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Characters of Affine Lie algebras and Kostka polynomials

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Reference. This talk is based on the following collaboration.

[1] G.Hatayama, A.N.Kirillov, A.Kuniba, M.Okado, T.Takagi and Y.Y, *Character Formulae of $U_q(\widehat{\mathfrak{sl}}_n)$-modules and Inhomogeneous Paths*, in preparation.

See also,
[2] A.Nakayashiki and Y.Y, *Kostka polynomials and Energy functions in Solvable lattice models*, Selecta Mathematica (to appear).

[3] A.Nakayashiki and Y.Y, *Crystallizing the spinon basis*, Comm. Math. Phys. **178**. 179-200 (1996).

1. The branching functions for $\widehat{\mathfrak{g}}/\mathfrak{g}$ and the Kostka polynomials

Let $\mathfrak{g}$ be a finite dimensional simple Lie algebra $(/\mathbb{C})$. Here, we will consider only the case $\mathfrak{g} = \mathfrak{sl}_{r+1}$. The affine Lie algebra $\widehat{\mathfrak{g}}$ is the unique central extension of the loop algebra of $\mathfrak{g}$

$$\widehat{\mathfrak{g}} = \mathfrak{g} \otimes \mathbb{C}[z, z^{-1}] \oplus \mathbb{C}d \oplus \mathbb{C}c.$$

where $d = zd/dz$ is the degree operator and c is the center.

$\widehat{\mathfrak{g}}$ is a Kac-Moody Lie algebra defined by the Cartan matrix $a = (a_{ij})$ as follows.

$$\widehat{\mathfrak{g}} = \langle\ e_i, h_i, f_i,\ (i = 0, \ldots, r = \mathrm{rank}\mathfrak{g})\ d,\ c \mid \text{relations}\ \rangle,$$

where the relations are

$$\begin{gathered}
[h_i, h_j] = 0, \quad [e_i, f_j] = \delta_{ij} h_i, \\
[h_i, e_j] = a_{ij} e_j, \quad [h_i, f_j] = -a_{ij} f_j, \\
[d, h_i] = 0, \quad [d, e_i] = \delta_{i0} e_i, \quad [d, f_i] = -\delta_{i0} f_i, \\
(ade_i)^{-a_{ij}+1} e_j = 0, \quad (i \neq j) \\
(adf_i)^{-a_{ij}+1} f_j = 0, \quad (i \neq j)
\end{gathered}$$

The finite dimensional algebra $\mathfrak{g} = \langle\ e_i, h_i, f_i,\ (i = 1, \ldots, r)\ \rangle$ naturally embedded in $\widehat{\mathfrak{g}}$ is called the classical part.

A vector $\Lambda = \sum l_i \Lambda_i \in \oplus_{i=0}^{r} \mathbb{C}\Lambda_i$ is called dominant integrable if $l_i \in \mathbb{Z}_{\geq 0}$. The integrable representation $V(\Lambda)$ of $\widehat{\mathfrak{g}}$ with highest weight Λ is defined as

$$V(\Lambda) = U(\widehat{\mathfrak{g}})v,$$

where v is an element such that

$$e_i v = 0, \quad h_i v = \lambda_i v, \quad (f_i)^{l_i+1} v = 0.$$

$V(\Lambda)$ is irreducible $\widehat{\mathfrak{g}}$-module. We are interested in the decomposition of $V(\Lambda)$ as $\mathfrak{g}$-module. Such a decomposition can be expressed by the branching coefficient $b_\lambda^\Lambda(q)$ defined by

$$\mathrm{Tr}_{V(\Lambda)}(q^{-d} z_1^{h_1} \cdots z_r^{h_r}) = \sum_\lambda b_\lambda^\Lambda(q) \mathrm{ch}_\lambda(z_1, \ldots, z_r).$$

Where $\mathrm{ch}_\lambda(z)$ is the character of the finite dimensional highest weight $\mathfrak{g}$-module with classical highest weight $\lambda = \sum_{i=1}^{r} l_i \Lambda_i$.

Let $\lambda = (\lambda_1, \ldots, \lambda_n)$ be a partition. We identify it with the dominant integrable weight $\lambda = \sum_{i=1}^{n-1} l_i \Lambda_i$ of $\mathfrak{sl}_n$ by $\lambda_i - \lambda_{i+1} = l_i$ $(1 \leq i \leq n-1)$. The Hall-Littlewood symmetric polynomial $P_\lambda(z_1, \ldots, z_n; q)$ is defined by

$$P_\lambda(z_1, \ldots, z_n; q) = \frac{1}{v_\lambda} \sum_{w \in \mathfrak{S}_n} w \left(z_1^{\lambda_1} z_2^{\lambda_2} \cdots z_n^{\lambda_n} \prod_{1 \leq i < j \leq n} \frac{z_i - q z_j}{z_i - z_j} \right),$$

where $q \in \mathbb{C}$ is a parameter, $v_\lambda = [m_1]![m_2]! \cdots$ for $\lambda = (1^{m_1} 2^{m_2} \ldots)$, $[n]! = [n][n-1] \cdots [2][1]$ and $[n] = (1-q^n)/(1-q)$. $m_\lambda(z) = P_\lambda(z; q=1)$ is the monomial symmetric function and $S_\lambda(z) = P_\lambda(z; q=0)$ is the Schur function (i.e. the character of irreducible $\mathfrak{gl}_n$-module).

For two partitions λ and μ, the Kostka polynomial $K_{\lambda,\mu}(q)$ is defined as follows

$$S_\lambda(z) = \sum_\mu K_{\lambda,\mu}(q) P_\mu(z; q).$$

If $|\lambda| = \sum_i \lambda_i \neq |\mu|$, $K_{\lambda,\mu}(q) = 0$. The following combinatorial formula is due to Lascoux-Schützenberger, $K_{\lambda,\mu}(q) = \sum_T q^{c(T)}$, where sum is taken over all the semi-standard tableau of shape λ and weight μ, $c(T) \in \mathbb{Z}_{\geq 0}$ is the charge.

Theorem. Let $k \in \mathbb{Z}_{\geq 0}$ and $\lambda = (\lambda_1, \ldots, \lambda_n) = \sum_{i=1}^{n-1} (\lambda_i - \lambda_{i+1}) \Lambda_i$ be a partition. Then the $\widehat{sl}_n / sl_n$ branching function $b_\lambda^{k\Lambda_0}(q)$ is given as a limit of the Kostka polynomials as

$$b_\lambda^{k\Lambda_0}(q) = \lim_{N \to \infty} q^{-\frac{1}{2} knN(N-1)} K_{\lambda^{(N)}, \mu^{(N)}}(q),$$

where $\lambda^{(N)} = (L + \lambda_1, \ldots, L + \lambda_n)$, $L = kN - |\lambda|/n$ and $\mu^{(N)} = (k^{nN})$. This Theorem was conjectured by A.N.Kirillov and proved in [2]. We have some formulas for more general partitions [1].

Example. $\widehat{\mathfrak{sl}_3} / \mathfrak{sl}_3$, level $k = 2$.

1) Branching function for $\lambda = 0$,

$$b_0^{2\Lambda_0}(q) = 1 + q^2 + 2q^3 + 4q^4 + 6q^5 + 14q^6 + 20q^7 + 39q^8 + \ldots,$$

Kostka polynomials,

$$\begin{aligned} K_{(4,4,4),(2^6)}(q) = \ & q^6(1 + q^2 + q^3 + 2q^4 + q^5 + 3q^6 + q^7 + 2q^8 + 2q^9 + q^{10} + q^{12}), \\ K_{(6,6,6),(2^9)}(q) = \ & q^{18}(1 + q^2 + 2q^3 + 3q^4 + 4q^5 + 9q^6 + \ldots), \\ K_{(8,8,8),(2^{12})}(q) = \ & q^{36}(1 + q^2 + 2q^3 + 4q^4 + 5q^5 + 12q^6 + \ldots). \end{aligned}$$

Kostka numbers are 16, 876 and 70279.

2. The Fermionic formula for Kostka polynomials and the spinon character formulas

The following formula for the Kostka polynomial (the fermionic formula) is due to Kirillov and Reshetikhin

Theorem. For partitions $\lambda = (\lambda_1, \dots, \lambda_n \geq 0)$ and $\mu = (\mu_1, \dots, \mu_L > 0)$ satisfying $|\lambda| = |\mu|$, we have

$$K_{\lambda,\mu}(q) = \sum_m q^{\bar{C}(m)} \prod_{a=1}^{n-1} \prod_{j=1}^{\infty} \begin{bmatrix} p_j^{(a)} + m_j^{(a)} \\ m_j^{(a)} \end{bmatrix},$$

where the sum extends over $m_j^{(a)} \in \mathbb{Z}_{\geq 0}$, $(j = 1, 2, \dots, a = 1, 2, \dots, n-1)$ such that

$$\sum_{j=1}^{\infty} j m_j^{(a)} = \sum_{i=a+1}^{n} \lambda_i, \quad (a = 1, 2, \dots, n-1),$$

$$\bar{C}(m) = n(\mu) - \sum_{i=1}^{L} \sum_{j=1}^{\infty} \min(\mu_i, j) m_j^{(1)} + \frac{1}{2} \sum_{a,b=1}^{n-1} \sum_{j,k=1}^{\infty} \min(j,k) C_{ab} m_j^{(a)} m_k^{(b)},$$

$$p_j^{(a)} = \delta_{a1} \sum_{i=1}^{L} \min(j, \mu_i) - \sum_{b=1}^{n-1} \sum_{k=1}^{\infty} \min(j,k) C_{ab} m_k^{(b)},$$

$n(\mu) = \sum_{i=1}^{\mu_1} = \mu_i'(\mu_i' - 1)/2$, C_{ab} is the Cartan matrix of $\mathfrak{sl}_n$ and

$$\begin{bmatrix} p+m \\ m \end{bmatrix} = \frac{[p+m]!}{[p]![m]!},$$

is the Gaussian polynomial (q-binomial coefficient). There exist similar formula for $\sum_\eta K_{\eta,\mu}(1) K_{\eta',\lambda}(q)$ [1].

By changing the summation variables, one can rewrite this formula to more convenient form to obtain small q-power terms. To make the idea clear, let us consider the case $\mathfrak{sl}_2$, $\lambda = (lN, lN)$ and $\mu = (l^{2N})$. (See [1] for more general statement.) In this simple case, we have

$$K_{\lambda,\mu}(q) = \sum_{m_i=0}^{\infty} q^{\bar{C}(m)} \prod_{j=1}^{\infty} \begin{bmatrix} p_j + m_j \\ m_j \end{bmatrix},$$

where

$$\bar{C}(m) = lN(2N-1) + \sum_{j,k=1}^{\infty} \min(j,k) m_j m_k - 2N \sum_{j=1}^{\infty} \min(j,l) m_j,$$

$$p_j = 2N \min(j,l) - 2 \sum_{k=1}^{\infty} \min(j,k) m_k,$$

and the sum extends over nonnegative integers $m_1, m_2, \dots$ such that

$$\sum_{j=1}^{\infty} j m_j = lN.$$

Now put

$$
\begin{aligned}
(m_1, m_2, \dots, m_{l-1}) &= (\tilde{m}_{l-1}, \dots, \tilde{m}_1), \\
(m_{l+1}, m_{l+2}, \dots) &= (\widehat{m}_1, \widehat{m}_2, \dots), \\
m_l &= N - \textstyle\sum_{j=1}^{\infty} \widehat{m}_j - \sum_{j=1}^{l} \tilde{m}_j,
\end{aligned}
$$

and

$$
\begin{aligned}
(p_1, p_2, \dots, p_{l-1}) &= (\tilde{p}_{l-1}, \dots, \tilde{p}_1), \\
(p_{l+1}, p_{l+2}, \dots) &= (\widehat{p}_1, \widehat{p}_2, \dots), \\
p_l &= 2M,
\end{aligned}
$$

Then we have

$$
\begin{aligned}
\bar{C}(m) &= lN(N-1) + \sum_{j,k=1}^{\infty} \min(j,k)\widehat{m}_j\widehat{m}_k + \sum_{j,k=1}^{l} \min(j,k)\tilde{m}_j\tilde{m}_k \\
&= lN(N-1) + \widehat{C}(\widehat{m}) + \tilde{C}(\tilde{m}),
\end{aligned}
$$

and

$$
\begin{aligned}
\tilde{p}_j &= 2M - 2\textstyle\sum_{k=1}^{l} \min(j,k)\tilde{m}_k, \qquad (1 \le j \le l-1) \\
\widehat{p}_j &= 2M - 2\textstyle\sum_{k=1}^{l} \min(j,k)\widehat{m}_k, \qquad (1 \le j) \\
p_l &= 2M = 2\textstyle\sum_{j=1}^{l} j\tilde{m}_j,
\end{aligned}
$$

Hence, we have

$$
q^{-lN(N-1)} K_{\lambda,\mu}(q) = \sum_M \left(\sum_{\tilde{m}} q^{\tilde{C}(m)} \prod_{j=1}^{l-1} \begin{bmatrix} \tilde{p}_j + \tilde{m}_j \\ \tilde{m}_j \end{bmatrix} \right) \begin{bmatrix} m_l + 2M \\ 2M \end{bmatrix} \left(\sum_{\widehat{m}} q^{\widehat{C}(m)} \prod_{j=1}^{\infty} \begin{bmatrix} \widehat{p}_j + \widehat{m}_j \\ \widehat{m}_j \end{bmatrix} \right)
$$

where the sum extends over nonnegative integers $\tilde{m}_j, \widehat{m}_j, M$ such that

$$
\sum_{j=1}^{l} j\tilde{m}_j = \sum_{j=1}^{\infty} j\widehat{m}_j = M,
$$

since

$$
\sum_{j=1}^{\infty} jm_j - lN = \sum_{j=1}^{\infty} j\widehat{m}_j - \sum_{j=1}^{l} j\tilde{m}_j = 0.
$$

By taking the limit $N \to \infty$, we have

$$
m_l = N - \sum_{j=1}^{\infty} \widehat{m}_j - \sum_{j=1}^{l} \tilde{m}_j \to \infty,
$$

hence

$$
b_0^{l\Lambda_0}(q) = \sum_{M=0}^{\infty} K^{(l)}_{(2^M),(1^{2M})}(q) \frac{1}{(q)_{2M}} K_{(2^M),(1^{2M})}(q).
$$

Here $K^{(l)}_{\lambda,\mu}(q)$ is a level l restricted version of $K_{\lambda,\mu}(q)$. This formula is the spinon character formula [1][3].