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AN INEQUALITY AMONG INFINITESIMAL CHARACTERS OF "M"

Dedicated to Professor Reiji Takahashi on the occasion of his seventieth birthday

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1. INTRODUCTION

Let G be a connected real semi-simple Lie group with finite center, and K a maximal compact subgroup of G. We denote by $\mathfrak{g}$ and $\mathfrak{k}$ the Lie algebras of G and K respectively. Let $G = KAN$ be an Iwasawa decomposition of G and M a centralizer of A in K.

A famous Parthasarathy's Dirac operator inequality in [P] (see also, [BW, Ku, VZ]) asserts that the length of the highest weight of a representation of $\mathfrak{k}$ occuring in the Harish-Chandra module of an irreducible unitary representation of G must be at least the eigenvalue of the Casimir operator of $\mathfrak{g}$.

In the present note, we shall give some simple inequality for the infinitesimal characters of irreducible representations of M. This inequality resembles the Dirac operator inequality in character. In fact, it relates to the discrete series of G via a lowest K-type.

Since the group M is, in general, considerably small in K it seems hard to expect any inequalities among characters of representations of M which are obtained by the restriction of representations of K. Moreover, a proper meaning of the group M is somewhat mysterious in spite of its structual definition is clear. In this sense, it is important to ask roles of M from various point of view. This is the aim of the study on a comparison among representations of M. In fact, we shall show that a "length" of the *dominant M-type* (see §2 for the precise definition) of the lowest K-type of a discrete series of G dominates all the other such dominant M-types which appear in a Weyl group-"orbit" of the lowest K-type. The inequality may have also a possibility to provide an information about a "scale" of parameters among various embedding

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of discrete series into non-unitary principal series induced from a minimal parabolic subgroup MAN. We shall lastly propose some questions concerning the inequality.

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2. Statement and proof

Assume that $\operatorname{rank}(K) = \operatorname{rank}(G)$. Then G contains a Cartan subgroup T which lies in K. Let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be a Cartan decomposition of $\mathfrak{g}$. Let $(\ ,\ ) = B|_{\mathfrak{p}\times\mathfrak{p}}$, where B is the Cartan-Killing form of $\mathfrak{g}$. Let $\Delta = \Delta(\mathfrak{g}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}})$ be the set of non-zero roots of the pair $(\mathfrak{g}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}})$, let $\Delta_{\mathfrak{k}}, \Delta_{\mathfrak{n}}$ be the set of compact, noncompact roots respectively, i.e. $\Delta_{\mathfrak{k}} = \Delta_{\mathfrak{k}}(\mathfrak{k}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}})$ and $\Delta_{\mathfrak{n}} = \Delta\backslash\Delta_{\mathfrak{k}}$. Let $W(\mathfrak{g}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}})$ be a Weyl group for the root system Δ. Let $\mathrm{ad} : \mathfrak{k} \to SO(\mathfrak{p}, (\ ,\ ))$ be the adjoint represenatation. If (σ, S) is the spin representation of $SO(\mathfrak{p}, (\ ,\ ))$ defined through the Clliford algebra of $\mathfrak{p}$, let L be the composition of σ and $\mathrm{ad}\ \mathfrak{k}|_{\mathfrak{p}}$. Since the dimension of $\mathfrak{p}$ is always even, we have an irreducible decomposition of σ ; $\sigma = \sigma^+ \oplus \sigma^-$, where $\sigma^{\pm}$ are the half-spin representations. Set $L^{\pm} = \sigma^{\pm} \circ \mathrm{ad}$. For any choice of positive roots $\Delta^+ \subset \Delta$ put $\Delta_{\mathfrak{k}}^+ = \Delta^+ \cap \Delta_{\mathfrak{k}}, \Delta_{\mathfrak{n}}^+ = \Delta^+ \cap \Delta_{\mathfrak{n}}, 2\delta = \langle\Delta^+\rangle, 2\delta_{\mathfrak{k}} = \langle\Delta_{\mathfrak{k}}^+\rangle$ and $2\delta_{\mathfrak{n}} = \langle\Delta_{\mathfrak{n}}^+\rangle$, where generally we write $\langle\Phi\rangle = \sum_{\alpha\in\Phi}\alpha$ for each subset $\Phi \subset \Delta$. Then it is known that the weights of (L, S) are of the form $\delta_{\mathfrak{n}} - \langle Q\rangle$, where $Q \subset \Delta_{\mathfrak{n}}^+$. We fix (L^+, S^+) so that $\delta_{\mathfrak{n}}$ is a weight of L^+. If $\lambda \in \mathfrak{t}_{\mathbb{C}}^*$ is a $\Delta_{\mathfrak{k}}^+$-dominant integral weight, τ_λ will denote the irreducible represenatation of $\mathfrak{k}_{\mathbb{C}}$ with highest weight λ. Each weight of L occurs with multiplicity one. In fact, Parthasarathy showed in [P] that

$$L^+ = \oplus_{\substack{s\in W^1,\\ \det s=1}} \tau_{s\delta-\delta_{\mathfrak{k}}}, \quad L^- = \oplus_{\substack{s\in W^1,\\ \det s=-1}} \tau_{s\delta-\delta_{\mathfrak{k}}},$$

where W^1 is a subgroup of $W(\mathfrak{g}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}})$ defined by

$$W^1 = \{s \in W(\mathfrak{g}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}})\,;\, s\Delta^+ \supset \Delta_{\mathfrak{k}}^+\}.$$

We now recall some well-known results on the discrete series of G (see e.g. [K]). Let $\mathcal{L}$ denote the weight lattice of T. Harish-Chandra has constructed a family of invariant eigendistributions $\Theta_{\lambda(\Delta^+)}$, where $\lambda \in \mathcal{L} + \delta$ and Δ^+ is a system of positive roots such that λ is Δ^+-dominant. When λ is regular, Δ^+ is uniquely determined by λ and $\Theta_{\lambda(\Delta^+)}$ gives the character of a discrete series $\omega(\lambda) = \omega(\lambda, \Delta^+)$. Any such character arises in this way. Further the lowest K-type of $\omega(\lambda)$ is given by $\tau_{\lambda-\delta_{\mathfrak{k}}+\delta_{\mathfrak{n}}}$.

Let $\alpha_1, \dots, \alpha_\ell$ be a fundamental sequence of positive non-compact roots (see, [KW]). It is known that we may associate to this sequence a canonical Iwasawa decomposition of $\mathfrak{g}$ as follows. Let $\mathfrak{a}$ be the maximal abelian subspace of $\mathfrak{p}$ given by $\mathfrak{a} = \sum_{j=1}^{\ell}\mathbb{R}(E_{\alpha_j} + E_{-\alpha_j})$, where E_α represents the root vector corresponding to the root $\alpha \in \Delta$. Form restricted roots with respect to $\mathfrak{a}$, and define an ordering on the restricted roots by the basis $E_{\alpha_1} + E_{-\alpha_1}, \dots, E_{\alpha_\ell} + E_{-\alpha_\ell}$. Let $\mathfrak{n}$ be the sum of the positive restricted root spaces. Then we have an Iwasawa decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}$. Let A and N be the analytic subgroups of G with Lie algebras $\mathfrak{a}$ and $\mathfrak{n}$ respectively, M for the centralizer of $\mathfrak{a}$ in K. Let $\mathfrak{m}$ be a Lie algebra of M.

Let (τ_μ, V_μ) be an irreducible representation of K (and of $\mathfrak{k}$) with highest weight μ, and let v_μ be a nonzero highest weight vector. We denote by σ_μ the restriction of an M-module $\tau_\mu(M)$ to the M-cyclic subspace generated by v_μ. Let H_μ be the subspace of V_μ in which σ_μ operates. For simplicity, we shall put the following assumption on G:

Assumption. *G has real rank one, or M is connected, or G/K is Hermitian symmetric.*

Under this assumption the representation σ_μ of M is *irreducible*. We call this representation σ_μ the *dominant M-type* of τ. The highest weight of σ_μ on the Cartan subalgebra $\mathfrak{h}_\mathfrak{m} = \mathfrak{t} \ominus \sqrt{-1}\sum_{j=1}^{\ell} \mathbb{R}H_{\alpha_j}$ of $\mathfrak{m}$, with the relative ordering, is $\mu|_{\mathfrak{h}_\mathfrak{m}}$, and v_μ is a highest weight vector. Let χ_μ be an corresponding infinitesimal character of σ_μ. We denote by Ω_M the Casimir element of M. Let $\Delta_\mathfrak{m} = \Delta(\mathfrak{m}, \mathfrak{h}_\mathfrak{m}) = \{\alpha \in \Delta;\ (\alpha, \alpha_j) = 0\}$ and put $2\rho_\mathfrak{m} = \langle \Delta_\mathfrak{m}^+ \rangle$. If $\mu \in \mathfrak{t}_\mathbb{C}^*$ the restriction of μ to $\mathfrak{h}_\mathfrak{m}$ is precisely given by

$$\mu|_{\mathfrak{h}_\mathfrak{m}} = \mu - \frac{1}{2}\sum_{i=1}^{\ell} \langle \mu, \alpha_i \rangle \alpha_i, \tag{2.1}$$

where we define $\langle\, ,\, \rangle$ by the formula $\langle \mu, \alpha \rangle = \frac{2(\mu, \alpha)}{|\alpha|^2}$, and $|\alpha| = \sqrt{(\alpha, \alpha)}$.

We recall some fact concerning virtual represenatations of K and its restriction to M. Let $R(K), R(M)$ be the representation rings of K, M respectively and let $\iota : R(K) \to R(M)$ denote the ring homomorphism induced by the natural restriction inherited from the inclusion $M \subset K$. If $\lambda \in it^*$ is $\Delta_\mathfrak{k}^+$-dominant integral, then $\tau_\lambda \otimes L^+$ (or $\tau_\lambda \otimes L^-$) integrates to a represenattion of K if and only if $\lambda + \delta_\mathfrak{n} \in \mathcal{L}$. We take a finite covering $p : \tilde{G} \to G$ such that $\delta_\mathfrak{n} \in \tilde{\mathcal{L}}$, the weight lattice of $\tilde{T} = p^{-1}(T)$. If $\tilde{K} = p^{-1}(K)$ then $R(K)$ is identified with the obvious subring of $R(\tilde{K})$. We now have the following characterization of $\ker \iota$ (see [M]) .

Lemma. *Suppose that* $\mathrm{rank}(K) = \mathrm{rank}(G)$. *Then we have*

$$\ker \iota = \{\tau \otimes (L^+ - L^-)\, ;\ \tau \in R(\tilde{K}) \quad \textit{and} \quad \tau \otimes L^\pm \in R(K)\}.$$

In other words, $\ker \iota$ *is freely generated by*

$$\{\eta_{\lambda-\delta_\mathfrak{k}} := \tau_{\lambda-\delta_\mathfrak{k}} \otimes (L^+ - L^-)\, ;\ \lambda - \delta_\mathfrak{k} \quad \textit{is} \quad \Delta_\mathfrak{k}^+ - \textit{dominant and} \quad \lambda - \delta_\mathfrak{k} + \delta_\mathfrak{n} \in \mathcal{L}\}.$$

This lemma leads us to focus on naturally the irreducible components of $\eta_{\lambda-\delta_\mathfrak{k}}$. Namely we look at the following irreducible decomposition of the tensor product:

$$\tau_{\lambda-\delta_\mathfrak{k}} \otimes \tau_{s\delta-\delta_\mathfrak{k}} = \tau_{\lambda+s\delta-2\delta_\mathfrak{k}} \oplus \sum_\mu \tau_\mu. \tag{2.2}$$

Note that the representation $\tau_{\lambda+s\delta-2\delta_\mathfrak{k}}$ occurs exactly once in this decomposition. We remark further that $\lambda + s\delta - 2\delta_\mathfrak{k} = \Lambda - (\delta - s\delta)$ for $s \in W^1$, where Λ is given by $\Lambda = \lambda - \delta_\mathfrak{k} + \delta_\mathfrak{n}$, the weight of the lowest K-type of $\omega(\lambda)$. The following theorem asserts that the value of the Casimir Ω_M at each dominant M-type of $\tau_{\lambda+s\delta-2\delta_\mathfrak{k}}$ is at most that of τ_Λ.

Theorem. *Let G be a connected real semi-simple Lie group with finite center. We retain the notation above. Suppose that λ is a regular dominant integral form in $\mathfrak{t}^*$ Put $\Lambda = \lambda - \delta_{\mathfrak{k}} + \delta_{\mathfrak{n}} (= \lambda - 2\delta_{\mathfrak{k}} + \delta)$. Then the inequality*

$$\chi_{\Lambda|_{\mathfrak{h}_{\mathrm{m}}}}(\Omega_M) \geq \chi_{\Lambda_Q|_{\mathfrak{h}_{\mathrm{m}}}}(\Omega_M) \tag{2.3}$$

holds for all $Q = Q_s = \delta - s\delta$ ($s \in W^1$). Here we put $\Lambda_Q = \Lambda - Q = \lambda - \delta_{\mathfrak{k}} + s\delta$.

Proof. Since $\chi_{\mu|_{\mathfrak{h}_{\mathrm{m}}}}(\Omega_M) = |\mu|_{\mathfrak{h}_{\mathrm{m}}} + \rho_{\mathrm{m}}|^2 - |\rho_{\mathrm{m}}|^2$ and we have a relation $\delta_{\mathfrak{k}} - \delta_{\mathfrak{n}}|_{\mathfrak{h}_{\mathrm{m}}} = \rho_{\mathrm{m}}$ (see (8.1) in [KW]) we see that

$$\begin{aligned}
\chi_{\Lambda|_{\mathfrak{h}_{\mathrm{m}}}}(\Omega_M) - \chi_{\Lambda_Q|_{\mathfrak{h}_{\mathrm{m}}}}(\Omega_M) =& |\Lambda|_{\mathfrak{h}_{\mathrm{m}}} + \rho_{\mathrm{m}}|^2 - |\Lambda_Q|_{\mathfrak{h}_{\mathrm{m}}} + \rho_{\mathrm{m}}|^2 \\
=& |\lambda|_{\mathfrak{h}_{\mathrm{m}}}|^2 - |\lambda|_{\mathfrak{h}_{\mathrm{m}}} - Q|^2 \\
=& (Q|_{\mathfrak{h}_{\mathrm{m}}}, 2\lambda - Q|_{\mathfrak{h}_{\mathrm{m}}}) \\
=& (2\lambda - Q, Q) - \sum_{i=1}^{\ell} \frac{(Q, \alpha_i)}{|\alpha_i|^2}(2\lambda - Q, \alpha_i).
\end{aligned}$$

We put here $\lambda = \delta + \mu$, where μ is a dominant integral form for Δ^+. Note the facts that $(2\delta - Q, Q) = (\delta + s\delta, \delta - s\delta) = |\delta|^2 - |s\delta|^2 = 0$ and $\frac{(2\delta, \alpha_i)}{|\alpha_i|^2} = \langle \delta, \alpha_i \rangle = 1$. It follows that the last expression of the formula above is turned to be

$$(2\mu, Q) - \sum_{i=1}^{\ell} (Q, \alpha_i)\Big\{1 - \frac{(Q, \alpha_i)}{|\alpha_i|^2} + \frac{(2\mu, \alpha_i)}{|\alpha_i|^2}\Big\}. \tag{2.4}$$

Let us denote Q as in the form

$$Q = \sum_{i=1}^{\ell} m_i \alpha_i + R. \tag{2.5}$$

Here $R = R_s$ is a sum of simple roots but it does not include any α_i. We note that m_i is a non-negative integer and $(R, \alpha_j) \leq 0$ for any j. Then we see that (2.4) can be rewritten as

$$(2\mu, R) - \sum_{i=1}^{\ell} \langle \mu, \alpha_i \rangle (R, \alpha_i) - \frac{1}{2} \sum_{i=1}^{\ell} \langle Q, \alpha_i \rangle |\alpha_i|^2 + \frac{1}{4} \sum_{i=1}^{\ell} \langle Q, \alpha_i \rangle^2 |\alpha_i|^2. \tag{2.6}$$

Since $(2\mu, R) \geq 0$, $\langle \mu, \alpha_i \rangle \geq 0$ and $(R, \alpha_i) \leq 0$ we finally obtain

$$\chi_{\Lambda|_{\mathfrak{h}_{\mathrm{m}}}}(\Omega_M) - \chi_{\Lambda_Q|_{\mathfrak{h}_{\mathrm{m}}}}(\Omega_M) \geq \frac{1}{4} \sum_{i=1}^{\ell} \langle Q, \alpha_i \rangle \Big(\langle Q, \alpha_i \rangle - 2 \Big) |\alpha_i|^2. \tag{2.7}$$

Since $\langle Q, \alpha_i \rangle \leq 0$ or ≥ 2, it is clear that the right hand is non-negative. In fact, it follows from the following simple observation; since $\langle Q, \alpha_i \rangle = \langle \delta - s\delta, \alpha_i \rangle = \langle \delta, \alpha_i \rangle - \langle \delta, s^{-1}\alpha_i \rangle = 1 - \langle \delta, s^{-1}\alpha_i \rangle$, it suffices to check the property that if $s^{-1}\alpha_i$ is negative (resp., positive) then $\langle \delta, s^{-1}\alpha_i \rangle \leq -1$ (resp., $\langle \delta, s^{-1}\alpha_i \rangle \geq 1$), but this is obviously true. This concludes the proof. □

Corollary. *We keep the notation and assumption of Theorem. Then the following equation holds.*

$$
(2.8)\qquad
\begin{aligned}
&\chi_{\Lambda|_{\mathfrak{h}_{\mathfrak{m}}}}(\Omega_M) - \chi_{\Lambda_Q|_{\mathfrak{h}_{\mathfrak{m}}}}(\Omega_M) \\
=&(2\lambda - 2\delta, R_s) - \sum_{i=1}^{\ell}\langle\lambda - \delta, \alpha_i\rangle(R, \alpha_i) + \frac{1}{4}\sum_{i=1}^{\ell}(\langle\delta, s^{-1}\alpha_i\rangle^2 - 1)|\alpha_i|^2,
\end{aligned}
$$

where R_s is defined by $Q = \delta - s\delta$ via the equation (2.5). □

Remarks. (1) By the corollary above, the equality in (2.3) holds if and only if $s \in W^1$ (or R_s or Q_s) and λ (or $\mu = \lambda - \delta$) satisfy the following conditions;

$$(R|_{\mathfrak{h}_{\mathfrak{m}}}, \mu) = 0, \quad \langle\delta, s^{-1}\alpha_j\rangle = \pm 1 \ (1 \leq j \leq \ell).$$

Here note that

$$(R|_{\mathfrak{h}_{\mathfrak{m}}}, \mu) = (\mu|_{\mathfrak{h}_{\mathfrak{m}}}, R) = (\mu, R) - \frac{1}{2}\sum_{i=1}^{\ell}\langle\mu, \alpha_i\rangle(R, \alpha_i).$$

(2) In view of the proof of Theorem it is clear that an inequality

$$|\Lambda|_{\mathfrak{h}_{\mathfrak{m}}} + \rho_{\mathfrak{m}}| \geq |\Lambda_Q|_{\mathfrak{h}_{\mathfrak{m}}} + \rho_{\mathfrak{m}}|$$

holds for any $Q = Q_s$ for $s \in W(\mathfrak{g}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}})$. It also shows that Theorem and Corollary remain true for any G without the assumption: $\mathrm{rank}(K) = \mathrm{rank}(G)$.

(3) Without Assumption, the irreducibility of the representation (σ_μ, H_μ) of M does not true in general. But if we regard (σ_μ, H_μ) as a representation of $\mathfrak{m}$ then the theorem remains true in an appropriate sense. Moreover, among the classical simple groups, Assumption can fail only for groups locally isomorphic to $SO(m, n)$.

We close this note by proposing some problems. Since the group M and its Lie algebra $\mathfrak{m}$ are defined via a choice of a fundamental sequence of positive non-compact roots, say $F = F(\alpha_1, \ldots, \alpha_\ell)$. For example, if G is of real rank one, then each non-compact simple root defines such a fundamental sequence (see, [KW]). Thus we denote it by $M(F)$ (resp. $\mathfrak{m}(F)$) for specifying the defining sequence F. It naturally comes to the following

Questions. *Put*

$$C(\Lambda) = \max_F \chi_{\Lambda|_{\mathfrak{h}_{\mathfrak{m}(F)}}}(\Omega_{M(F)}).$$

(1) *It is conjectured that the inequalities $C(\Lambda) \geq \chi_\sigma(\Omega_{M(F)})$ would hold for all irreducible representations σ appearing in the restriction $\iota(\tau)$ of τ to $M(F)$. Here τ represents an irreducible summand in $\eta_{\lambda-\delta_{\mathfrak{k}}}$.*
(2) *Which $\chi_{\Lambda|_{\mathfrak{h}_{\mathfrak{m}(F)}}}(\Omega_M)$ does attain the maximam value $C(\Lambda)$? Describe the condition in terms of F. Moreover, when does $\chi_\sigma(\Omega_{M(F)})$ attain the maximam?*

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