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# On the Pompeiu problem and its equivalent forms

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The Romanian mathematician, Dimitru Pompeiu in [5] posed the probably most famous long standing open problem of Integral Geometry, the so called Pompeiu problem. In **MathSci Disc** (the CD-ROM version of the Mathematical Reviews) one can find from 1940 to 12/1994 about 307 item containing the name "Pompeiu". Because he did not invent any other significant problem, we can be sure that all the items what we found are references to partial results on the Pompeiu problem. A more sophisticated search can produce about 500 item in close connection with it. This is an extraordinary number for a single problem. Sometimes whole branches of Mathematics does not have more than 200 papers. The reason of this richness is that the problem has connection with many different areas of mathematics. For example in harmonic analysis, overdetermined Neumann problems, Lie group representation theory and even it has connections with combinatorics, as I showed in [4].

It is impossible to refer even for the most significant results and methods in connection with the Pompeiu problem. The results and methods mentioned in this note naturally cannot be regarded as a short survey. It strongly mirrors my personal taste and present knowledge. (The situation changes in almost every week.)

First we state the problem explicitly, in its most popular form. Let $f$ be a continuous function and let $\Omega$ be a bounded open domain with connected Lipschitzian boundary in the $d$ dimensional real vector space $R^d$ having the property that the integral of the function $f$ is zero on every congruent copy of the domain $\Omega$. If $f$ must be zero, then we say that the domain $\Omega$ has the Pompeiu property.

The only known bounded open domains which fails to have the Pompeiu property are the balls. The long standing conjecture of Raleigh, Pompeiu and Schiffer that there are no other $\Omega$'s failing the Pompeiu property.

On the other hand, assume that $\Omega$ is the unit ball with radius 1 and $J_{d/2-1}(\lambda) = 0$ for a $\lambda > 0$, where $J_\nu$ is the Bessel function with parameter $\nu$. If $\omega \in S^{d-1}$, the integral of the function $\cos\lambda\langle\omega, x\rangle$, where $S^{d-1}$ is the unit sphere and $\langle\, .\, ,\, .\, \rangle$ is the standard inner product, vanish on every ball with radius 1.

**Conjecture 1** *(Schiffer) Let $\Omega \subseteq R^d$ be an open domain with the previous assumptions such that the overdetermined Neumann problem $\Delta u + \lambda u = 0$ in $\Omega$, $u|_{\partial\Omega} = const.$, $\partial u/\partial\nu|_{\partial\Omega} = 0$ has an eigenvalue $\lambda > 0$; then $\Omega$ must be a disk.*

The following equivalent form states:

**Conjecture 2** *If the null variety (that is the zero set of the Fourier transform of the characteristic function) of $\Omega$ contains a sphere then $\Omega$ must be a ball.*

The previous two conjectures are known to be equivalent to the Pompeiu problem. Investigating the equivalent forms is important, because it broadens the range of the possible devices to attack the problem.

The asymptotic behavior of the null variety of strictly convex sets was investigated by Toshiyuki Kobayashi in [6]. He shows that from the asymptotic behavior of the null variety of strictly convex sets one can reconstruct the set itself (up to translation). In higher dimensions from the null variety he deduces the with function and the ratio of the Gaussian curvatures of the antipodal points (having parallel support planes) on the boundary. It is a long standing conjecture in convex geometry due to Bonnensen and Fenchel that this data determines the convex domain up to parallel displacements.

He also investigate the perturbation of $\Omega$ in [7]. He shows that in an sophisticated metric if a domain $\Omega$ having the Pompeiu problem is close to the unit ball then it must be a ball. The proof and even the exact statement is very sophisticated, so we do not state the latter explicitly.

Interesting result of Brown and Kahane [8], which state that "long" convex domains have the Pompeiu property in the plane.

**Theorem.** (Brown–Kahane) Assume that $\Omega$ is a convex domain in $R^2$. If

$$2 \min_{\omega \in S^1} h(\Omega, \omega) \leq \max_{\omega \in S^1} h(\Omega, \omega),$$

hold, where $h$ is the support function of $\Omega$, then the null variety of $\Omega$ cannot contain hypersphere with radius $\alpha$ for any $\alpha \in C \setminus \{0\}$. (A hypersphere with radius $\alpha$ is an algebraic surface in $C^d$ defined by the equation $z_1^2 + \ldots + z_d^2 - \alpha = 0$.)

I proved in [3] and [10] that the null variety of a non zero distribution with compact support cannot contain hypersphere with radius $\alpha$ in its null variety if $\alpha$ is not real.

And last but not least from our point of view one of the most important result is due to S. M. Williams.

**Theorem.** (Williams) *Assume that $\Omega \subset R^d$ is a bounded domain with connected Lipschitzian boundary not having the Pompeiu property. Then $\partial\Omega$ is analytic.*

This result is quite interesting, because it shows the strange nature of the problem. Usually the weaker the differentiability conditions, the harder the problem (see for example Hilbert's fourth and fifth problems). But for the Pompeiu problem the opposite holds!

# 1 My work in the field

I have a far more simple proof for a slightly different version of Kobayashi's perturbation theorem [7]. The proof was part of my Ph.D. dissertation [10], and [3]. In [3] I use a new method, which is a combination of the perturbation method and an extremum method, which has very fruitful applications in some problems of higher dimensional $X$-ray reconstruction of convex bodies. These methods are presented in [1, 2, 10].

One of my result [4] shows that the Pompeiu problem has connection even with combinatorics.

Discrepancy is the measure how uniform is the distribution of points in the unit ball. We measure this discrepancy using congruent copies of convex bodies.

Let $\mathcal{P}_n = \{P_1, \ldots, P_n \in B^d\}$ be an $n$ element point set. $\Pi_n$ denote the set of at most $n$ element point sets in the unit ball $B^d$. For any Borel set $B \in \mathcal{B}(R^d)$, let

$$Z_0(B, \mathcal{P}_n) = card\{P_j \in \mathcal{P}_n : P_j \in B\},$$

and

$$\mu_0(B) = \mu(B \cap B^d).$$

Let $K$ be a convex body. $M(d)$ and $H(d)$ denote the group of rigid motions and similarities (that is the group generated by $M(d)$ and dilations) of $R^d$, respectively. Let

$$\mathcal{D}_n = \inf_{\mathcal{P}_n \in \Pi_n} \sup_{\gamma \in M(d)} \left| Z_0(\gamma^{-1}K, \mathcal{P}_n) - n \cdot \mu(\gamma^{-1}K \cap B^d) \right|$$

and

$$\mathcal{H}_n = \inf_{\mathcal{P}_n \in \Pi_n} \sup_{\gamma \in H(d)} \left| Z_0(\gamma^{-1}K, \mathcal{P}_n) - n \cdot \mu(\gamma^{-1}K \cap B^d) \right|$$

be defined on the usual way. We call the function $\mathcal{D}_n$ the rotation discrepancy, and $\mathcal{H}_n$ the similarity discrepancy with respect to the convex body $K$.

**Theorem.** (József Beck) *For every convex body $K$ there exists a real number $\mu_K > 0$ that*

$$\mathcal{H}_n(\mu \cdot K) \geq c(K, \mu) \cdot n^{\frac{1}{2} - \frac{1}{2d}}$$

*holds for every* $0 < \mu \leq \mu_0(K)$.

**Theorem.** (T. Ódor) *Let* $K$ *be a convex body with* $(d+1)/2$ *times differentiable boundary and positive Gauss curvature having the Pompeiu property. Then there exists a positive real number* $\nu_0(K) > 0$ *such that*

$$\mathcal{D}_n(\nu \cdot K) \geq c(K,\nu) \cdot n^{\frac{1}{2}-\frac{1}{2d}}$$

*holds for every* $0 < \nu \leq \nu_0(K)$.

So, for using a smaller group we "have to pay" by assuming the Pompeiu property, and with some not so essential differentiability conditions.

József Beck has kindly informed me about his following (unpublished) result:

**Theorem.** (József Beck) *Assume that* $d = 4k+1$ *for a positive integer* $k \in Z_+$. *Then for an infinite sequence of* $n_i \in N$ *the rotation discrepancy of the lattice* $n^{-1/d}Z^d$ *with respect to a ball with radius* $r$ *is not greater than* $C \cdot n_i^{\frac{1}{2}-\frac{1}{2d}} / \log^c n_i$, *where* $c$, $C \in R_+$ *are suitable positive constants.*

His proof uses elementary properties of Bessel functions and rational approximation arguments.

If we could extend the result of J. Beck to convex bodies $\Omega$ not having the Pompeiu property then together with my previous result, at least in dimensions $d = 4k+1$ and for convex $\Omega$'s the Pompeiu problem is equivalent to the usual lower bound $C \cdot n^{\frac{1}{2}-\frac{1}{2d}}$, that is equivalent to a combinatorial problem.

## 2 The extremum method

The hart of the extremum method is the following simple Lemma, which converts the highly nonlinear inverse problems (originally $X$-ray problems) to problems of operator theory and/or spherical harmonics.

Informally, we have two different domains, $\Omega_0$ and $\Omega$. We know that the weight of $\Omega_0$ and $\Omega$ are equals for a certain weight $\alpha$. Using some monotonity assumption with respect to $\alpha$ we can deduce, that the weight with respect to an other weight $\beta$ of $\Omega$ is strictly smaller than the weight of $\Omega_0$. Changing the role of the domains, we get a contradiction, so $\Omega = \Omega_0$.

Because of its importance we state our key lemma explicitly. Let $\alpha : R^d \to R$ and $\beta : R^d \to R$ be continuous functions, and assume that $\beta$ is positive.

Assume that the function $\alpha/\beta(x)$ is strictly monotone increasing on every half line $\omega \cdot R_+$ and constant on the boundary of the domain $\Omega_0$, which is star shaped with respect to the origin $O$.

Let $\gamma$ be an arbitrary continuous function. Let $V_\gamma(\Omega)$ denote the volume of $\Omega$ with respect to the weight $\gamma$, that is

$$V_\gamma(\Omega) = \int_\Omega \gamma(t)\, dt.$$

Let $\mathcal{F}_\beta(\Omega_0)$ be the set of star shaped open domains with respect to the origin $O$ and with their volume $V_\beta(\Omega)$ not smaller than $V_\beta(\Omega_0)$.

**Extremum Lemma.** *For every* $\Omega \in \mathcal{F}_\beta(\Omega_0)$ *for which* $\Omega \neq \Omega_0$, *the inequality* $V_\alpha(\Omega) > V_\alpha(\Omega_0)$ *holds.*

**Remark.** *It is clear that a similar statement is true for strictly monotone decreasing* $\alpha/\beta$.

To apply the Extremum Lemma, we have to construct the weight functions $\alpha$ and $\beta$. To do so, we use continuity properties of the inverse of generalized Radon transforms on the sphere in their defining parameters. At last, our problem is reduced to an image characterization problem, which we can solve using the standard theory of Fredholm operators. In the case of the Pompeiu problem we can make the construction of $\alpha$ and $\beta$ using elementary properties of spherical harmonics.

To prove a variant of the local Pompeiu problem we use only this step that our body $K$ is close to a ball, in some highly analytic metric [3, 10].

## 3 Applications in technology and conclusions

In my approach the Pompeiu problem is a kind of $X$-ray reconstruction problem. $X$-ray reconstruction problems in general have very important applications in several areas of technology and science. For example $X$-ray reconstruction methods are used in computerized tomography, cardiology, robotic vision, pattern recognition, radio astronomy and radar target shape estimation. The field also important from theoretical point of view because several open problems can be stated as (generalized) $X$-ray reconstruction problem.

Among these problems the Pompeiu problem is probably on of the most significant, so it is reasonable to try to apply the new methods in $X$-ray reconstruction to attack it.

One other interesting feature of the applied methods that they work without essential modification in non-compact rank 1 symmetric spaces. At least to my knowledge from the point of view of group representation theory it would be interesting and important to try to generalize them to higher non-compact symmetric spaces, but this seems to be not trivial, because even the most fundamental equivalent forms are not established in the higher rank case.

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