + \lambda'_i \equiv 1 \mod 2$ *then their stable branching coefficients are the same :* $m([\lambda], \mu) = m([\lambda'], \mu)$. *We have*

$$\sum_{[\lambda] \in \mathbb{Z}_2^m} m([\lambda], \mu) = 2 \dim \sigma_\mu^{SO(m,\mathbb{C})}.$$

Remark 4.3 Note that our terminology slightly differs from Sato's in this case. In fact, in his notation, $[\lambda] = [\lambda']$ and the multiplication of 2 in the right hand side does not appear. But we prefer the existence of 2.

Anyway, the multiplicity $m(\tau_\lambda^{GL(m,\mathbb{C})}, \sigma_\mu^{SO(m,\mathbb{C})})$ depends only on the class $[\lambda]$ if $\min\{\lambda_i - \lambda_{i+1}\}_i$ is sufficiently large.

4.2 K-types of $L(\sigma)$

Hereafter we assume that $m \leq n$ and denote by σ an irreducible representation of $\widehat{O}(m)$.

We shall use information of branching rule of the restriction $GL(m, \mathbb{C}) \downarrow O(m, \mathbb{C})$ to get Bernstein degree. Recall that

$$L(\sigma)|_{\widetilde{K}} \simeq (\sigma\text{-isotipic component of } \mathbb{C}[M_{n,m}]) \otimes \det{}^{m/2}.$$

On the other hand, it is well-known that there is a $GL_n \times GL_m$-duality

$$\mathbb{C}[M_{n,m}] \simeq \bigoplus_\lambda \tau_\lambda^{(n)} \boxtimes \tau_\lambda^{(m)} \qquad (\text{as } GL(n, \mathbb{C}) \times GL(m, \mathbb{C})\text{-module}).$$

Therefore, if we know the dimension of the space

$$\mathrm{Hom}_{O(m,\mathbb{C})} \left(\sigma, \tau_\lambda^{(m)}|_{O(m,\mathbb{C})}\right) \simeq \mathrm{Hom}_{GL(m,\mathbb{C})} \left(\mathrm{Ind}_{O(m,\mathbb{C})}^{GL(m,\mathbb{C})} \sigma, \tau_\lambda^{(m)}\right),$$

which we denote by $\mathrm{mult}_\sigma(\lambda)$, then we get

$$L(\sigma)|_{\widetilde{K}} \simeq \bigoplus_{l(\lambda) \leq m,n} \mathrm{mult}_\sigma(\lambda)\, \tau_\lambda \otimes \det{}^{m/2}, \tag{4.1}$$

$$P(L(\sigma); x^2) = x^{-|\lambda_0|} \sum_{l(\lambda) \leq m,n} \mathrm{mult}_\sigma(\lambda)\ (\dim \tau_\lambda)\, x^{|\lambda|}, \tag{4.2}$$

where λ_0 is the highest weight of the minimal K-type of $L(\sigma)$. To relate $P(L(\sigma);x^2)$ to $P(L(\mathbf{1}_m);x^2)$ we have to know the asymptotic behavior of branching function $\mathrm{mult}_\sigma(\lambda)$.

In order to use Theorem 4.2, we reformulate the above arguments via the restriction $GL(m,\mathbb{C}) \downarrow SO(m,\mathbb{C})$. The decomposition of $\mathbb{C}[M_{n,m}]$ becomes

$$\begin{aligned}\sum_{\lambda\in\mathcal{P}_m} \tau_\lambda^{GL_n} \boxtimes \left(\tau_\lambda^{GL(m,\mathbb{C})}|_{SO(m,\mathbb{C})}\right) &= \sum_{\lambda\in\mathcal{P}_m} \tau_\lambda^{GL_n} \boxtimes \left(\sum_\mu m(\lambda,\mu)\sigma_\mu^{SO(m,\mathbb{C})}\right) \\ &= \sum_\mu \left(\sum_{\lambda\in\mathcal{P}_m} m(\lambda,\mu)\tau_\lambda^{GL_n}\right) \boxtimes \sigma_\mu^{SO(m,\mathbb{C})}.\end{aligned}$$

We in fact need a decomposition with respect to $O(m,\mathbb{C})$ instead of $SO(m,\mathbb{C})$. Let us consider it in the following three cases.

Case 1. m is odd. In this case, $\sigma = (\sigma_\mu^{SO(m,\mathbb{C})};\det^\delta)$ $(\delta = 0,1)$, and we have

$$L(\sigma)|_{\widetilde{K}} \simeq \sum_{\lambda\in\mathcal{P}_m,|\lambda|\equiv\delta(\mathrm{mod}\,2)} m(\lambda,\mu)\tau_\lambda^{GL_n} \otimes \det{}^{m/2}. \tag{4.3}$$

Case 2. m is even. In this case, $\sigma \in \widehat{O}_m$ is characterized by the restriction to $SO(m,\mathbb{C})$. In fact, we know

$$\sigma|_{SO(m,\mathbb{C})} = \begin{cases} \sigma_{\mu^+}^{SO(m,\mathbb{C})} \oplus \sigma_{\mu^-}^{SO(m,\mathbb{C})} & \mu^\pm = (\mu_1,\mu_2,\cdots,\pm\mu_{m/2})\ (\mu_{m/2}>0) \quad \text{Case 2-A} \\ \sigma_\mu^{SO(m,\mathbb{C})} & \mu = (\mu_1,\mu_2,\cdots,\mu_l,0,\cdots,0) \quad \text{Case 2-B} \end{cases}$$

Case 2-A. In this case, we have $m(\lambda,\sigma_{\mu^+}^{SO(m,\mathbb{C})}) = m(\lambda,\sigma_{\mu^-}^{SO(m,\mathbb{C})}) = m(\lambda,\sigma)$ and we have

$$\begin{aligned}&\left(\sum_\lambda m(\lambda,\mu^+)\tau_\lambda^{GL_n}\right) \boxtimes \sigma_{\mu^+}^{SO(m,\mathbb{C})} \oplus \left(\sum_\lambda m(\lambda,\mu^-)\tau_\lambda^{GL_n}\right) \boxtimes \sigma_{\mu^-}^{SO(m,\mathbb{C})} \\ &= \left(\sum_\lambda m(\lambda,\mu^+)\tau_\lambda^{GL_n}\right) \boxtimes \left(\sigma_{\mu^+}^{SO(m,\mathbb{C})} \oplus \sigma_{\mu^-}^{SO(m,\mathbb{C})}\right) \\ &= \left(\sum_\lambda m(\lambda,\mu^+)\tau_\lambda^{GL_n}\right) \boxtimes \sigma.\end{aligned}$$

Therefore, we get

$$L(\sigma)|_{\widetilde{K}} = \sum_{\lambda\in\mathcal{P}_m} m(\lambda,\mu^+)\tau_\lambda^{GL_n} \otimes \det{}^{m/2}. \tag{4.4}$$

Case 2-B. In this case, σ is determined by $\sigma|_{SO(m,\mathbb{C})} = \sigma_\mu^{SO(m,\mathbb{C})}$ and its determinant. So we get

$$L(\sigma^0) \oplus L(\sigma^1)|_{\widetilde{K}} = \sum_{\lambda\in\mathcal{P}_m} m(\lambda,\mu)\tau_\lambda^{GL_n} \otimes \det{}^{m/2}, \tag{4.5}$$

where $\sigma^\delta = (\sigma_\mu^{SO(m,\mathbb{C})};\det^\delta)$.

4.3 Associated cycle and Bernstein degree of $L(\sigma)$

The knowledge of K-type decomposition (4.3), (4.4), and (4.5), and the asymptotic formula of the branching coefficient $m(\lambda, \mu)$ by Sato (Theorem 4.2) enable us to calculate Bernstein degree of $L(\sigma)$.

Theorem 4.4 *Let $m \le n$ and take $\sigma \in \widehat{O}(m)$. Then the associated cycle of $L(\sigma)$ is given by*

$$\mathcal{AC}\,(L(\sigma)) = \dim \sigma \cdot \overline{\mathcal{O}}_m.$$

As a consequence, Bernstein degree is given by,

$$\operatorname{Deg} L(\sigma) = \dim \sigma \cdot \deg \overline{\mathcal{O}}_m = \dim \sigma \cdot \operatorname{Deg} L(\mathbf{1}_m).$$

PROOF. We omit the proof. Details will appear in [20]. Q.E.D.

Even in the case $m \ge n$, if $L(\sigma)$ is non-trivial, we get some formula of Bernstein degree of $L(\sigma)$ via stable branching coefficients.

For example, let us assume that m is odd. For $\sigma = (\sigma_\mu^{SO(m,\mathbb{C})}; \det^\delta)$, we get

$$\operatorname{Deg} L(\sigma) + \operatorname{Deg} L(\sigma \otimes \det) = \sum_{[\lambda] \in \mathbb{Z}_2^n} m([\lambda], \mu)$$

Note that, since $m \ge n$, $\deg \overline{\mathcal{O}}_n = 1$. If $m \ge 2n$, this formula coincides with $\dim \tau_\lambda$ (holomorphic discrete series case). However, for $n < m < 2n$, any exact formula does not seem to be known.

If m is even, we can calculate similarly and get a similar summation formula using stable branching coefficients. Though, again, there is no closed formula.

Problem 4.5 For $2n > m > n$, give an exact value of

$$\sum_{l(\lambda) \le n, [\lambda] \in \mathbb{Z}_2^n} m([\lambda], \mu).$$

5 Further results

In this article, we only consider the dual reductive pair $Sp(2n, \mathbb{R}) \times O(m)$. However, our arguments is applicable for the other dual pairs $G_1 \times G_2$, where G_1 is non-compact hermitian symmetric type, and G_2 is compact. Here we list up such irreducible pairs.

(D, ι)	G	(G_1, G_2)	
$(\mathbb{R}, 1)$	$Sp(2nm, \mathbb{R})$	$(Sp(2n, \mathbb{R}), O(m))$	
$(\mathbb{C}, \bar{\ })$	$Sp(2nm, \mathbb{R})$	$(U(p,q), U(m))$	$n = p + q$
$(\mathbb{H}, \bar{\ })$	$Sp(2nm, \mathbb{R})$	$(O^*(2r), Sp(m))$	$n = 2r$

Also, some of our arguments will go for hermitian symmetric, non-compact simple group of type E_7. Details will appear elsewhere [20].

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