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# Affine Weyl groups and Painlevé equations

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It is known by the work of Kazuo Okamoto in 80's that the Painlevé equations $P_{II}, P_{III}, \dots, P_{VI}$ have symmetries under the affine Weyl groups of type $A_1^{(1)}$, $C_2^{(1)}$, $A_2^{(1)}$,$A_3^{(1)}$,$D_4^{(1)}$, respectively. The general scheme of irreducibility (to measure the transcendency of solutions), developed by Hiroshi Umemura, is also related closely with this sort of symmetry of nonlinear differential equations. A key to the internal relationship between irreducibility and symmetry is the simple fact that *the classical solutions should be transformed among themselves by Bäcklund transformations.* In this talk, I will discuss some algebraic properties of Painlevé equations which can be seen from their symmetries with respect to affine Weyl groups. Also, I will touch upon some conjectures on special algebraic solutions of Painlevé equations. The main part of this talk is based on a joint work with Yasuhiko Yamada.

**1.** Let us consider the case of the fourth Painlevé equation

$$(P_{IV}) \qquad y'' = \frac{1}{2y}(y')^2 + \frac{3}{2}y^3 + 4ty^2 + 2(t^2 - a)y + \frac{b}{y}$$

for the unknown function $y = y(t)$, where $' = d/dt$ and $a, b \in \mathbb{C}$ are parameters.

One surprising fact is that this equation $P_{IV}$ is equivalent to the following system of differential equations for unknown functions $f_0$, $f_1$, $f_2$:

$$(1) \qquad \begin{aligned} f_0' + f_0(f_1 - f_2) &= \alpha_0, \\ f_1' + f_1(f_2 - f_0) &= \alpha_1, \\ f_2' + f_2(f_0 - f_1) &= \alpha_2, \end{aligned}$$

with normalization

$$(2) \qquad f_0 + f_1 + f_2 = 3x,$$

where $' = d/dx$ and $\alpha_0, \alpha_1, \alpha_2 \in \mathbb{C}$ are parameters with $\alpha_0 + \alpha_1 + \alpha_2 = 3$.

This system (1) is in fact transformed into equation $P_{IV}$ by the change of variables $f_1 = -cy$, $x = -t/c$, $c = \sqrt{-3/2}$ (together with an appropriate change of parameters).

**2.** We now consider the affine Weyl group

$$W = \langle s_0, s_1, s_2 \mid s_i^2 = 1,\ s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1} \rangle \tag{3}$$

of type $A_2^{(1)}$ acting on the parameters $\alpha_0, \alpha_1, \alpha_2$ by

$$s_i(\alpha_i) = -\alpha_i, \quad s_i(\alpha_j) = \alpha_j + \alpha_i \quad (j = i \pm 1) \tag{4}$$

as usual. (The indices are understood as elements of $\mathbb{Z}/3\mathbb{Z}$.) One can extend this action of $W$ to the variables $f_0$, $f_1$, $f_2$ in terms of the Bäcklund transformations

$$s_i(f_{i-1}) = f_{i-1} + \frac{\alpha_i}{f_i}, \quad s_i(f_i) = f_i, \quad s_i(f_{i+1}) = f_{i+1} - \frac{\alpha_i}{f_i} \tag{5}$$

for $i = 0, 1, 2$, so that the action of $W$ should commute with the derivation $' = d/dx$ acting on the variables $f_0$, $f_1$, $f_2$.

**3.** We introduce the triple of $\tau$-functions $\tau_0$, $\tau_1$, $\tau_2$ for our system (1) by

$$f_i = \frac{\tau'_{i+1}}{\tau_{i+1}} - \frac{\tau'_{i-1}}{\tau_{i-1}} + x \qquad (i = 0, 1, 2) \tag{6}$$

with a certain normalization of overall factors. Then the Painlevé equation for these variables are given by the Hirota bilinear equations

$$(D_x^2 - xD_x - \frac{\alpha_i - \alpha_{i+1}}{3})\ \tau_i \cdot \tau_{i+1} = 0 \qquad (i = 0, 1, 2). \tag{7}$$

The Bäcklund transformations described above lift also to the level of $\tau$-functions:

$$s_i(\tau_i) = \frac{\tau_{i+1}\tau_{i-1}}{\tau_i} f_i = \frac{1}{\tau_i}(D_x + x)\tau_{i+1} \cdot \tau_{i-1} \tag{8}$$

and $s_i(\tau_j) = \tau_j$ $(i \neq j)$, for $i, j = 0, 1, 2$.

**4.** Our system (7) can be understood as a certain similarity reduction of the modified KP hierarchy. From this fact, one can show in particular that the so-called *Okamoto polynomials*, which were introduced earlier by Okamoto as $\tau$-functions for rational solutions of the fourth Painlevé equations, are in fact expressible in terms of the 3-reduced Schur functions.

## References


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