表現論シンポジウム報告集, 1997
pp.101-108

On Some Degenerate Principal Series of $SO(p,2)$

Takashi FUJIMURA, University of Tokyo

In [1], Howe and Tan considered some degenerate principal series of $O(p,q)$, $U(p,q)$ and $Sp(p,q)$ in two steps: first decompose them into K-types, then write down the actions of the noncompact part. Now the reducibility and the unitarizability of the irreducible constituents are deduced. Using the same method, we can do something similar with another kind of parabolic subgroup for the $SO(p,2)$ case($p > 4$).

1 Construction of the representation

Let $G = SO(p,2), K = S(O(p) \times O(2))$(maximal compact subgroup) and $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ the corresponding Cartan decomposition of the complexified Lie algebra.

G acts from left on $\mathbb{R}^{p+2} \oplus \mathbb{R}^{p+2}$ (standard representation), and we define:

$$
\begin{aligned}
I_2 &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \\
v_0 &= \begin{pmatrix} I_2 \\ 0_{p-2,2} \\ I_2 \end{pmatrix} \in \mathbb{R}^{p+2} \oplus \mathbb{R}^{p+2}, \\
\mathbf{X} &= Gv_0, \\
S &= \left\{ p \in G \;\middle|\; \exists X = X(p) \in GL(2,\mathbb{R}) \quad s.t. \quad pv_0 = \begin{pmatrix} X \\ 0 \\ X \end{pmatrix} \right\}.
\end{aligned}
$$

S is a parabolic subgroup of G. For any complex number c, we define a character χ_c of S:

$$\chi_c(p) = |\det X(p)|^c \quad (\forall p \in S).$$

We study the Harish-Chandra module of:

$$S^c(\mathbf{X}) = \left\{ F \in C^\infty(\mathbf{X}) \;\middle|\; F(gpv_0) = \chi_c(p)F(gv_0) \quad (\forall g \in G, \forall p \in S) \right\}.$$

This is a G-module by left translation. We can interpret this as a (generalized) principal series representation such as follows. In the Langlands decomposition

$S = MAN$, we can take as A:

$$A = \left\{ a = \exp H(h) \;\middle|\; H(h) = \left(\begin{array}{cc|c|cc} & & & h & \\ & & & & h \\ \hline & & & & \\ \hline h & & & & \\ & h & & & \end{array} \right), \; h \in \mathbb{R}, \text{zeros elsewhere} \right\},$$

which is one dimensional. For any complex number c', we define a linear functional $\nu_{c'}$ on the Lie algebra of A:

$$\nu_{c'}(H(h)) = c'h \quad (\forall h \in \mathbb{R}).$$

Then, since

$$\chi_c(man) = \exp(2ch) \quad (\forall m \in M, \forall a = \exp H(h) \in A, \forall n \in N),$$

we get

$$S^c(\mathbf{X}) \cong \mathrm{Ind}_{MAN}^{G}(\mathbf{1}_M \otimes \exp \nu_{c'} \otimes \mathbf{1}_N).$$

Here we take $2c = -(c' + p - 1)$ and $\mathbf{1}_M, \mathbf{1}_N$ denote the trivial representations. The induction on the right hand side is shifted to preserve unitarity.

2 Stiefel harmonics theory

In Howe-Tan's case[1], the module is identified with a space of functions on $SO(p)/SO(p-1) \simeq S^{p-1}$, and the theory of spherical harmonics is used for further analysis. In our case we have instead $SO(p)/SO(p-2)$, a Stiefel manifold. But we have at hand the Stiefel version of harmonics theory by Ton-That[2] and Gelbart[3]. We quote the results on $SO(p,\mathbb{C})/SO(p-2,\mathbb{C})$.

Theorem 1 *We identify $\mathbf{X}_p = SO(p,\mathbb{C}) \begin{pmatrix} I_2 \\ 0_{p-2,2} \end{pmatrix}$ with $SO(p,\mathbb{C})/SO(p-2,\mathbb{C})$. The space of functions on $\mathbf{X}_p$ is an $SO(p,\mathbb{C})$ module by left translation. We denote by $\mathcal{P}$ the space of polynomials on $\mathbb{C}^p \oplus \mathbb{C}^p$, with coordinates $x_i, y_i (1 \le i \le p)$. Then,*
(a) The space of $SO(p,\mathbb{C})$-finite vectors in $C^\infty(\mathbf{X}_p)$ is

$$A(\mathbf{X}_p) = \{\text{restriction to } \mathbf{X}_p \text{ of a polynomial} \in \mathcal{P}\}.$$

(b) $A(\mathbf{X}_p) \cong \mathcal{H}$ by restriction. Here,

$$\mathcal{H} = \{h \in \mathcal{P} \mid \Delta_{x^2} h = 0, \Delta_{y^2} h = 0, \Delta_{xy} h = 0\},$$

$$\Delta_{x^2} = \sum_{i=1}^{p} \frac{\partial^2}{\partial x_i^2}, \quad \Delta_{y^2} = \sum_{i=1}^{p} \frac{\partial^2}{\partial y_i^2}, \quad \Delta_{xy} = \sum_{i=1}^{p} \frac{\partial^2}{\partial x_i \partial y_i}.$$

(c) If we denote by $\mathcal{J}$ the space of $SO(p,\mathbb{C})$ invariant polynomials, then

$$\mathcal{P} \cong \mathcal{J} \otimes \mathcal{H}.$$

(d) $\mathcal{J}$ is generated by $\xi_{x^2}, \xi_{y^2}, \xi_{xy}$. Here,

$$\xi_{x^2} = \sum_{i=1}^{p} x_i^2, \quad \xi_{y^2} = \sum_{i=1}^{p} y_i^2, \quad \xi_{xy} = \sum_{i=1}^{p} x_i y_i.$$

(e) $\mathcal{H}$ has also a $GL(2,\mathbb{C})$ action by right translation. Then, for $p > 4$, we have the decomposition into irreducibles as $SO(p,\mathbb{C}) \times GL(2,\mathbb{C})$ modules(multiciplity free):

$$\mathcal{H} \cong \bigoplus_{\substack{l_1, l_2 \in \mathbb{Z} \\ l_1 \geq l_2 \geq 0}} \mathcal{H}_{(l_1,l_2)},$$

$$\mathcal{H}_{(l_1,l_2)} \cong \sigma^p_{(l_1,l_2,0,\ldots,0)} \otimes \rho^2_{(l_1,l_2)},$$

where $\sigma^p_{(l_1,l_2,0,\ldots,0)}$ is the irreducible representation of $SO(p,\mathbb{C})$ of highest weight $\overbrace{(l_1, l_2, 0, \ldots, 0)}^{[p/2]}$ and $\rho^2_{(l_1,l_2)}$ is the irreducible representation of $GL(2,\mathbb{C})$ of highest weight (l_1, l_2).

3 K-types and $\mathfrak{p}$-action

Now we turn to the study of $S^c(\mathbf{X})$. Using the "compact picture" of the induced representation and by complexifying, we get

$$S^c(\mathbf{X}) \cong C^\infty(\mathbf{X}_p)$$

as $SO(p,\mathbb{C}) \times SO(2,\mathbb{C})$-modules. The action on the right hand side is given by

$$((k_1, k_2)\, h)\,(x) = h(k_1^{-1} x k_2)$$

$$(\forall (k_1, k_2) \in SO(p,\mathbb{C}) \times SO(2,\mathbb{C}),\ \forall h \in C^\infty(\mathbf{X}_p)).$$

So, the Harish-Chandra module is

$$\mathcal{H} \cong \bigoplus_{l_1,l_2} \mathcal{H}_{(l_1,l_2)}.$$

Once we fix a Cartan subgroup of $GL(2,\mathbb{C})$, we have the weight space decomposition:

$$\mathcal{H}_{(l_1,l_2)} = \bigoplus_{m=0}^{l_1-l_2} \mathcal{H}_{(l_1,l_2),m}$$

where $\mathcal{H}_{(l_1,l_2),m}$ has weight$(l_1 - m, l_2 + m)$ for the Cartan subgroup of $GL(2,\mathbb{C})$.

Twisting the $SO(2,\mathbb{C})$-action by Cayley transform, we conclude that

$$S^c(\mathbf{X})_{K-finite} \underset{\cong}{\overset{\Phi}{\longleftarrow}} \bigoplus_{l_1,l_2,m} \mathcal{H}_{(l_1,l_2),m}$$

is the irreducible decomposition as $SO(p) \times SO(2)$-modules. We call $\mathcal{H}_{(l_1,l_2),m}$ as a "K-type". (To be more precise, a "K-type" should be an irreducible $S(O(p) \times O(2))$-module, so we really have to consider pairs of $\mathcal{H}_{(l_1,l_2),m}$. But since this does not affect the results, we will ignore it.)

To understand the $(\mathfrak{g}, K)$-module structure of $S^c(\mathbf{X})$, we have to compute the $\mathfrak{p}$-action to find out which of the "K-types" are matched together. This is the next key proposition.

Proposition 2 *For* $(s_1, s_2, s_3) \in \{\pm(1,0,0), \pm(1,0,1), \pm(0,1,-1), \pm(0,1,0)\}$, *there exist linear maps*

$$T^{s_1,s_2,s_3}_{(l_1,l_2),m} : \mathfrak{p} \otimes \mathcal{H}_{(l_1,l_2),m} \to \mathcal{H}_{(l_1+s_1,l_2+s_2),m+s_3}$$

satisfying the followings:

- *They do not depend on c.*
- *They are not zero maps(when the target space is not zero).*
- *For* $\forall X \in \mathfrak{p}, \forall h \in \mathcal{H}_{(l_1,l_2),m}$,

$$\begin{aligned}
X(\Phi(h)) = \quad & (c - l_1) && \left(\Phi(T^{(1,0,0)}_{(l_1,l_2),m}(X \otimes h)) + \Phi(T^{(1,0,1)}_{(l_1,l_2),m}(X \otimes h))\right) \\
+ \quad & (c - l_2 + 1) && \left(\Phi(T^{(0,1,-1)}_{(l_1,l_2),m}(X \otimes h)) + \Phi(T^{(0,1,0)}_{(l_1,l_2),m}(X \otimes h))\right) \\
+ \quad & (c + l_1 + p - 2) && \left(\Phi(T^{(-1,0,-1)}_{(l_1,l_2),m}(X \otimes h)) + \Phi(T^{(-1,0,0)}_{(l_1,l_2),m}(X \otimes h))\right) \\
+ \quad & (c + l_2 + p - 3) && \left(\Phi(T^{(0,-1,0)}_{(l_1,l_2),m}(X \otimes h)) + \Phi(T^{(0,-1,1)}_{(l_1,l_2),m}(X \otimes h))\right).
\end{aligned}$$

4 Irreducible Constituents

The previous proposition gives us a criterion of reducibility, in the sense that we can move from one "K-type" to another by the action of an element of $\mathfrak{p}$ if and only if the corresponding coefficient is non zero.

We can easily deduce the following lemma.

Lemma 3 *If an irreducible constituent of* $S^c(\mathbf{X})$ *contains* $\mathcal{H}_{(l_1,l_2),m_0}$ *for some* l_1, l_2 *and* m_0, *then it contains all of* $\mathcal{H}_{(l_1,l_2)} = \bigoplus_{m=0}^{l_1-l_2} \mathcal{H}_{(l_1,l_2),m}$.

So, we can forget the parameter m.

The formula in the proposition is rewritten as:

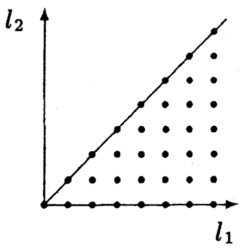
$$\begin{aligned} X(\Phi(h)) = \quad & A^{+0}(l_1,l_2)\Phi(h_X^{+0}) + A^{-0}(l_1,l_2)\Phi(h_X^{-0}) \\ & + A^{0+}(l_1,l_2)\Phi(h_X^{0+}) + A^{0-}(l_1,l_2)\Phi(h_X^{0-}) \end{aligned}$$

with

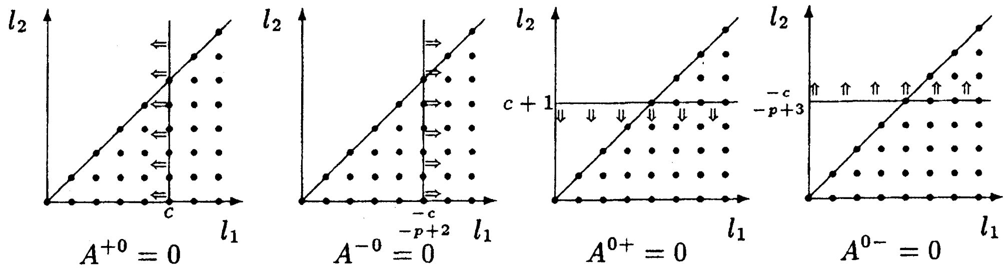
$$\begin{aligned} A^{+0}(l_1,l_2) &= c - l_1, \quad A^{-0}(l_1,l_2) = c + l_1 + p - 2, \\ A^{0+}(l_1,l_2) &= c - l_2 + 1, \quad A^{0-}(l_1,l_2) = c + l_2 + p - 3, \end{aligned}$$

$$\begin{aligned} h_X^{+0} &\in \mathcal{H}_{(l_1+1,l_2)}, \quad h_X^{-0} \in \mathcal{H}_{(l_1-1,l_2)}, \\ h_X^{0+} &\in \mathcal{H}_{(l_1,l_2+1)}, \quad h_X^{0-} \in \mathcal{H}_{(l_1,l_2-1)}. \end{aligned}$$

We identify $\mathcal{H}_{(l_1,l_2)}$ with the point (l_1,l_2) in a plane. Thus, the K-types are represented by integral points in the cone $l_1 \geq l_2 \geq 0$.



The four lines $A^{\pm 0}(l_1,l_2) = 0$ and $A^{0\pm}(l_1,l_2) = 0$ are "potential barriers" which are obstructions to the transitions.



If a barrier meets an integral point, it prevents the K-types in the integral point from crossing the barrier in one direction, yielding a submodule and a quotient module.

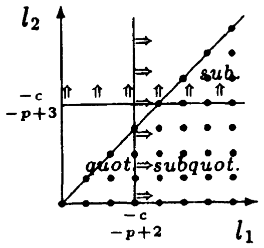
All that remains is to determine the positions of the barriers according to the value of c.

Theorem 4 *The decomposition of the Harish-Chandra module of $S^c(\mathbf{X})$ is*

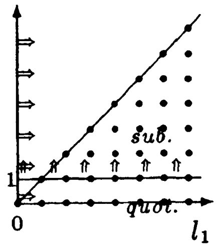
(0) $c \notin \mathbb{Z}$: The barriers never cross the K-types, so it is irreducible.

From now on, assume $c \in \mathbb{Z}$.

(1)$c \leq -p+1$: *1 finite dimensional quotient, 1 subquotient, 1 submodule.*

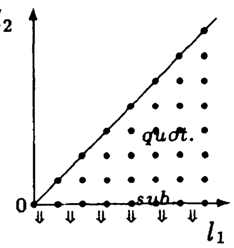


(2)$c = -p+2$: *1 quotient, 1 submodule.*

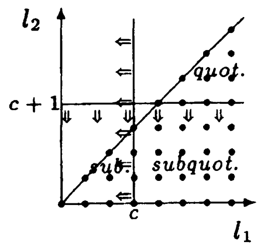


(3)$-p+3 \leq c \leq -2$: *irreducible.*

(4)$c = -1$: *1 submodule, 1 quotient.*



(5)$c \geq 0$: *1 finite dimensional submodule, 1 subquotient, 1 quotient.*



5 Unitarizability

We discuss the unitarizability of the irreducible constituents.

On each K-type, K-invariant hermitian inner products are unique up to some constants. So, if we fix a standard one on each, a given K-invariant hermitian inner product is determined by the set of constants

$$\left\{ c_{(l_1,l_2),m} \middle| \mathcal{H}_{(l_1,l_2),m} \text{is contained in the constituent} \right\}.$$

The condition that the inner product is G-invariant gives us some constraints. By the fact that the normalized induction preserves unitarity, the terms involving the standard inner products cancel, leaving some recursive relations between the $c_{(l_1,l_2),m}$'s. It follows that $c_{(l_1,l_2),m}$ is independent of m, and we can once more forget about m. We will write $c_{(l_1,l_2)}$ instead of $c_{(l_1,l_2),m}$.

Now the recursive relations become

$$\begin{aligned} A^{+0}(l_1,l_2)c_{(l_1+1,l_2)} + \overline{A^{-0}(l_1,l_2)}c_{(l_1-1,l_2)} &= 0 \\ A^{0+}(l_1,l_2)c_{(l_1,l_2+1)} + \overline{A^{0-}(l_1,l_2)}c_{(l_1,l_2-1)} &= 0 \end{aligned}$$

with bar denoting complex conjugation. So, once we fix one of the $c_{(l_1,l_2)}$'s, the rest is determined automatically.

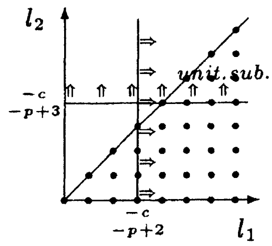
The condition that the inner product is positive definite is that all the $c_{(l_1,l_2)}$'s are positive real numbers. This is equivalent to $c' \in \mathbb{R}$ or $c' \in i\mathbb{R}$ (recall $c' = -2c - p + 1$, the parameter for the normalized induction). The case $c' \in i\mathbb{R}$ is unitary induction("unitary axis"). The case $c' \in \mathbb{R}$ needs case-by-case checking according to the value of c'(or c).

Here is the result.(The numbers in parentheses correspond to the previous theorem.)

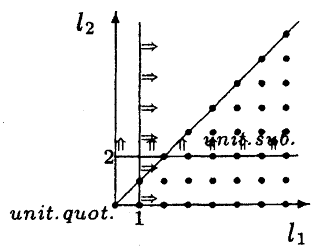
Theorem 5 *The Harish-Chandra module of $S^c(\mathbf{X})$ is irreducible and unitary if and only if $c' = -2c - p + 1 \in i\mathbb{R}$(unitary axis) or (0),(3) $-p + 2 < c < -1$.*

When it is reducible(i.e.$c \in \mathbb{Z}$),

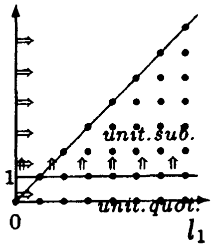
(1)$c \leq -p$: 1 non-unitary finite dimensional quotient, 1 non-unitary subquotient, 1 unitary submodule.



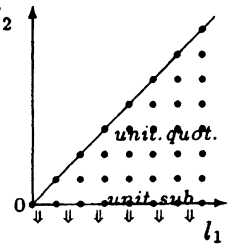
(1bis)$c = -p + 1$: 1 unitary 1-dimensional quotient, 1 non-unitary subquotient, 1 unitary submodule.



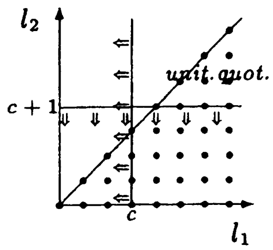
(2) $c=-p+2$: 1 unitary quotient, 1 unitary submodule.



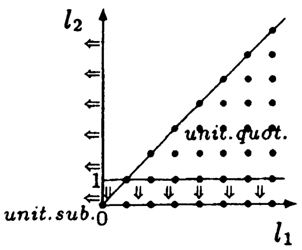
(4) $c=-1$: 1 unitary submodule, 1 unitary quotient.



(5) $c \geq 1$: 1 non-unitary finite dimensional submodule, 1 non-unitary subquotient, 1 unitary quotient.



(5bis) $c=0$: 1 unitary 1-dimensional submodule, 1 non-unitary subquotient, 1 unitary quotient.



References

[1] R.Howe and E.Tan: Homogeneous functions on light cones: the infinitesimal structure of some degenerate principal series representations, *Bull.Amer.Math.Soc.* 28 (1993) 1-74

[2] T.Ton-That: Lie group representations and harmonic polynomials of a matrix variable, *Trans.Amer.Math.Soc.* 216 (1976), 1-46

[3] S.Gelbart: A theory of Stiefel harmonics, *Trans.Amer.Math.Soc.* 192 (1974), 29-50