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CONSTANT TERMS OF SPHERICAL FUNCTIONS

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Let G/H be a reductive symmetric space of H-C class with involutive automorphism σ. Let τ be a representation of K on V. A continuous function $f : G/H \to V$ is called τ-spherical if

$$f(kx) = \tau(k)f(x) \qquad (k \in K, x \in G/H).$$

Let $C^\infty(G/H, \tau)$ be the space of all τ-spherical functions in $C^\infty(G/H, V)$.

Let $\xi : G/H \to \mathbb{R}^+$ be defined by

$$\xi(gH) = ||\log a|| \qquad (g = kah \in KAH)$$

Definition 1. We define Ξ on G by

$$\Xi(g) = \int_K e^{-\rho(A(gk))}dk$$

It will be used to characterize the exponential growth order as asymptotic behaviers.

Lemma 1. *The function Ξ is analytic and satisfies the following properties:*

(1)

$$0 < \Xi(g) = \Xi(g^{-1}) \leq 1 \qquad (g \in G).$$

(2) Let L be a compact subset of G. Then there is a constant C

$$\Xi(y_1 x y_2) \leq C\, \Xi(x) \qquad (x \in G, y_1, y_2 \in L)$$

We extend the function Ξ on symmetric spaces as follows.

Definition 2. We put

$$\Theta(x) = (\Xi(x\sigma(x)^{-1}))^{1/2} \qquad (x \in G/H).$$

Θ will be used to characterize the Schwartz space on G/H.

We define a norm on G/H

$$||x||_\sigma = ||x\sigma(x)^{-1}||^{1/2}.$$

Typeset by $\mathcal{A}_{\mathcal{M}}\mathcal{S}$-TEX

For real r, we denote by $C_r(G/H)$ the space of continuous functions $f : G/H \to \mathbb{R}$ whose value

$$||f||_r = \sup_{x \in G/H} ||x||_r^{-r} |f(x)|$$

are finite.

Let $P = L_P N_P$ be a parabolic subgroup of G satisfying its Levi part $L_P = M_P A_P$ is σ-stable. Put $d_P(g) = |\det(g\sigma(g)^{-1})|_{\mathfrak{n}_P}|^{1/4}$ $(g \in L_P/H_P)$. Let $\mathfrak{a}_P$ be the Lie algebra of A_P. Put $\mathfrak{a}_P^+ = \{X \in \mathfrak{a}_P : \alpha(X) > 0, \forall \alpha \in \Sigma(P)\}$ and $A_P^+ = \exp \mathfrak{a}_P^+$. For given an ideal I of codimention finite of algebra $D(G/H)$, we define

$$C_I^\infty(G/H) = \{f \in C^\infty : Df = 0, \ \forall D \in I\}.$$

For any $r \in \mathbb{R}$, we set $\mathcal{E}_{I,r}^\infty(G/H) = C_P^\infty(G/H) \cap C_r(G/H)$.

Proposition 2. *For any ideal I of codimention finite of $D(G/H)$ and $\zeta \in X(P, I)$, there existe an integer d_ζ such that:*

(1) For real r and any $f \in \mathcal{E}_{I,r}^\infty(G/H)$, there is a function on $G : x \mapsto p_\zeta(f, x)$ into the space $P_{d_\zeta}(\mathfrak{a}_P)$ of polynomial functions on $\mathfrak{a}_P$ of degree $\leq d_\zeta$ and such that we have the following asymptotic behavier

$$f(x \exp tX) \sim \sum_{\zeta \in X(P,I)} p_\zeta(f, x, \exp tX) e^{t\zeta(X)}$$

for every $X \in \mathfrak{a}_P^+$ when $t \mapsto \infty$.

(2) For any $r \in \mathbb{R}$, there is a real number $s \geq r$ such that $f \mapsto p_\zeta(f)$ is a continuous linear map from $\mathcal{E}_{I,r}^\infty$ into $C_s^\infty \otimes P_{d_\zeta}(\mathfrak{a}_P)$, eqivariant for the left actions of G on $\mathcal{E}_{I,r}^\infty$ and C_s^∞.

We denote by $\mathcal{A}(G/H, \tau)$ the space of all $f \in C^\infty(G/H, \tau)$ satisfying the conditions: For $\forall D \in U(\mathfrak{g})$, $\exists C > 0$ and $\exists d \in \mathbb{N}$

$$|(l(D)f)(x)| \leq C(1 + \xi(x))^d \Theta(x) \qquad (\forall x \in G/H).$$

For an ideal I of finite codimention of $D(G/H)$, define a closed subspace of $\mathcal{A}(G/H, \tau)$, $\mathcal{A}_I(G/H, \tau) = \{f \in \mathcal{A}(G/H, \tau) : Df = 0, \forall D \in I\}$.

Theorem 3. *Let I be an ideal of finite codimention of $D(G/H)$. The ideal I_P of $D(L_P/H_P)$ generated by $\mu_P(I)$ is of finite codimention in $D(L_P/H_P)$. For given $f \in \mathcal{A}_I(G/H, \tau)$, there exists a unique element $f_P \in \mathcal{A}_{I_P}(L_P/H_P, \tau|_M)$ such that*

$$\lim_{\substack{a \to \infty \\ P}} \{d_P(ma) f(ma) - f_P(ma)\} = 0 \qquad (ma \in L_P/H_P)$$

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