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Factorization of invariant differential equations

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We will give a remark on differential equations which are satisfied by Heckman-Opdam's hypergeometric function associated with a root system.

Example

We start from the Gauss hypergeometric differential equation,

$$z(1-z)\frac{d^2w}{dz^2}+\{c-(a+b+c)z\}\frac{dw}{dz}-ab=0,$$

where a, b, c are complex numbers such that $c \neq 0, -1, -2, \ldots$. The Gauss hypergeometric function

$$F(a,b;c;z)=\sum_{k=0}^{\infty}\frac{(a)_k(b)_k}{k!(c)_k}z^k$$

gives a unique solution of the hypergeometric differential equation which is analytic at the origin and $w(0)=1$. Here we put

$$(\lambda)_k=\lambda(\lambda+1)\cdots(\lambda+k-1).$$

If $b=0$, then the hypergeometric differential equation factorizes;

$$\left(z(1-z)\frac{d}{dz}+\{c-(a+c)z\}\right)\frac{dw}{dz}=0$$

and the hypergeometric function $F(a,0;c;z)\equiv 1$ gives a unique solution of

$$\frac{dw}{dz}=0$$

up to a constant multiple.

By a change of variable, the Gauss hypergeometric function becomes a Heckman-Opdam's hypergeometric function associated with the root system A_1.

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Goal Generalize this example for arbitrary root systems.

Notice that the hypergeometric differential equation should be replaced by a system of differential equations and the factorization of the differential operator in the above example should be replaced by a factorization in the sense of D-module.

Commuting family of invariant differential operators

Let us consider the case of the root system A_{n-1}. Let $n \geq 2$ be an integer and $k \in \mathbb{C}$. We define

$$L(k) = \sum_{i=1}^{n} \left(z_i \frac{\partial}{\partial z_i} \right)^2 + k \sum_{1 \leq i < j \leq n} \frac{z_i + z_j}{z_i - z_j} \left(z_i \frac{\partial}{\partial z_i} - z_j \frac{\partial}{\partial z_j} \right).$$

For $k = 1/2$, 1 or 2, $L(k)$ is the radial part of the Laplace-Beltrami operator on Riemannian symmetric space $GL(n, \mathbb{F})/O(n, \mathbb{F})$, where $\mathbb{F} = \mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$ respectively.

There exists a commuting family of differential operators $L_1, \ldots, L_n$ such that

$$\begin{aligned} L_1 &= \sum_{i=1}^{n} z_i \frac{\partial}{\partial z_i}, \\ L_2 &= L(k) + \frac{k^2}{12}(n^3 - n), \\ L_m &= \sum_{i=1}^{n} \left(z_i \frac{\partial}{\partial z_i} \right)^m + \text{(lower order terms)}, \end{aligned}$$

which are called the *Sekiguchi operators.* For $k = 1/2$, 1 or 2, they are radial parts of invariant differential operators on the corresponding symmetric space.

Constant coefficients case

For $k = 0$, the associated eigenvalue problem is very simple. We put $p_m(x) = \sum_i x_i^m$ $(1 \leq m \leq n)$. It is well known that $\{p_1, \ldots, p_n\}$ gives an algebraically independent set of generator of symmetric polynomials in x. For $\lambda \in \mathbb{C}^n$ we consider the system of differential equations,

$$p_m(\partial)u = p_m(\lambda)u \quad (1 \leq m \leq n). \tag{1}$$

If λ is regular, that is $\lambda_i \neq \lambda_j$ for $i \neq j$, then the above system has $n!$ linearly independent solutions $\exp\langle \sigma\lambda, x \rangle$ $(\sigma \in S_n)$. If λ is not regular, then there still exist $n!$ linearly independent solutions, but logarithmic terms occur.

Conversely, we start from the vector space

$$\operatorname{span}\{\exp\langle\sigma\lambda, x\rangle\,;\,\sigma \in S_n\}. \tag{2}$$

If λ is not regular, then this space is a proper subspace of the solution space of (1). Let I_λ denote the set of polynomials that vanish on the set $\{\sigma\lambda\,;\,\sigma \in S_n\}$. Then the space (2) is characterized as a solution space of the system,

$$p(\partial)u = 0, \quad p \in I_\lambda.$$

We give a description of a generator of the ideal I_λ. Let $0 = n_0 < n_1 < \cdots < n_L = n$ be a sequence of integers and $\mu \in \mathbb{C}^L$. Define $\lambda \in \mathbb{C}^n$ by

$$\lambda_i = \mu_j \text{ for } n_{j-1} < i \leq n_j\,(j = 1, \ldots, L).$$

For $1 \leq j \leq L$ we put $m_j = n - n_j + n_{j-1} + 1$. Assume that $\mu \in \mathbb{C}^L$ is regular. Then $\{\prod_{l=1}^{m_j}(x_{i_l} - \mu_j)\}$, where j and i_k vary so that $j = 1, \ldots, L$ and $1 \leq i_1 < \cdots < i_{m_j} \leq n$, gives a set of generators of the ideal I_λ.

k deformation case

Now we consider the case $k \neq 0$. The commuting family of differential operators can be constructed by means of the Cherednik operator, there are solutions of the eigenvalue problem that are given by the Harish-Chandra series, and there exists a unique solution that is analytic at the origin and has value 1 there, that is Heckman-Opdam's hypergeometric function.

We have a result that is similar to the constant coefficient case. There are cumbersome *rho* shifts, which can be understood in two ways; a) characteristic exponents of the hypergeometric function, b) weights of an induced module for the graded affine Hecke algebra (cf. Opdam[4]). (The latter is valid for $k > 0$ and λ belongs the weight lattice.)

We review on construction of $\{L_m\}$ by means of the Cherednik operator. Define

$$D_i = z_i\frac{\partial}{\partial z_i} + k\sum_{i<j}\frac{z_j}{z_i - z_j}(1 - \sigma_{ij}) + k\sum_{i>j}\frac{z_i}{z_i - z_j}(1 - \sigma_{ij}) + \left(\frac{n+1}{2} - i\right)k,$$

where $\sigma = \sigma_{ij} \in S_n$ acts on a function f in $z = (z_1, \ldots, z_n)$ by $\sigma_{ij}f(z) = f(z_{\sigma(1)}, \ldots, z_{\sigma(n)})$. Then

$$L_m = \sum_{i=1}^{n} D_i^m|_{S_n\text{-invariant functions}} \quad (1 \leq m \leq n)$$

give the Sekiguchi operators.

Put $z_i = e^{x_i}$ and consider eigenvalue problems on $x \in \mathbb{R}^n$;

$$L_m u = p_m(\lambda) u \quad (1 \leq m \leq n)$$

for $\lambda \in \mathbb{C}^n$. Heckman and Opdam proved that for $k \neq -1/2, -3/2, \dots$ there exists a unique S_n-invariant analytic solution $F(\lambda, k; x)$ such that $F(\lambda, k; 0) = 1$. For $k = 1/2$, 1 or 2, it is the radial part of the zonal spherical function on the corresponding symmetric space.

If λ is a degenerate eigenvalue, then $F(\lambda, k; x)$ satisfies more differential equations. Let $0 = n_0 < n_1 < \cdots < n_L = n$ be a sequence of integers and $\mu \in \mathbb{C}^L$. Define $\lambda \in \mathbb{C}^n$ by

$$\lambda_i = \mu_j + (-(n+1)/2 + n_j + n_{j-1} + 1 - i)k \text{ for } n_{j-1} < i \leq n_j \; (j = 1, \dots, L).$$

Then $u(x) = F(\lambda, k; x)$ satisfies

$$\prod_{l=1}^{m_j} \left(D_{i_l} + \left(-\frac{n+1}{2} + i_l \right) k - \mu_j + (m_j - n_{j-1} - l)k \right) \Bigg|_{S_n\text{-invariant functions}} u = 0,$$

where $m_j = n - n_j + n_{j-1} + 1$, $1 \leq i_1 < \cdots < i_{m_j} \leq n$ for $j = 1, \dots, L$.

For $k = 1/2$, 1 or 2, they are radial parts of differential operators on the corresponding symmetric space that characterize the image of the Poisson transform from degenerate boundary components ([5]).

Example

We give an example of the case $n_1 = 1$, $n_2 = n$. We put $\mu_2 = \nu$ and $\mu_1 = -(n-1)\nu$ so that $\sum_i \lambda_i = 0$. Then

$$\lambda = \left(-(n-1)\nu - \frac{n-1}{2}k, \nu + \frac{n-1}{2}k, \dots, \nu + \frac{-n+3}{2}k \right).$$

If $k = 0$, then

$$\{(x_i - \nu)(x_j - \nu)\,;\, 1 \leq i < j \leq n,\, x_1 + \cdots x_n\}$$

gives a set of generator of the ideal I_λ. The k-deformation of $x_i x_j$ for $1 \leq i < j \leq n$ is given by

$$\left(x_i + k\left(-\frac{n+1}{2} + i \right) - \nu \right) \left(x_j + k\left(-\frac{n+1}{2} + j \right) - \nu - k \right).$$

Substituting the Cherednik operator to x and restricting to symmetric functions, it gives a differential operator

$$z_i z_j \frac{\partial^2}{\partial z_i \partial z_j} - \frac{k}{2}\frac{z_i + z_j}{z_i - z_j}\left(z_i \frac{\partial}{\partial z_i} - z_j \frac{\partial}{\partial z_j} \right) + \left(\nu + \frac{k}{2} \right)\left(z_i \frac{\partial}{\partial z_i} + z_j \frac{\partial}{\partial z_j} \right).$$

By a change of variables these operators become differential operators that are satisfied by Appell's F_D, which was shown by J. Sekiguchi.

We can generalize the above result for arbitrary root systems. For the root system BC_n, hypergeometric systems of Debiard-Gaveau and Korányi-Yan can be obtained by the construction by means of the Cherednik operator.

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