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Fourier multipliers on Non-Riemannian Symmetric Spaces

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Abstract

Restricting to K-invariant functions we prove $\mathbf{L}^p$ boundedness of Hörmander-Michlin multipliers on Non-Riemannian Symmetric Spaces using Flensted-Jensen duality.

1 Introduction

In accordance with the Riemannian case one could try looking at H-invariant convolutors but unfortunately this does not work. The reason is that H is infinite so H-invariant functions cannot belong to any $\mathbf{L}^p$ space. So if we apply the convolutor to a function with small support then the resulting function will be almost H-invariant and hence not in $\mathbf{L}^p$. Instead one may look at what operators correspond to multipliers preserving $\mathbf{L}^2$. It turns out that if we restrict our attention to multipliers acting on K-invariant functions then on the dual side these are the convolutors. The problem is that K-invariant $\mathbf{L}^p$ functions on G/H corresponds to K^d-invariant $\mathbf{L}^p$ functions on $H^d\backslash G^d$. In this paper we show that in spite of this we can use multiplier theory from the Riemannian case to prove a Hörmander-Michlin theorem.

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2 Notation

For more details about the general theory of non-Riemannian symmetric spaces see [8] and [9]. Let G be a noncompact semisimple connected Lie group with finite center and an involution σ. Let H be an open subgroup of G^σ. So G/H is a semisimple symmetric space. Assume that we have a Cartan involution θ commuting with σ and denote by K the fixed point set for θ. Corresponding to the involutions we have the following decompositions

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{q} = \mathfrak{k} \oplus \mathfrak{p}. \tag{1}$$

Since $\sigma \circ \theta = \theta \circ \sigma$, we also have

$$\mathfrak{g} = \mathfrak{h} \cap \mathfrak{k} \oplus \mathfrak{h} \cap \mathfrak{p} \oplus \mathfrak{q} \cap \mathfrak{k} \oplus \mathfrak{q} \cap \mathfrak{p}. \tag{2}$$

This splits into eigenspaces for $\sigma \circ \theta$

$$\mathfrak{g}_+ = \mathfrak{h} \cap \mathfrak{k} \oplus \mathfrak{q} \cap \mathfrak{p},$$

$$\mathfrak{g}_- = \mathfrak{h} \cap \mathfrak{p} \oplus \mathfrak{q} \cap \mathfrak{k}.$$

We can now define the non-compact Riemannian form G^d/K^d of G/H. First we set

$$\mathfrak{g}^d = \mathfrak{h} \cap \mathfrak{k} \oplus i(\mathfrak{h} \cap \mathfrak{p}) \oplus i(\mathfrak{q} \cap \mathfrak{k}) \oplus \mathfrak{q} \cap \mathfrak{p}. \tag{3}$$

Then we let G^d be the real form of $G_{\mathbf{C}}$(the complexification of G) with Lie algebra $\mathfrak{g}^d$. The subgroup $K^d = G^d \cap H_{\mathbf{C}}$ corresponding to $\mathfrak{h} \cap \mathfrak{k} \oplus i(\mathfrak{h} \cap \mathfrak{p})$ is a maximal compact subgroup. Hence the symmetric space G^d/K^d is Riemannian. We will also need $H^d = G^d \cap K_{\mathbf{C}}$ with Lie algebra $\mathfrak{h}^d = \mathfrak{h} \cap \mathfrak{k} \oplus i(\mathfrak{q} \cap \mathfrak{k})$. By partial holomorphic extension we obtain a map $f \mapsto f^r$ mapping $\mathcal{C}^\infty(K; G/H)$ into $\mathcal{C}^\infty(H^d; G^d/K^d)$, i.e. it takes K-finite smooth functions on G/H to H^d-finite smooth functions on G^d/K^d. Since we are only interested in the most continuous part we take a maximal split θ-invariant Cartan subspace $\mathfrak{b}$ of $\mathfrak{q}$, i.e. such that the intersection $\mathfrak{a} = \mathfrak{b} \cap \mathfrak{p}$ is maximal Abelian in $\mathfrak{q} \cap \mathfrak{p}$. We have two rootsystems $\Sigma(\mathfrak{a}, \mathfrak{g})$ and $\Sigma(\mathfrak{a}, \mathfrak{g}_+)$ with respective Weyl group W and $W_{K\cap H}$. We will also use the quotient $\mathcal{W} = W/W_{K\cap H}$. Set $m_\alpha^+ = \dim \mathfrak{g}_\alpha \cap \mathfrak{g}_+$ and $m_\alpha^- = \dim \mathfrak{g}_\alpha \cap \mathfrak{g}_-$, where $\mathfrak{g}_\alpha$ is the root space related to $\alpha \in \Sigma(\mathfrak{a}, \mathfrak{g})$. If we define

$$\mathfrak{b}^r = \mathfrak{b} \cap \mathfrak{p} + i(\mathfrak{b} \cap \mathfrak{k}). \tag{4}$$

then $\mathfrak{b}^r$ is maximal split for $\mathfrak{g}^d$. We have the generalized Cartan decompositions

$$G = K\bar{A}^+H, \quad G^d = H^d\bar{A}^+K^d.$$

Here A^+ corresponds to the positive roots in $\Sigma(\mathfrak{a}, \mathfrak{g}_+)$.

3 Multipliers

Let ψ be a function in $\mathbf{PW}^*(\mathfrak{b}_{\mathbf{C}}^*)^{\mathcal{W}}$, i.e. a $\mathcal{W}$-invariant entire function of exponential type with slow growth, then by van den Ban, Flensted-Jensen and Schlichtkrull [4] this function has an operator $M_\psi : \mathcal{C}_c^\infty(K; G/H) \mapsto \mathcal{C}_c^\infty(K; G/H)$ associated to it. Such operators are special cases of what they call multipliers, which are operators

$$M : \mathcal{C}_c^\infty(K; G/H) \mapsto \mathcal{C}_c^\infty(K; G/H) \tag{5}$$

that are equivariant under the action of $\mathfrak{g}$, K and $\mathbf{D}(G/H)$ and restricts continuously to $\mathcal{C}_c^\infty(G/H)^\mu$ for all μ in $\hat{K}$. The connection to multipliers is that the Fourier transform of $M_\psi(f)$ is $\psi(\lambda)\hat{f}(\pi)$ on $v \in \mathcal{V}_\pi$ of type λ. Here $\mathcal{V}_\pi$ denotes the space of H-fixed distribution vectors for the irreducible unitary representation π. To construct M_ψ they use partial holomorphic extension defining it by

$$(M_\psi f)^r = f^r * F, \quad f \in \mathcal{C}_c^\infty(K; G/H) \tag{6}$$

where F is the K-biinvariant distribution with spherical transform ψ. By the Paley-Wiener theorem, the assumptions on ψ implies that $F \in \mathcal{E}'(K\backslash G/K)$. Hence the convolution is in the image of $\mathcal{C}_c^\infty(K;G/H)$ under the isomorphism $(.)^r$, so M_ψ is well defined.

We shall be interested in $\mathbf{L}^p$-multipliers for K-biinvariant functions on G/H. Let ψ be a $\mathcal{W}$-invariant function on the support of the Plancherel measure. Let $\mathbf{L}^p(K\backslash G/H)$ be the space of K-invariant $\mathbf{L}^p$-functions on G/H. We can repeat the construction above, referring to the isomorphism $\mathbf{L}^p(K\backslash G/H) \cong \mathbf{L}^p(H^d\backslash G/K^d)$ instead. That is if F maps $\mathbf{L}^p(H^d\backslash G^d/K^d)$ to itself, the operator M_ψ will be well defined and map $\mathbf{L}^p(K\backslash G/H)$ to itself. We also have to show that $\widehat{M_\psi f}v = \psi(\lambda)\hat{f}v$, where v is an H-fixed distribution of type λ. This may be done as in [4] by referring to a paper by Delorme and Flensted-Jensen [5], where a related result is proved, but for the convenience of the reader we will give the proof, which in our case is simpler as we only look at the trivial K-type.

Proof. We want to show that

$$< \pi(M_\psi f)v, v' >= \psi(\lambda) < \pi(f)v, v' >,$$

for v' with trivial K-action and $v \in \mathcal{V}_\pi$ of type λ. As $M_\psi f$ is K-invariant we can rewrite the LH-side as, using P_1 to denote the projection to the trivial isotypic component,

$$\int_{K\backslash G/H} < P_1\pi(x)v, M_\psi f(x)v' > dx,$$

which by duality is, denoting $\phi(x) = P_1\pi(x)v$

$$\int_{H^d\backslash G^d/K^d} < \phi^r(x), f^r * F(x)v' > dx.$$

As the integrand is right K^d-invariant this equals

$$\int_{H^d\backslash G^d} < \phi^r(x), f^r * F(x)v' > dx.$$

We now write the convolution as a double integral and substitute this into the last expression

$$\int_{H^d\backslash G^d}\int_{H^d\backslash G^d/K^d} f^r(y) < \phi^r(x), \int_{H^d} F(y^{-1}hx)dhv' > dydx,$$

which we rewrite using the left H^d-invariance of ϕ^r as

$$\int_{H^d\backslash G^d/K^d} f^r(y)\int_{G^d} < \phi^r(x), F(y^{-1}x)v' > dxdy.$$

This becomes after a change of variables

$$\int_{H^d\backslash G^d/K^d} f^r(y)\int_{G^d} < \phi^r(yx), F(x)v' > dxdy.$$

We now use the right K-invariance of F to obtain

$$\int_{G^d/K^d} F(x) < \int_{H^d\backslash G^d/K^d} \int_{K^d} f^r(y)\phi^r(ykx)dkdy, v' > dx.$$

Let

$$\Phi(x) =< \int_{H^d\backslash G^d/K^d} \int_{K^d} f^r(y)\phi^r(ykx)dkdy, v' > .$$

We then observe that Φ is bi-K^d-invariant and it is also an eigenfunction for $\mathbf{D}(G^d/K^d)$ with eigenvalue λ and hence it must be a constant multiple of the spherical function $\phi_{-i\lambda}$. Thus it remains to prove that this constant is equal to $< \pi(f)v, v' >$. But this constant is just

$$\Phi(e) =< \int_{H^d\backslash G^d/K^d} \int_{K^d} f^r(y)\phi^r(yk)dkdy, v' >,$$

which as ϕ^r is right-K^d-invariant is

$$< \int_{H^d\backslash G^d/K^d} f^r(x)\phi^r(x)dx, v' >=< \int_{K\backslash G/H} f(x)\phi(x)dx, v' > .$$

Taking into account the left-K-invariance of f and ϕ this completes the proof. $\square$

Let $\mathcal{T}$ be the strip $\mathfrak{a}^* + i\mathrm{Conv}(\mathcal{W}\rho)$.

Theorem 1. *Let ψ be a holomorphic function in the strip $\mathcal{T}$ and continuous up to the boundary, satisfying the following estimate*

$$|\nabla^i\psi(\lambda)| \leq C(1+|\lambda|)^{-i}, \quad \lambda \in \bar{\mathcal{T}}, i < \left[\frac{n}{2}\right]+1, \tag{7}$$

then M is a bounded operator on $L^p(K\backslash G/H)$ for $1 < p < \infty$.

Remark 1. *Before the proof it may be useful to recall that Anker [1] has proved that this assumption implies $\mathbf{L}^p$-boundedness in the Riemannian case.*

Remark 2. *In the Riemannian case the analytic continuation is necessary since the spherical functions are in $\mathbf{L}^p$ for $p > \frac{2\rho}{Im\lambda+\rho}$ and ψ is the spherical transform of a K-invariant distribution, but in the non-Riemannian case the problem arises that although we can make a similar construction with Eisenstein integrals of trivial K-type, these may have zeros and hence may cause the function ψ to have poles. See section 4.*

Proof. By construction the corresponding operator on the dual side is convolution with the inverse spherical transform of ψ which is a K-biinvariant function F. By Anker if we divide F into two parts, F_0 concentrated close to the origin and $F_\infty = F - F_0$. Then these parts have the following properties

- $\int_{|x|\geq 2|y|} |F_0(y^{-1}x) - F_0(x)|dx \leq C$,
- $F_\infty \in \mathbf{L}^1(G^d/K^d)$.

He also shows that F_0 is bounded on $\mathbf{L}^2(G^d/K^d)$ but this we will not use. Let f be in $\mathbf{L}^p(K\backslash G/H)$ then by duality f^r will be in $\mathbf{L}^p(H^d\backslash G^d/K^d)$ and we have the following commutative diagram

$$\begin{array}{ccc}
\mathbf{L}^p(K\backslash G/H) & \xrightarrow{\cong} & \mathbf{L}^p(H^d\backslash G^d/K^d) \\
{\scriptstyle M}\downarrow & & \downarrow{\scriptstyle *F} \\
\mathbf{L}^p(K\backslash G/H) & \xrightarrow{\cong} & \mathbf{L}^p(H^d\backslash G^d/K^d).
\end{array}$$

Hence we want to prove that $f^r * F \in \mathbf{L}^p(H^d\backslash G^d/K^d)$. But considering the convolution as a K^d-invariant function on $H^d\backslash G^d$, we obtain by the second condition above:

$$\left(\int_{H^d\backslash G^d} |f^r * F_\infty(x)|^p dx\right)^{1/p} =$$

$$\begin{aligned}
&= \left(\int_{H^d\backslash G^d} \left|\int_{G^d/K^d} f^r(xy^{-1})F_\infty(y)dy\right|^p dx\right)^{1/p} \\
&\leq \left(\int_{H^d\backslash G^d} |f^r(xy^{-1})|^p dx\right)^{1/p} \int_{G^d/K^d} |F_\infty(y)|dy \\
&\leq C\left(\int_{H^d\backslash G^d} |f^r(x)|^p dx\right)^{1/p}.
\end{aligned}$$

For the local part we first use a covering lemma to reduce to functions with support in the unit ball of A, which is a space of homogeneous type and so we may use Hardy spaces. So assume that f is an atom with support in a ball of radius t. Let V_t denote the part of $H^d\backslash G^d$ where $|a| \geq 2t$ then with $\delta(a) = \prod_{\alpha\in\Sigma^+(\mathfrak{a},\mathfrak{g})} \sinh^{m_\alpha^+}(\log a)\cosh^{m_\alpha^-}(\log a)$ (see Flensted-Jensen [7])

$$\int_{V_t} |f^r * F_0(x)|dx =$$

$$\int_{V_t}\left|\int_A\int_{H^d} f^r(a')F_0(a'^{-1}h^{-1}x)\delta(a')da'dh\right| dx \leq$$

$$\int_A |f^r(a')|\delta(a')da' \int_{|a|\geq 2t}\int_{H^d} \left|F_0(a'^{-1}h^{-1}a) - F_0(h^{-1}a)\right| \delta(a)dadh.$$

To be able to use our first condition on F_0 we need a lemma.

Lemma 1. *If $h \in H^d$ and $a \in a$ then the following inequality is true in G^d/K^d,*

$$|ha| \geq |a|. \tag{8}$$

Proof. Since $\mathfrak{a} \in \mathfrak{p}\cap\mathfrak{q}$ it is orthogonal to $\mathfrak{h}^d$. This implies that the orbits $A.o$ and $H^d.o$ are orthogonal at the origin in G^d/K^d, hence by the triangle inequality (negative curvature) $d(0, a\cdot 0) + d(0, h\cdot 0) \geq d(h\cdot 0, a\cdot 0)$. □

For the second part we use the $\mathbf{L}^2$-boundedness of the operator and the assumption $t \leq 1$. □

Remark 3. *By the same argument we also obtain the same result for multipliers satisfying* $\mathbf{L}^2$*-conditions instead, since by Anker[1] the corresponding operator fulfills the same conditions.*

Example 1. (for more information about the material in this section see [11]) Let $G = SL(2, \mathbf{R})$ this group has two involutions $\theta : g \mapsto (g^T)^{-1}$ and $\sigma : g \mapsto IgI$ where

$$I = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

The first is associated the hyperbolic plane and to the second we have two symmetric spaces G/G^σ and G/H, where $G^\sigma = \{$ diagonal matrices in $G\}$ is the fix point set of σ and H is the subset of all positive diagonal matrices. We will be interested in the space G/H. Clearly $\theta \circ \sigma = \sigma \circ \theta$, hence we are in the situation of our earlier study. The generalized Cartan decomposition in this case is $G = KAH$ where

$$H_s = \begin{pmatrix} e^{s/2} & 0 \\ 0 & e^{-s/2} \end{pmatrix},$$

$$A_t = \begin{pmatrix} \cosh t/2 & \sinh t/2 \\ \sinh t/2 & \cosh t/2 \end{pmatrix},$$

$$K_\phi = \begin{pmatrix} \cos \phi/2 & \sin \phi/2 \\ -\sin \phi/2 & \cos \phi/2 \end{pmatrix}.$$

In this case we can improve our result a little because instead of restricting our attention to K-invariant function we can look at functions of a fixed K-type. The reason is that this corresponds to picking Fourier coefficients. Since M is assumed to preserve K-types we can take away the K action and then after we have applied M we may put it back, hence reducing to the K-invariant case. We want to find the Riemannian non-compact form X^r of $X = G/H$. For this we first introduce $\mathfrak{g}^d$ which is

$$\begin{aligned} \mathfrak{g}^d &= \mathfrak{k}^d(= i\mathfrak{h}) \oplus \mathfrak{h}^d(= i\mathfrak{k}) \oplus \mathfrak{a} \\ &= \mathbf{R}\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \oplus \mathbf{R}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\ &= \begin{pmatrix} it & z \\ \bar{z} & -it \end{pmatrix}, \end{aligned}$$

where $t \in \mathbf{R}$ and $z \in \mathbf{C}$. This implies that $G^d = SU(1,1)$. We also obtain $K^d = S(U(1) \times U(1))$ and

$$H^d = \begin{pmatrix} \cosh s & i \sinh s \\ -i \sinh s & \cosh s \end{pmatrix}. \tag{9}$$

So $X^r = SU(1,1)/S(U(1) \times U(1))$, i.e the hyperbolic disc. As the maximal split Cartan subspace we take $\mathfrak{a}$ and we see that $\mathfrak{b}^r = \mathfrak{a}$ and $\mathfrak{a}_{\mathbf{C}} = \mathbf{C}$. To introduce the Fourier transform we need some representation spaces. Put $t^{s,\epsilon} = |t|^s sgn^\epsilon t$.

Let $D_{(s,\epsilon)}$ be the space of C^∞ functions ϕ on $\mathbf{R}$ such that $t \mapsto \phi(-\frac{1}{t})t^{s-1,\epsilon}$ is also infinitely differentiable. When s is a positive integer the case of interest to us will be $\epsilon \equiv s+1$ and we will denote this space D_s. If we use the notation

$$g = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}, \tag{10}$$

then we can define a representation of G on $D_{(s,\epsilon)}$ as follows

$$T_{(s,\epsilon)}(g)\phi(t) = \phi\left(\frac{\alpha t + \gamma}{\beta t + \delta}\right)(\beta t + \delta)^{s-1,\epsilon}. \tag{11}$$

The operator

$$B_{(s,\epsilon)}\phi(t) = \int (t - t_1)^{-s-1,\epsilon}\phi(t_1)dt_1$$

is an intertwining operator between $T_{(s,\epsilon)}$ and $T_{(-s,\epsilon)}$. For s imaginary the representation is irreducible but for s a positive integer and $\epsilon \equiv s+1$ there are two invariant subspaces with intersection $E_s = KerB_{(s,\epsilon)}$. When s is imaginary this representation is unitary for the scalar product

$$(\psi, \phi) = \int \psi(t)\overline{\phi(t)}dt.$$

For s a positive integer we obtain a unitary representation by letting $T_{(s,\epsilon)}$ act on the factor space D_s/E_s equipped with the scalar product

$$(\tilde{\psi}, \tilde{\phi})_s = (A_s\psi, \phi),$$

where $A_s = 2^{-s}\pi^{\frac{1}{2}}\Gamma(\frac{1-s}{2})(\frac{d}{dt})^s$. The H-invariant distribution vectors are $\theta_{(s,\epsilon,\nu)} = t^{\frac{s-1}{2},\nu}$. In our present situation the type of these vectors are determined by s, since $\mathbf{D}(G/H)$ is generated by the Laplacian

$$\Delta = -\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}^2 + \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}^2 \tag{12}$$

and

$$T_{(s,\epsilon)}(\Delta) = \frac{1}{4}(1 - s^2) \cdot I. \tag{13}$$

The Fourier transform

$$F_{(s,\epsilon,\nu)}f(t) = \int_X f(x)(\delta t - \gamma)^{(\frac{s-1}{2},\nu)}(-\beta t + \alpha)^{(\frac{s-1}{2},\nu+\epsilon)}dx. \tag{14}$$

maps $C_c^\infty(X)$ to D_χ or D_s/E_s. Since $K = \mathbf{S}^1$ the K-types are functions f such that $f(kaH) = e^{in\phi}f(aH)$. Even if this is not directly related to the discrete series one can show that for functions of a fixed K-type the discrete series is empty, see Appendix 1 in [6]. This explains why it suffices to look at the dual, which is only related to the (most) continuous part. If we integrate such functions against $e^{-in\phi}$ we obtain a K-invariant function to which we may apply Thm 1. Then we define a new function from the resulting K-invariant function by defining it to be of the same K-type as the original function. As a result such a multiplier M will map K-finite $\mathbf{L}^p$-functions to K-finite functions with norm only depending on the number of K-parts of the given function. The Plancherel formula for X

$$\int_X |f(x)|^2 dx = \frac{1}{16\pi^2} \sum_{s=1}^{\infty} s i_s^{-1} \|F_s f\|_s^2 + \frac{1}{64\pi^2} \sum_{\epsilon,\nu=0,1} \int_0^{\infty} \rho \tanh\left(\pi \frac{\rho + i\epsilon}{2}\right) \|F_{i\rho,\epsilon,\nu} f\|^2 d\rho,$$

where $i_s = 2^{-s}\pi^{-1/2}\Gamma(\frac{1+s}{2})\left(\cos(\frac{\epsilon\pi}{2}) + 1\right)$, should be compared with the Plancherel formula for X^r [10]

$$\int_{X^r} |f(x)|^2 dx = \frac{1}{2\pi} \int_{K/M} \int_{\mathbf{R}_+} |\hat{f}(\lambda, kM)|^2 \lambda \tanh(\frac{\pi\lambda}{2}) d\lambda dkM. \qquad (15)$$

When we neglect the discrete part we observe that the formulas are of the same kind and hence it is not so surprising that the behaviour of the multipliers should be similar.

Two examples of multipliers are $\psi(s) = e^{-t(1+\epsilon-s^2)^{\frac{1}{2}}}$ and $\psi'(s) = 4/(\frac{3}{2} - s^2)$, the first one is almost the spherical transform of the Poisson Kernel for the Riemannian form and the second one is the inverse of $\frac{1}{8}I + \Delta$. They clearly satisfies the conditions of Thm 1. The reason that we are not considering the inverse of Δ itself is that then the function would have a pole at $s = \pm 1$, also to avoid poles when we differentiate we include the ϵ shift in the first example. Another example of a multiplier connected to the first example is the spherical transform of the Heat kernel for the Riemannian form, $\psi(s) = e^{-t(1-s^2)}$, this satisfies the conditions of thm 1 and is entire, but it does not belong to the space $\mathbf{PW}^*(\mathfrak{b}_{\mathbf{C}}^*)^{\mathcal{W}}$ as it is not of exponential type.

4 Necessity of holomorphic extension in some rank-one cases

We shall now prove that for pseudo Riemannian symmetric spaces of split rank one with $\mathcal{W} = \{1\}$ the multipliers extend holomorphically as in the Riemannian case.

Theorem 2. *If M_ψ is a (p,p)-multiplier then $\psi(\lambda)$ has a holomorphic extension to the strip $\mathfrak{a}^* \pm i(|\frac{1}{p} - \frac{1}{p'}|\rho)$.*

Remark 4. *Note that the extension is only made around $\mathfrak{a}^*$ and hence, in general this is different from the set occuring in theorem 1.*

Proof. The idea of the proof is the same as in the Riemannian setting. First we prove that if the operator is bounded on $\mathbf{L}^p$ then it is also bounded on $\mathbf{L}^{p'}$, then we show that there is a function in $\mathbf{L}^{p'}$, which is holomorphic in the given strip and such that the operator acts by multiplication with ψ on it. For Riemannian symmetric spaces we can use spherical functions, in the present situation we use Eisenstein integrals. Let η_w be the w-component of $\eta \in \mathbf{C}^{\mathcal{W}}$(Of course under the assumptions above $\mathbf{C}^{\mathcal{W}} = \mathbf{C}$ and so η is just a constant, but as Lemma 2 and 3 below are true in general we give the full definition.) Using this we define a function on X by setting

$$\tilde{\eta}_\lambda(x) = \begin{cases} a^{i\lambda-\rho}\eta_w & \text{if } x \in HwP \\ 0 & \text{if } x \notin \bigcup_{w\in\mathcal{W}} HwP \end{cases}$$

Where P is the parabolic subgroup corresponding to the positive roots in $\Sigma(\mathfrak{a},\mathfrak{g})$. The Eisenstein integral is now defined by

$$E(\eta,\lambda)(x) = \int_K \tilde{\eta}_\lambda(x^{-1}k)dk.$$

The normalized Eisenstein integral is obtained by a transformation of η

$$E^0(\eta,\lambda) = E(C(1,\lambda)^{-1}\eta,\lambda).$$

It is normalized by its asymptotic behaviour

$$E^0(\eta,\lambda)(aw^{-1}) \sim a^{\lambda-\rho}\eta_w,$$

where $a \in A^+(correspondingto\Sigma^+(\mathfrak{a},\mathfrak{g})), w \in \mathcal{W}$ and Reλ is strictly dominant.

Lemma 2. *If a multiplier M is bounded on $\mathbf{L}^p$ then it is also bounded on the dual space $\mathbf{L}^{p'}$.*

Proof. We want to show that the dual of $.*F$ is $((.\circ\theta)*F)\circ\theta$. The result then follows because $f\circ\theta \in \mathbf{L}^p \Leftrightarrow f \in \mathbf{L}^p$.

$$\int_{H^d\backslash G^d} f * F(x)g(x)dx = \int_{G^d}\int_A f(a)F(a^{-1}x)g(x)\delta(a)dadx$$

Here we have written the convolution as an integral over H^d and A. This in turn is equal to

$$\int_A f(a)\int_{G^d} F(x)g(ax)dx\delta(a)da = \int_A f(a)((g\circ\theta)*F)(\theta(a))\delta(a)da.$$

This equality follows easily using $\theta a = a^{-1}$. Finally as the convolution is right K-invariant this is just (as K is theta invariant)

$$\int_{H^d\backslash G^d} f(x)((g\circ\theta)*F)(\theta(x))dx$$

□

The next step is to prove that M acts on the normalized Eisenstein integrals as multiplication by ψ.

Lemma 3. $(E^0(\eta,\lambda))^r * F = \psi(\lambda)(E^0(\eta,\lambda)))^r$

Proof. We write the convolution as

$$(E^0)^r * F(x) = \int_{K^d}\int_{B^r} (E^0)^r(xkb^{-1})F(b)dbdk.$$

Using the fact that the K-invariant Eisenstein integrals are joint eigenfunctions of $\mathbf{D}(G/H)$ and hence by duality that $(E^0)^r$ is a joint eigenfunction of $\mathbf{D}(G^d/K^d)$, we obtain

$$(E^0)^r * F(x) = \int_{B^r} \phi_\lambda(b^{-1})F(b)db(E^0)^r(x),$$

which is what we wanted to show. $\square$

So far we have not used the assumptions but for the remaining steps we will use them, even if it seems unlikely that they should be needed.

Lemma 4. *If we multiply the normalized Eisenstein integrals with a suitable polynomial, $p(\lambda)$, as in Lemma 14 [2], so that they become holomorphic in the strip with endpoints $\mathfrak{a} \pm i(\frac{1}{p} - \frac{1}{p'})\rho$, then for λ in the same set we obtain $\mathbf{L}^{p'}$-functions.*

Proof. As we assume that G/H has rank one and $\mathcal{W} = 1$ we can write $E^0(\lambda)(a) = \Phi_\lambda(a) + C^0_-(\lambda)\Phi_{-\lambda}(a)$. To estimate the Φ-functions we use theorem 9.1 in [3], which gives the estimate

$$|p(\lambda)\Phi_\lambda(a)| \leq C(1+|\lambda|)^{degp}a^{(-\mathrm{Im}\lambda-\rho)},$$

for $a \in A$ s.t. $\alpha(\log a) > \epsilon$ when $\alpha \in \Sigma^+(\mathfrak{a},\mathfrak{g})$. By this, together with the holomorphisity of $p(\lambda)E^0(\lambda)$ and the trivial estimate $\delta(a) \leq e^{2\rho(\log a)}$ the $\mathbf{L}^{p'}$-boundedness follows. Actually one can prove the lemma by direct computation because as we will see below the Eisenstein integral is just a hypergeometric function. $\square$

By the identity $\mathbf{L}^p(K\backslash G/H) = \mathbf{L}^p(H^d\backslash G^d/K^d)$ the lemma also shows that $(E^0)^r \in \mathbf{L}^{p'}(H^d\backslash G^d/K^d)$. To finish the proof of the theorem we have to be able to divide by $(E^0)^r$, at least evaluated at some point. But the normalized Eisenstein integral is an eigenfunction for the radial part of the Laplacian, which by Helgason [10] is

$$\delta^{-\frac{1}{2}} \circ (\mathcal{L}_A \circ \delta^{\frac{1}{2}} - \mathcal{L}_A(\delta^{\frac{1}{2}})),$$

with $\mathcal{L}_A$ denoting the Laplace operator on A. Inserting the expression $\delta(a) = \sinh^{m_+}(\log(a))\cosh^{m_-}(\log(a))$ and rewriting the result, using the change of variable $z = -\sinh^2(\log a)$, we obtain the hypergeometric equation.

$$z(1-z)f'' + (c-(a+b+1)z)f' - abf = 0,$$

where $a = \frac{\rho+\lambda}{2}, b = \frac{\rho-\lambda}{2}$ and $c = \frac{m_-+1}{2}$. Thus the normalized Eisenstein integrals are just some constant(depending on λ) times the unique solution which is regular at the origin and takes the value one there. As the partial holomorphic extension of a function and the function itself are equal on A, the same is true for $(E^0)^r$. $\square$

Remark 5. *By Flensted-Jensen [6] we can determine the discrete series. Set $\alpha = \frac{m_+-1}{2}$ and $\beta = \frac{m_--1}{2}$. The discrete series consists of the points $\{i\eta | \eta = |\beta| - (\alpha+1+2m) > 0, m \in \mathbf{N}\}$. Hence it is nonempty iff $m_- > \rho + 1$. This shows that for symmetric spaces of that kind, there exists (p,p)-multipliers for G^d/K^d which are not multipliers for G/H, if $2 \geq p > \frac{\rho}{\beta}$. For example we have the hyperbolic spaces $SO_o(p,1)/SO_o(p-1,1)$ with $p > 3$.*

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